SEVERAL PLANE PROBLEMS OF BRITTLE FRACTURE MECHANICS FOR CRACKS WITH INTERACTING EDGES

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ABSTRACT

To simulate the phenomena of contact fracture a contact problem with friction for cracked bodies is formulated. The consideration of this problem is given for the case of subsurface cracks. The problem is studied by means of a method which is applicable to a wide family of plane problems with cracks. In addition to the mentioned above different problems for cracked elastic plane and semiplane with free and loaded boundaries are considered. These problems are formulated in terms of inequalities and equations. Partial or full crack edges interacting is possible. The solutions are obtained in the form of uniformly valid asymptotic expansions. For various stressed states the size and the location of the intervals of crack edge interactions, stress intensity factors in the apexes of the cracks, the dependencies of the problem parameters on the distribution of the external stresses, and the crack location, orientation, and size are obtained.

KEYWORDS

Semiplane, crack, stress intensity factor, contact problem, lubrication.

METHOD OF INVESTIGATION

The following problems for elastic bodies with straight cracks under the assumption of the absence of friction stresses between crack edges are considered: 1. Stressed plane with cracks; 2. Stressed plane containing a main crack surrounding a number of microcracks; 3. Stressed semiplane with loaded (free) boundary containing subsurface cracks; 4. Plane contact problem with friction for bodies containing subsurface cracks; 5. Directions of crack propagation in a stressed plane.

For these situations the relations between the strain jumps of the crack edges u_n and v_n ($u_n=u_n^+-u_n^-$ and $v_n=v_n^+-v_n^-$ - tangential and normal to the n-th crack edges direction strain jumps), normal p_n and tangential τ_n contact stresses between crack edges and acting on the crack external normal p_n^0 and tangential τ_n^0 stresses are given by the equations (see Fig.1)

$$\int_{-l_n}^{l_n} \frac{g_n'(t)\,dt}{t-x_n} + \sum_{k=1,k\neq n}^{N} \int_{-l_k}^{l_k} [g_k'(t) K_{nk}(t,x_n) + \overline{g_k'}(t) L_{nk}(t,x_n)]\,dt = \pi P_n(x_n),\ -l_n \le x_n \le l_n\ (n=1,2,\ldots,N). \tag{1}$$

$$g_n(x_n) = -\frac{iE}{4(1-\nu^2)}[u_n(x_n)+iv_n(x_n)],\ P_n(x_n) = p_n(x_n)+p_n^0(x_n)-i\tau_n^0(x_n), \tag{2}$$

$$K_{nk}(t,x_n) = \frac{1}{2}e^{i\alpha_k}\left(\frac{1}{T_k-X_n}+\frac{e^{-2i\alpha_n}}{\overline{T_k}-\overline{X_n}}\right),\ T_k = te^{i\alpha_k}+z_k^0,\ X_n = x_ne^{i\alpha_n}+z_n^0,$$
$$L_{nk}(t,x_n) = \frac{1}{2}e^{-i\alpha_k}\left[\frac{1}{\overline{T_k}-\overline{X_n}}-e^{-2i\alpha_n}\frac{T_k-X_n}{(\overline{T_k}-\overline{X_n})^2}\right]. \tag{3}$$

$(\tau_n(x_n)\equiv 0$ because of the absence of the friction stresses between the crack edges). Here N - total crack number; x_n and l_n - local coordinate and semilength of the n-th crack; E and ν - Young's modulus and Puasson's coefficient of the material (The given equations describe the state of plane deformation.) To describe the plane stressed state it is sufficient to replace ν in all of the formulas by $-0.5\nu/(1-\nu)$.); α_n and z_n^0 - angle of inclination and the coordinates of the center of the n-th crack, $z_n^0=x_n^0+iy_n^0$; i - imaginary unit. The overline sign means the operation of taking the conjugate of the given expression.

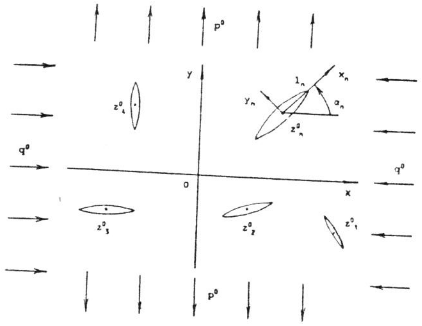


Fig.1

On the edges of each subsurface crack the following conditions, written in the form of alternative equalities and inequalities, take place:

$$p_n(x_n)=0,\ v_n(x_n)>0;\quad p_n(x_n)\le 0,\ v_n(x_n)=0. \qquad (4)$$

In the subsurface crack's apexes the following conditions are valid:

$$g_n(\pm l_n)=0. \qquad (5)$$

In the crack's apexes opened to the surface the conditions depend on the particular problem (for example, see Kudish (1984)).

The following assumptions are common for Problems 1,3-5: the crack sizes are small in comparison with distances between the cracks, with distances between cracks and the semiplane boundary, and with the region size of the load acting on the boundary of the semi-plane. For Problem 2 it is supposed that the microcracks' sizes are much smaller than the distances between them and the distances between them and the main crack.

The above assumptions allow to introduce in (1)-(3) a small parameter δ_0 and to get the following asymptotic expansions for the kernels K_{nk}, L_{nk} and the given terms of the equations (1)-(3):

$$[\,K_{nk}(t,x_n),L_{nk}(t,x_n)\,]=\sum_{j+m=0;j,m\ge 0}^{\infty}\delta_0^{j+m}x_n^j t^m[\,K_{nkjm},L_{nkjm}\,], \qquad (6)$$

$$[\,p_n^0(x_n),\tau_n^0(x_n)\,]=\sum_{m=0}^{\infty}(\delta_0 x_n)^m[\,p_{nm},\tau_{nm}\,]. \qquad (7)$$

The expressions (6),(7) give an opportunity to search for the problem solution in the form of uniformly valid asymptotic expansions:

$$[\,v_n,u_n,p_n\,]=\sum_{m=0}^{\infty}\delta_0^m[\,v_{nm},u_{nm},p_{nm}\,]. \qquad (8)$$

The main difficulty in the solution is the analysis of the system of alternative equalities and inequalities following from (4),(8):

$$\sum_{m=0}^{\infty}\delta_0^m p_{nm}(x_n)=0,\quad \sum_{m=0}^{\infty}\delta_0^m v_{nm}(x_n)>0;$$
$$\sum_{m=0}^{\infty}\delta_0^m p_{nm}(x_n)\le 0,\quad \sum_{m=0}^{\infty}\delta_0^m v_{nm}(x_n)=0. \qquad (9)$$

To surmount this difficulty all possible cases (for example, $v_{n0}(x_n)>0$, $v_{n0}(x_n)=0$, $v_{n1}(x_n)>0$, $v_{n1}(x_n)=0$, etc.) are considered under assumption $\delta_0<<1$. Under certain conditions, some contact regions between crack edges can appear. The boundaries b_{nk} of these regions can be found in the form of the following expansions:

$$b_{nk}=\sum_{m=0}^{\infty}\delta_0^m\beta_{nkm},\quad k=1,2,\ldots,K. \qquad (10)$$

The equations which are valid at the ends of the crack contact regions

$$p_n(b_{nk})=0,\quad k=1,2,\ldots,K. \qquad (11)$$

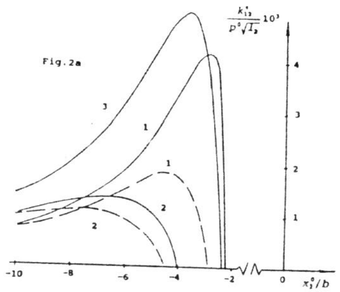
are used to determine the constants β_{nkm}.

As a result of the solution, the normal $k_{1n}^{\pm}$ and shear $k_{2n}^{\pm}$ stress intensity factors at the apexes $x_n=\pm l_n$ of the n-th crack can be obtained according to the formula

$$k_{1n}^{\pm}+ik_{2n}^{\pm}=\mp\lim_{x_n\to\pm l_n}\frac{E}{4(1-\nu^2)}\sqrt{\frac{l_n^2-x_n^2}{l_n}}\,[\,v_n'(x_n)+iu_n'(x_n)\,], \qquad (12)$$
$$n=1,2,\ldots,N$$

The expressions for $k_{1n}^{\pm}$ and $k_{2n}^{\pm}$ can be also represented in the form of uniformly valid expansions for $\delta_0<<1$.

Problem 1. Let us introduce a small parameter $\delta_0=l_0/b\ll 1$ (l_0 - semilength of the maximum size crack, b - minimum distance between crack centers), $p_n^0-i\tau_n^0=-\frac{1}{2}p^0[1+\eta+(1-\eta)\exp(-2i\alpha_n)]$, $\eta=q^0/p^0$ (q^0 and p^0 -stresses at the infinity acting in the x- and y-directions. Some solutions of the problem for stress intensity factors $k^+_{12}(x^0_2)$ (graphs ——) and $k^-_{12}(x^0_2)$ (graphs - -) occuring under stresses $p^0=1$, $q^0=0$ in the case of two cracks: $z^0_1=(0,0)$, $y^0_2/b=-1$, $\alpha_1=0$, $\alpha_1=\pi/2$, $\delta_1=\delta_2=0.4$ (curves 1) and in the case of three cracks: $z^0_1/b=(0,1)$, $z^0_3/b=(0,-1)$, $\alpha_1=\alpha_3=0$, $\alpha_2=\pi/2$, $\delta_1=\delta_3=0.4$, $\delta_2=0.2$, $y^0_2/b=-2$ (curves 2) and $y^0_2=0$ (curve 3) are given in Fig.2.



Problem 2. Let us introduce a small parameter $\delta_0=l_{max}/l_0\ll 1$ (l_{max} and l_0 - semilength of the maximum size micro- and macrocrack, $p_n^0-i\tau_n^0=-\frac{1}{2}p^0[\eta_p+\eta_q+(\eta_p-\eta_q)\exp(-2i\alpha_n)]$, $\eta_p=p^0/q$, $\eta_q=q^0/q$, ($q=p^0$ for $p^0\neq 0$, $q=q^0$ for $p^0=0$); p^0 and q^0 -

stresses at the infinity acting in the y- and x-directions. The stress intensity factors $k^{\pm}_{11}$ and $k^{\pm}_{21}$ for one microcrack with $\alpha_1=0$, $\delta_1=0.1$, $y^0_1/l_0=0.25$ (curve 1) and $y^0_1=0.5$ (curve 2) are presented in Fig. 3a,c and for $\alpha_1=\pi/2$ in Fig.3b. For the given around macrocrack microcrack distribution (Fig.4) and stressed state $p^0=1$, $q^0=0$ the calculated macrocrack stress intensity factors $k^+_{10}(x^0_1)$ (graphs ——) and $k^-_{10}(x^0_1)$ (graphs - -) are shown in Fig.5. The presented results are obtained for the following microcrack parameters: $\delta_n=0.1$ ∀n, $y^0_1=-y^0_3=0.25l_0$, $y^0_2=-y^0_4=0.5l_0$ and (1) $x^0_2-x^0_1=0.5l_0$, (2) $x^0_2-x^0_1=0$, (3) $x^0_2-x^0_1=x^0_4-x^0_3=0.5l_0$, $x^0_3-x^0_1=0$, (4) $x^0_2-x^0_1=x^0_4-x^0_3=x^0_3-x^0_1=0.1l_0$

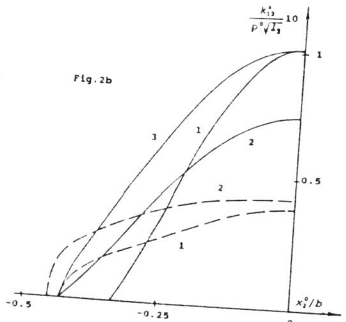


Fig.2b

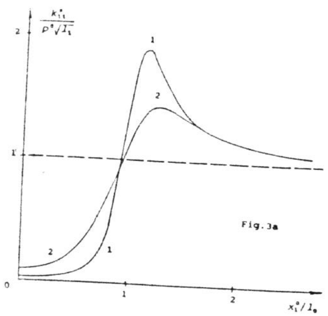


Fig.3a

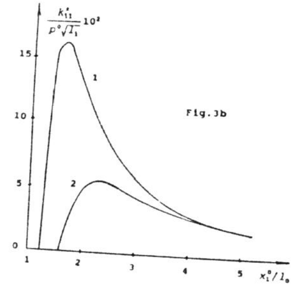


Fig.3b

Problem 3. Let us introduce a small parameter $\delta_0=l_0/d\ll 1$ (l_0 - semilength of the maximum size crack; $d=\min(d_0,d_1,b)$, d_0 and d_1 - minimum distances between crack centers and between crack centers and the semiplane boundary, b - semiwidth of the interval to which the external pressure p(x) and friction stresses τ(x) are applied (in the case of a loaded semiplane boundary d=b). The external stresses acting on the n-th crack edges are of the following form (Kudish, 1986,1987)

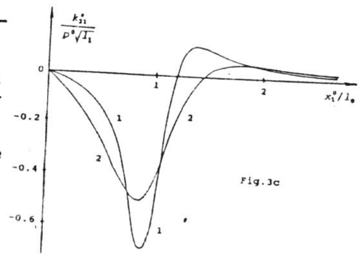


Fig.3c

$$p^0_n-i\tau^0_n=-\frac{1}{\pi}\int_a^c[p(t)D_n(t,x_n)+\tau(t)G_n(t,x_n)]dt-\frac{1}{2}q_0(1-e^{-2i\alpha_n}),$$

$$D_n(t,x_n)=\frac{1}{2}\left[-\frac{1}{t-X_n}+\frac{1}{t-\overline{X}_n}-e^{-2i\alpha_n}\frac{\overline{X}_n-X_n}{(t-\overline{X}_n)^2}\right], \qquad (13)$$

$$G_n(t,x_n)=\frac{1}{2}\left[\frac{1}{t-X_n}+\frac{1-e^{-2i\alpha_n}}{t-\overline{X}_n}-e^{-2i\alpha_n}\frac{t-X_n}{(t-\overline{X}_n)^2}\right],$$

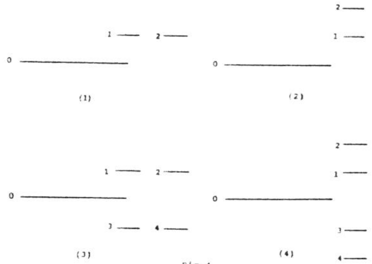


Fig.4

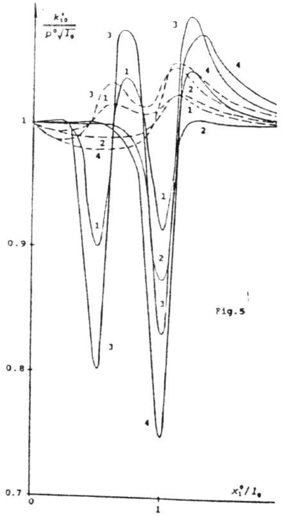


Fig.5

where q^0 - stresses at the infinity acting on the semiplane in the x- direction, [a,c] - interval of acting the stresses p(x) and τ(x). Let c=-a= b, $\delta_0=0.1$. For the cases of $\tau(x)=-\lambda\tau_0(1-x^2/b^2)^{1/2}$, $\lambda=0.1$, $p(x)\equiv q^0=0$, $y^0_n/b=-0.2$, $\delta_0=0.1$, $\alpha_n=0$ (curve 1) and $\alpha_n=\pi/2$ (curve 2) the stress intensity factors $k^+_{2n}(x_{0n})$ and $k^+_{1n}(x_{0n})$ are shown in Fig.6a,b. For the cases of

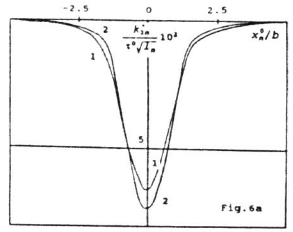


Fig.6a

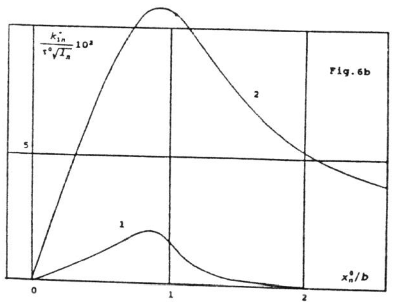


Fig.6b

$p(x)=\frac{1}{2}p_0(1-x^2/b^2)^{-1/2}$, $\tau(x)=-\lambda p(x)$, $y^0_n/b=-0.2$, $\alpha_n=\pi/2$: (1) $q^0=0$, $\lambda=0.1$; (2) $q^0=0$, $\lambda=0.2$; (3) $q^0/p_0=-0.015708$, $\lambda=0.1$; (4) $q^0/p_0=0.0314$, $\lambda=0.1$the stress intensity factors $k^+_{2n}(x^0_n)$ and $k^+_{1n}(x^0_n)$ are shown in Fig.7a,b.

Problem 4. The parameters p^0_n,τ^0_n and $\delta_0\ll 1$ will be defined as in (12) (see Problem 3). In comparison with Problem 3, the difference is that now the pressure p(x) and friction stresses τ(x) are unknown. Under Hertzian assumptions and the assumption that $\tau(x)=-\lambda p(x)$ (λ - friction coefficient), the pressure is determined by the equations:

$$\frac{1-\nu^2}{\pi E}\int_a^c p(t)\ln\frac{1}{|x-t|}dt+\frac{(1-2\nu)(1+\nu)}{E}\int_a^x\tau(t)dt- \qquad (14)$$

$$\frac{1}{4\pi}\sum_{k=1}^{N}\int_{-l_k}^{l_k}[v'_k(t)W^r_k(t,x)-u'_k(t)W^i_k(t,x)]dt=\delta_0-f(x),$$

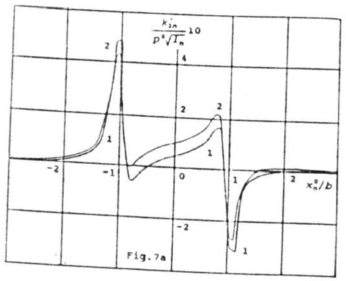


Fig.7a

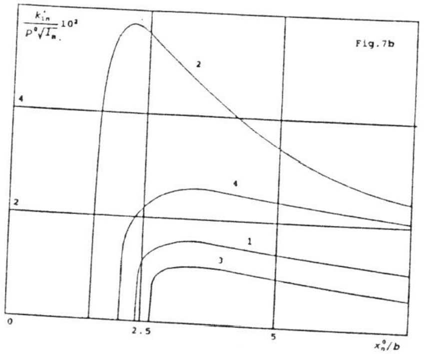


Fig.7b

$$W_k(t,x)=i\exp(-i\alpha_k)\frac{\overline{T_k}-T_k}{\overline{T_k}-x},\quad W_k^r=\mathrm{Re}W_k,\quad W_k^i=\mathrm{Im}W_k;\quad \int_a^c p(t)\,dt=P, \qquad (15)$$

where δ^0 - bar settling, f(x) - shape of the bar bottom, P -external load acting on the bar.

If the pressure in the contact region is bounded, it is necessary to add to (14),(15) the following conditions

$$p(a)=p(c)=0. \qquad (16)$$

The solution for p(x),a and c can be found for $\delta_0 \ll 1$ in the form of uniformly valid asymptotic expansions. For $\tau(x)=-\Lambda p(x)$ it can be shown that $p(x)=p_0\cos\pi\gamma(b_0+x/b)^{1/2-\gamma}(b_0-x/b)^{1/2+\gamma}+O(\delta_0^2)$, $b_0=(1-4\gamma^2)^{-1/2}$, $\gamma=1/\pi\ \arctan\Lambda/\pi$ ($p_0=2P/(\pi b)$, P - total load per unit of length). Some solutions for $y^0_n/b=-0.2$, $\alpha_n=\pi/2$, $\delta_0=0.1$ and: (1) $q^0=0$, $\Lambda=0.1$; (2) $q^0=0$, $\Lambda=0.2$; (3) $q^0/p_0=0.04$, $\Lambda=0.1$; (5) $q^0/p_0=0.02$, $\Lambda=0.2$; (4) $q^0/p_0=-0.01$, $\Lambda=0.1$; (5) $q^0/p_0=-0.03$, $\Lambda=0.2$ are represented in Fig.8.

Problem 5. In general the crack growth rate depends on the stress state near the crack apex and the crack size as well as on the crack orientation. The latter is extremely important for the analysis of the fatigue crack growth which determines the body's contact fatigue.

The angle $\Theta_{*n}^{\pm}$ determining the inclination of the crack growth satisfies the following equality:

$$\Theta^{\pm}_{*n}=2\arctan\frac{k^{\pm}_{1n}-\sqrt{(k^{\pm}_{1n})^2+8(k^{\pm}_{2n})^2}}{4k^{\pm}_{2n}}. \qquad (17)$$

Iterative calculations of the fatigue crack growth taking into account the changes of the growth direction

$$\alpha_n^{(k+1)}=\alpha_n^{(k)}+\Theta_{*n}^{(k)},\quad \alpha_n^{(0)}=\alpha_n \qquad (18)$$

(the angle $\Theta_{*n}^{(k)}$ can be determined by (17), k=0,1,... number of iterations) lead to a polygonal trajectory of the crack growth. Nevertheless, in the limit this process lead (actually after several iterations) to $\alpha_n=\lim_{k\to\infty}\alpha_n^{(k)}$ which satisfies to the equation

$$k^{\pm}_{2n}=0. \qquad (19)$$

The equation (19) follows from the equality $\lim_{k\to\infty}\Theta_{*n}^{(k)}=0$. A set of experimental data (Parton and Morozov, 1985; Finkel,1977; Solncev and Morozov, 1978; Finkel,1970) confirms that during the growth, cracks tend to orient in order to satisfy the equation (19).

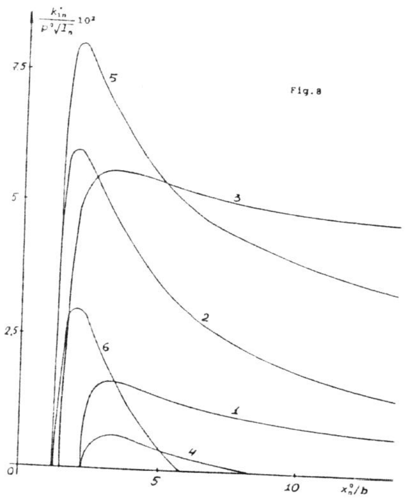


Fig.8

Let us consider a plane containing N straight cracks and try to determine the limit cracks' orientation in the above mentioned meaning. Using the expressions for $p_n^0(\alpha_n),\tau_n^0(\alpha_n)$ (see Problem 1) and

$$k^{\pm}_{2n}=-\tau^0_n-\delta_0^2 c^i_{n001}+\dots,\quad c^i_{n001}=\mathrm{Im}c_{n001},\quad \delta_n=\frac{l_k}{d_o},$$

$$c_{n001}=-\frac{1}{2}\sum_{k=1}^{N}\left(\frac{\delta_k}{\delta_0}\right)^2[p_k^0\theta(-p_k^0)A_{nk01}-\tau_k^0B_{nk01}], \qquad (20)$$

the angle α_n can be found in the form (A_{nk01} and B_{nk01} - corresponding coefficients of asymptotic expansions of the following kernels $A_{nk}=\overline{K_{nk}+L_{nk}}$, $B_{nk}=\overline{i(K_{nk}-L_{nk})}$)

$$\alpha_n=\sum_{k=0}^{\infty}\delta_0^k\alpha_{nk} \qquad (21)$$

From equations (20) for α_{nk} (see (21)) the following equations can be received ($\eta=q^0/p^0$, see Problem 1)

$$\tau^0_n(\alpha_{n0})=0,\quad \alpha_{n1}=0,\quad \dots,\quad (n=1,2,\dots,N);\ \eta\neq 1, \qquad (22)$$

$$c^i_{n001}(\alpha_{10},\dots,\alpha_{n0},\dots,\alpha_{N0})=0,\quad \dots,\quad (n=1,2,\dots,N);\ \eta=1, \qquad (23)$$

For $\eta\neq1$ from (22) we get two values of α_{n0}: $\alpha_{n0}=0$ and $\alpha_{n0}=\pi/2$. Hence, for nonsymmetric stressed state ($\eta\neq1$) there are two marked-out directions of crack orientation, the angle between which is with accuracy $O(\delta_0^2)$ equal to $\pi/2$. It might be an explanation for the experimentally stated curves of crack developing in different materials (see Finkel,1977,pp.150,168,175; Solncev and Morozov, 1978,pp.36,37,40-42) in nonsymmetric stressed state.

To determine N angles α_{n0} for symmetric stressed state ($\eta=1$) it is necessary to consider the system of N nonlinear equations (23). The solution of this system depends on the crack relative location and size and, in general, might be non-unique. The described mechanism of crack developing can be an explanation of the radiant fracture of the non-hardened glass during or after impact (Solncev and Morozov,1978).

The possibility of some limit crack orientations can be treated as a model of the branching process.

The considered Problems 1-4 can be solved in an analogous way by taking into consideration the friction between the interacting crack edges or liquid containing the crack voids. Besides, in the case of absence of the crack edge interactions, the problem for unbounded three-dimensional elastic space containing differently oriented and located circular plane cracks can be solved using a similar procedure.

The problem for lubricated elastic bodies taking into account a lubricant which can penetrate into cavities of surface opened cracks is considered in authors paper (Kudish,1984).

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