SLIP-LINES IN THE CORNER POINT OF THE MEDIA-SEPARATING BOUNDARY

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ABSTRACT

A question on the initial development of plastic zone in the vicinity of corner point of the boundary separating two homogeneous isotropic media with different elastic constants is considered under the conditions of plane static symmetrical problem (plane deformation). The plastic zone is modelled by two rectilinear cuts - slip-lines, emerging from the corner point. The corresponding boundary problem of the theory of elasticity is reduced to Wiener-Hopf functional equation. The exact analytical solution of the equation is obtained and the stress intensity factor at the end of the slip-line is determined. The length of the slip-lines and the angle of their inclination to the media-separating boundary (the directions of the initial slip-lines development) are found on the base of obtained solution and proposed by Cherepanov selection principle.

KEYWORDS

Medium, cut, slip, stress, energy, dissipation.

Formulation of the Problem. Let media-separating boundary of the region, composed of two homogeneous isotropic parts with different elastic constants, has a corner point. The region is supposed symmetrical relatively to the bisector of this angle. Let us consider the symmetrical relatively to the mentioned bisector a problem on the initial development of plastic zone in the vicinity of the given corner point. We shall model the plastic zone by two straight slip-lines, emerging from the corner point.

Since the length of the slip-lines is small in comparison with the sizes of the region, we come to the plane static symmetrical problem of the theory of elasticity for the piece-homogeneous plane with the media-separating boundary in the form of two rays, having a common initial point, including two straight slip-lines, emerging from the last one.

To one of the parts (region 1), composing the plane, corresponds an angle 2α, Young's modulus E_1 and Poisson's ratio ν_1 and to the other (region 2) - elastic constants E_2, ν_2.

At the infinity an asymptotic is realized, representing itself a satisfying to the condition of stresses attenuation asymptotically the largest solution of analogous problem without slip-lines (problem A). The mentioned solution, which won't be presented here, is built by a singular solution method (Cherepanov, 1979; Parton and Perlin, 1981) and is determined up to arbitrary constant C. This constant is supposed as given according to the formulation of the slip-lines problem under consideration. It characterizes the intensity of external field and is found from the solution of the external problem.

We are to find the length ℓ of the slip-lines and the angle β of their inclination to the media-separating boundary.

Let the slip-lines be located in the region 1. Limiting by consideration of the half-plane $\beta-\alpha \le \theta \le \pi-\alpha+\beta$ $(r,\theta$ - polar coordinates, the pole coincides with the corner point and the polar axis is directed along the slip-line), let us present the boundary conditions

$$\theta=\beta,\ \langle\sigma_\theta\rangle=\langle\tau_{r\theta}\rangle=0,\ \langle u_\theta\rangle=\langle u_r\rangle=0 \tag{1}$$

$$\theta=\beta-\alpha;\ \theta=\pi-\alpha+\beta,\ \tau_{r\theta}=0,\ u_\theta=0$$

$$\theta=0,\ \langle\sigma_\theta\rangle=\langle\tau_{r\theta}\rangle=0,\ \langle u_\theta\rangle=0$$

$$\theta=0,\ r<\ell,\ \tau_{r\theta}=\tau_1;\ \theta=0, r>\ell,\ \langle u_r\rangle=0 \tag{2}$$

$$\theta=0,\ r\to\ell-0,\ \langle\partial u_r/\partial r\rangle\sim -q K_{II}/\sqrt{2\pi(\ell-r)} \tag{3}$$

$$\theta=0,\ r\to\ell+0,\ \tau_{r\theta}\sim K_{II}/\sqrt{2\pi(r-\ell)}$$

$$\theta=0,\ r\to\infty,\ \tau_{r\theta}=Cg r^{\lambda}+o(1/r)$$

$$q=4(1-\nu_1^2)/E_1,\ g=g_1(\alpha,æ_1,æ_2,e,\lambda)\sin\lambda(\alpha-\beta)+$$
$$+g_2(\alpha,æ_1,æ_2,e,\lambda)\sin(\lambda+2)(\alpha-\beta),\ æ_{1,2}=3-4\nu_{1,2}$$
$$e=(1+\nu_2)/(1+\nu_1)e_0,\ e_0=E_1/E_2$$

Here $\sigma_\theta, \tau_{r\theta}, \sigma_r$ are stresses; u_θ, u_r - displacements; $\langle a\rangle$ - jump of the value a; $\tau_1=\tau_S$, if $Cg>0$ (awkward expressions for $g_{1,2}$ are not presented), $\tau_1=-\tau_s$, if $Cg<0$ (τ_s - shear yield point); K_{II} - stress intensity factor at the end of the slip-line, subjected to determining; λ - the only within the interval $]-1;0[$ root of equation

$$\Delta(-\lambda-1)=0,\ \Delta(z)=\delta_2\delta_3+(q_1q_2\sin^2 z\pi-\delta_1\delta_4- \tag{4}$$
$$-\delta_2\delta_3)e+\delta_1\delta_4e^2,\ \delta_1=\sin 2z\alpha+z\sin 2\alpha$$
$$\delta_2=æ_1\sin 2z\alpha-z\sin 2\alpha,\ \delta_3=\sin 2z(\pi-\alpha)-z\sin 2\alpha$$
$$\delta_4=æ_2\sin 2z(\pi-\alpha)+z\sin 2\alpha,\ q_{1,2}=1+æ_{1,2}$$

The equation (4) is a characteristic equation of the canonical singular problem A.

The solution of the problem thus formulated is the sum of the solutions of the two following problems. The first one differs due to the fact that instead of the first condition (2) we have

$$\theta=0,\ r<\ell,\ \tau_{r\theta}=\tau_1-Cgr^{\lambda} \tag{5}$$

and at the infinity stresses are attenuated as $O(1/r)$ (particularly, in the third formula (3) the first addend is absent). The second problem is the problem A (it means its solution, mentioned above). Since the solution of the second problem is known, we are to deal only with the solution of the first one.

<u>Solution of the Wiener-Hopf Fuctional Equation</u>. Making use of the Mellin transform with the parameter p to the equations of equilibrium, the condition of simultaneous deformations, Hooke's law, conditions (1) and taking into account both the second condition (2) and the condition (5), we come to the Wiener-Hopf functional equation (Noble, 1962; Ufland, 1967)

$$\Phi^+(p)+\sum_{j=1}^{2}\tau_j/(p+p_j)=\mathrm{ctg}\,p\pi\, G(p)\,\Phi^-(p) \tag{6}$$
$$(-\varepsilon_1<\mathrm{Re}\,p<\varepsilon_2)$$

$$\Phi^+(p)=\int_1^\infty \tau_{r\theta}(\rho\ell,0)\rho^p d\rho,\ \Phi^-(p)=1/q\int_0^1 \langle\partial u_r/\partial r\rangle_{\substack{r=\rho\ell\\ \theta=0}}\rho^p d\rho$$

$$G(p)=\mathrm{tg}\,p\pi[\Delta_0(p)+\Delta_1(p)e+\Delta_2(p)e^2]/[2\Delta(p)]$$
$$\Delta_0=\delta_3[(q_1^2-4\delta_8)\delta_9-2\delta_6\delta_{10}],\ \Delta_1=\delta_{10}(2\delta_3\delta_6+$$
$$+2\delta_4\delta_5+q_1q_2\delta_{11})-\delta_9[\delta_3(q_1q_2-4\delta_8)-4\delta_4\delta_7-$$
$$-q_1q_2\delta_{12}],\ \Delta_2=-2\delta_4(\delta_5\delta_{10}+2\delta_7\delta_9)$$
$$\delta_5=\sin 2p\beta+p\sin 2\beta,\ \delta_6=æ_1\sin 2p\beta-p\sin 2\beta$$
$$\delta_7=\sin^2 p\beta-p^2\sin^2\beta,\ \delta_8=æ_1\sin^2 p\beta+p^2\sin^2\beta$$
$$\delta_9=\cos 2p(\alpha-\beta)-\cos 2(\alpha-\beta),\ \delta_{10}=\sin 2p(\alpha-\beta)+$$
$$+p\sin 2(\alpha-\beta),\ \delta_{11}=\cos 2p(\pi-\alpha+\beta)-\cos 2(\pi-\alpha+\beta)$$

$$\sigma_{12} = \sin 2p(\pi-\alpha+\beta) + p\sin 2(\pi-\alpha+\beta)$$

$$\tau_2 = -Cgl^{\lambda}, \quad p_1 = 1, \quad p_2 = \lambda+1$$

(ε_j are sufficiently small positive numbers).

The solution of the equation (6) is as follows (Kipnis, 1981, 1986; Kipnis and Cherepanov, 1989):

$$\Phi^+(p) = -K^+(p)G^+(p)\sum_{j=1}^{2}[\tau_j/(p+p_j)]\{1/[K^+(p)G^+(p)] - \tag{7}$$

$$-1/[K^+(-p_j)G^+(-p_j)]\} \quad (\mathrm{Re}\, p<0)$$

$$\Phi^-(p) = pG^-(p)/K^-(p)\sum_{j=1}^{2}\tau_j/[(p+p_j)K^+(-p_j)\times$$

$$\times G^+(-p_j)] \quad (\mathrm{Re}\, p>0)$$

$$\exp\left\{1/(2\pi i)\int_{-i\infty}^{i\infty}[\ln G(z)]/(z-p)dz\right\} = \begin{cases} G^+(p), & \mathrm{Re}\, p<0 \\ G^-(p), & \mathrm{Re}\, p>0 \end{cases}$$

$$K^{\pm}(p) = \Gamma(1\mp p)/\Gamma(1/2 \mp p)$$

($\Gamma(z)$ - Euler's gamma-function).

Making use of (3), (7) and Abel's type theorem (Noble, 1962), we get

$$K_{II} = \sqrt{2}g\Gamma(\lambda+3/2)/[G^+(-\lambda-1)\Gamma(\lambda+2)]Cl^{\lambda+1/2} - \tag{8}$$

$$-\sqrt{\pi}/[\sqrt{2}G^+(-1)]\tau_1\sqrt{l}$$

<u>The Length of the Slip-Lines and the Angle of Their Inclination to the Media-Separating Boundary</u>. Equating the stress Intensity factor to its critical value - slip toughness, which determines the resistance of the material to the development of the slip-lines in it and can be considered as the given constant of it, we obtain the equation for determining of the length of the slip-lines.

Let us assume that the resistance of the material to the development of the slip-lines in it is neglectly small. Then by means of (8) we obtain the following formula for determining of the length of the slip-lines:

$$l = \mathcal{D}(|C|/\tau_s)^{-1/\lambda}, \quad \mathcal{D} = \{2|g|\Gamma(\lambda+3/2)G^+(-1)/ \tag{9}$$

$$/[\sqrt{\pi}\Gamma(\lambda+2)G^+(-\lambda-1)]\}^{-1/\lambda}$$

The angle of the inclination of the slip-lines to the media-separating boundary we shall determine on the base of the selection principle (Cherepanov, 1974). According to this principle, among all possible directions of the slip-line development the direction corresponding to the largest value of the rate of energy dissipation is the only one realized.

Since on the slip-line is valid the first condition (2), according to the selection principle and on the base of (7), (9), among all possible values β as the angle of inclination of the slip-lines to the media-separating boundary to be found that one must be selected, which gives the largest value of the function (the parameter C is supposed as a positive increasing or a negative decreasing function on time)

$$V(\beta) = \left(\tau_1\int_0^l \langle u_n\rangle_{\theta=0}dr\right)^{\cdot} = QW(\beta)F \tag{10}$$

$$Q = \pi/(\lambda+2)\{2\Gamma(\lambda+3/2)/[\sqrt{\pi}\Gamma(\lambda+2)]\}^{-2/\lambda}$$

$$W = |g|^{-2/\lambda}[G^+(-1)]^{-2/\lambda-2}/[G^+(-\lambda-1)]^{-2/\lambda}$$

$$F = q|C|^{-2/\lambda-1}\dot{C}\,\mathrm{sign}\,C/(4\tau_s^{-2/\lambda-2})$$

The point in (10) denotes differentiation in time.

The analysis from the point of view of the largest value of the function $V(\beta)$ and the corresponding function in the case of the slip-lines located in region 2 by various values of the angle α and elastic constants $e_0>1, \nu_1, \nu_2$ allows to make the following conclusions (below the Poisson's ratios are supposed to be fixed).

The slip-lines are located in the region 2.

Let us assume that e_0 are fixed. At $0<\alpha<\alpha_1$ ($\alpha_n = \alpha_n(e_0, \nu_1, \nu_2)$, n = 1,2,3) the slip-lines develop at an angle to the media-separating boundary, which diminishes with the growth of α. At $\alpha_1 \leq \alpha < \alpha_2$ they develop along the media-separating boundary. If the case $\alpha = \alpha_2$ is realized, then from the corner point there emerge four slip-lines, two of which are located at the media-separating boundary, and the other two - at an angle to it. At $\alpha_2 < \alpha < \pi/2$ and $\pi/2 < \alpha < \alpha_3$ the slip-lines and the media-separating boundary again constitute the angle that diminishes when the growth of α, and at $\alpha_3 \leq \alpha < \pi$ the slip-lines develop along the boundary.

Let us assume that e_0 increases. The region $]0;\alpha_1[$ of values of α, at which the initial inclination of the slip-lines from the media-separating boundary takes place, diminishes. At fixed α the angle of the initial inclination of the slip-lines from the media-separating boundary diminishes. Subject to diminishing is the value of α, beginning from which the slip-lines do not deviate from the media-separating boundary.

Let's present some values of the angle of the inclination of the slip-lines to the media-separating boundary, when the value of the angle α is equal to 5°, 30°, 45°, 60°, 120° ($\nu_1 = 0{,}333$, $\nu_2 = 0{,}250$). At $e_0 = 2$ the corresponding values of the angle of the inclination are: 37°, 6°, 88°, 73°, 12°. If in this case the angle α is equal or greater than 135°, then the slip-lines are located along the media-separating boundary.

At $e_0 = 5$ the values of the angle of the inclination are equal 29°, 0°, 86°, 72°, 8°. If the angle α is equal or greater than 130°, the slip-lines are located along the media-separating boundary.

At $e_0 = 10$ the values of the angle of the inclination will be as follows: 20°, 0°, 85°, 72°, 6°. If the angle α is equal or greater than 125°, the slip-lines are located along the media-separating boundary.

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