ON STRESS STATE OF FIBROUS COMPOSITE WITH CRACK IN MATRIX

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ABSTRACT

The paper is devoted to analytical investigation of stress- strain state of fibrous composite with matrix crack for the plane and space problems. At infinity composite is subjected to axial stretching coinciding with fibre direction.The crack is perpendicular to load and axis of fibre penetrates through its center. Fibre is treated as unidimensional elastic beam. Its contact with matrix is accomplished along the line with plane problem and on cylindrical area in the case of space problem. In the plane cases the questions of interaction of fibre are touched upon. The asymptotic method is permitting to split the stress-strain state of matrix on components is used. Their determination can be limited by simpler problems.

KEYWORDS

Fibrous, composite, stress state, matrix, ,crack, analitical method.

INTRODUCTION

Plane problems on influence of strengthening elements on stress state of isotropic plate with crack were considered by Bloom et al.(1966), Cherepanov et al.(1969), Greif et al.(1965), Sanders (1959) by means of analitical and numerical methods. It was shown, that effect of strengthening element was essential, when it was not far from crack end.Maximum decrease of stress near crack top takes place, when crack stretches over strengthening element slightly, thus, study of interaction of strengthening element and symmetric about it crack is of great interest. Such case was considered Pavlenko et al.(1981a) for orthotropic plate. Up to the present time such space problems were practically not investigated, as they contain considerable mathematical difficulties. Besides ,the model of unidimensional elastic inclusion in combination with model of contact along line is not applicable immediately in space problems for solids with elastic inclusions,having small cross-sections (Sternberg, 1970).

In this paper the plane and space problems on stress state of fibrous composite with crack in matrix are considered. Composite is stretched along fibres, crack is perpendicular to the load action direction and axis of fibre passes through its center. The questions on interaction of fibres are touched upon in the plane problem. In the space problem the attention is concentrated on one inclusion with constant circular cross-section. It is assumed, that matrix is orthotropic in common case, the main directions of anisotrophy coincide either with Dekart coordinate axises x,y or with cylindrical r,θ,z.

ON THE METHOD OF INVESTIGATION

During the investigation of complicated problems of anisotropic and composite materials one has to deal with the necessity of simplification of corresponding differential equation system. However, perturbation method allow to obtain well-founded approximate equations and to estimate the field of application for different geometrical or physical hypotheses (Manevich et al., 1982, 1991; Pavlenko 1979, 1980, 1981). The stress-strain state of anysotropic solid is splitted on components having various properties. The determination of the each from the components is reduced to successive decision of boundary problems of potential theory. The corresponding recurrent processes connected with boundary conditions, are constructed. Two types of components are distinguished in the plane problem for orthotropic solid. If the main directions of anisotrophy coincide with Dekart coordinate axises x,y, then axis displacement u, corresponding axis tension σ_1 and the stress tangential component τ_1 play the decisive role and is determined from the equations in the first approximation:

$$E_1 u_{xx} + G u_{yy} = 0$$

$$\sigma = E_1 u_x \,, \quad \tau_1 = G u_y \tag{1}$$

The displacement v, stress σ_2 and component of tangential stress τ_2 are the main components of stress state of second type, they are determined in the first approximation out of the equations

$$G v_{xx} + E_2 v_{yy} = 0$$

$$\sigma_2 = E_2 v_y \,, \quad \tau_2 = G v_x \tag{2}$$

The complete tangential stress $\tau=\tau_1+\tau_2$. Here u, v are the com- ponents of displacement,E_1, E_2 are modulus of elasticity along the main directions x,y, G is a modul of shear, indexes x,y mean differentiation on corresponding coordinate. After the decision of equation (1) the boundary conditions for displacement u and stress σ_1 are satisfied, but discrepancies for displacement v and tangential stress τ arise, these discr- epancies are taken off by the solution of the equation (2). However, discrepancies for u and σ_1 are resulted, but they have higher order. These discrepancies are taken off by the solution of second approximation equation of stress state for the first type. On every stage it is necessary to integrated only equations (1), (2).

Equations (1),(2) are the generalization of applied engineering practice model. In this model one of the components of displacement is assumed equal to zero and, consequently, boundary conditions for this component and for tangential stresses cannot be satisfied. In so far as the system order (1),(2) is equal to order of initial system then all boundary conditions can be fulfilled approximately. Besides that, the decision of the first approximate equation allows to makeing more precise rate and consequently the limits of it application can be determined. Corresponding space problems of orthotropic solids mechanics in Dekart coordinats x,y,z at each step are reduced to subsequently boundary problems solution of potential theory. Two types of stress-srtrain state stand out in orthotropic solid with cylindrical anisotrophy in the conditions of axial symmetry.Determination of them comes to subsequent solution at the each stage equation

$$E_3 \frac{\partial^2 w}{\partial z^2} + G \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) = 0 \tag{3}$$

$$E_1 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) - E_2 \frac{u}{r^2} + G \frac{\partial^2 u}{\partial z^2} = 0 \tag{4}$$

In the first approximate corresponding stresses are given by formulars:

$$\sigma_3 = E_3 \, \partial w / \partial z \,, \quad \tau_1 = G \, \partial w / \partial r \tag{5}$$

$$\sigma_1 = E_1 \, \partial u / \partial r \,, \quad \sigma_2 = E_2 \, u / r \,, \quad \tau_2 = G \, \partial u / \partial z \,; \qquad \tau = \tau_1 + \tau_2 \tag{6}$$

The proposed method allowed to extend essentially the scope of problems, which can be solved analytically. Generalization of the method is carried out by Kagadiy et al. (1992) for viscoelastic material.

THE PLANE PROBLEM

Let us consider stress-strain state of fibrous composite with crack in matrix. At infinity composite is subjected to uniaxis codirected with fibrous stretching with intensivity σ_0.The crack is perpendicular to load and stretches for equal length on both sides of fibre through its center. Fibre is treated as unidimensional elastic beam. The contact scheme passes along the line. Matrix is orthotropic, the main directions of anisotrophy coincide with Dekart coordinate axises x_1 , y_1 . It is assumed, that neighbouring fibres do not influence on stretching along fibre on the infinite orthotropic plate with symmetric crack. Fibre is placed along axis x, crack is placed on the line ($x_1 = 0, -l \leq y_1 \leq l$). The method described above is used for the solution.At the first stage we come to integration of equayion (1) with the following boundary conditions

$$\sigma_1 = 0 \quad (x_1 = 0, \; -l \leq y_1 \leq l) \,, \quad \sigma_1 = \sigma_0 \quad (|x_1| \to \infty)$$

$$u = 0 \quad (x_1 = 0, \; y_1 = 0; \; x_1 = 0, \; y_1 \geq l) \,, \quad u = u_s \; (y_1 = 0)$$

where u is displasement of fibrous. Equation (1) satisfies, if the function F will be introduced:

$$\sigma_1 = F_y \,, \quad \tau = -F_{x_1} \tag{7}$$

Component v of displacement vector equals zero with $y_1 = 0$ (from the conditions of symmetry) then $\tau=\tau_1$ along fibre. Stress in fibre must be $\sigma_1 E/E_1$, and beam effort P* for cross-section $x_1 = 0$ is given by formular (from the requirement of equilibrium)

$$P^* = \sigma_1 A \frac{E}{E_1} + 2 \int_0^{x_1} + \tau_1 \, dx_1 = A \frac{t}{E_1} \frac{\partial F}{\partial y_1} - 2t \int_0^{x} \frac{\partial F}{\partial x_1} = A \frac{E}{E_1} \frac{\partial F}{\partial y_1} - 2tF \tag{8}$$

where E,A - are moduls of elasticity and area of fibre cross- section, t is thickness of matrix. Undimensional variabilities are introduced later.

$$x=\sqrt{\frac{G}{E_1}}\,\frac{x_1}{\ell},\quad y=\frac{y_1}{\ell},\quad F=\sigma_0\ell\varphi,\quad u=\frac{\sigma_0\ell}{\sqrt{E_1G}}\psi,\quad \sigma_1=\frac{\partial\varphi}{\partial y},\quad \tau_1=-\sigma_0\sqrt{\frac{G}{E_1}}\,\frac{\partial\varphi}{\partial x} \qquad (9)$$

we shall consider the field $y\geqslant 0$, as the problem is symmetric. From the equations (1),(7),(9) it follows, that

$$\varphi_x=\psi_y\ ,\quad \varphi_y=-\psi_x$$

Last equalities are the Koshy-Ryman equations, so as function

$$\Phi(z)=\varphi+i\psi \qquad \left(i=\sqrt{-1}\ ,\ z=x+iy\right)$$

is the analytical function of complex variable z. From the equality (8) it follows, that

$$\varphi_y-\mu\varphi=P \qquad (10)$$

where $\mu=2t\ell E_1/EA$, $P=P^{*}E_1/\sigma_0 EA$

As far as the stress σ_1 is zero on crack, then

$$\varphi=0 \qquad (x=0,\ 0\leqslant y\leqslant 1) \qquad (11)$$

From the condition u=0 in the point (x=0,y=0) and on the line (x=0,y$\geqslant$1) we obtain:

$$\psi=0 \qquad (x=0\ ,\ y=0)\ ,\ (x=0\ ,\ y\geqslant 1) \qquad (12)$$

Thus, the decision of the problem is reduced to definition of analytical in upper semi-surface function $\Phi(z)$, which meets the conditions (10)-(12). Function $\Phi(z)$ must meet the additional condition (from the condition at infinity):

$$\Phi(z)\backsim -iz \qquad (z\rightarrow\infty) \qquad (13)$$

As a result we obtain

$$\Phi(z)=-\frac{P}{\mu}-i\sqrt{z^2+1}+e^{-i\mu z}\int\limits_{i\infty}^{z}\left(\frac{2P}{\pi}\ln\frac{\sqrt{\tau^2+1}-1}{\tau}+\frac{\tau i+C}{\sqrt{\tau^2+1}}\right)e^{i\mu\tau}d\tau \qquad (14)$$

Under the conditions (11), (12) it follows that $\Phi(0)=0$, $\Phi(i)=0$. It leads to equation system for P and C, decision of which is as follows:

$$P=\frac{\pi\mu}{2}\,\frac{B_1K_1+B_2K_0}{BK_0+B_1\int\limits_{\mu}^{\infty}K_0(s)ds} \qquad (15)$$

$$C=\frac{BK_1-B_2\int\limits_{\mu}^{\infty}K_0(s)\,ds}{BK_0+B_1\int\limits_{\mu}^{\infty}K_0(s)\,ds} \qquad (16)$$

where

$$B(\mu)=\frac{\pi}{2}\int\limits_{0}^{\mu}\left[I_0(s)-L_0(s)\right]ds\ ,$$

$$B_1(\mu)=\frac{\pi}{2}\left[I_0(\mu)-L_0(\mu)\right],\quad B_2=\frac{\pi}{2}\left[I_1(\mu)-L_1(\mu)\right]$$

$I\nu(x)$, $K\nu(x)$ are modify Bessel functions, $L\nu(x)$ is Struve function. Function $\Phi(z)$ (14), displacement u, stresses σ_1 and τ_1 are defined by functions P and C with formula (9). In particular:

$$\tau_1=-\left(\sigma_0\sqrt{G/E_1}\right)\partial\varphi/\partial x=-\left(\sigma_0\sqrt{G/E_1}\right)\mathrm{Re}\,\Phi'(z)$$

On the line (x=0)

$$\mathrm{Re}\,\Phi'(z)=\omega(y) \qquad (y<1)\ ;\quad \mathrm{Re}\,\Phi'(z)=0 \qquad (y>1)$$

where

$$\omega(y)=\frac{C}{\sqrt{1-y^2}}-\frac{2P}{\pi}e^{\mu y}\int\limits_{y}^{1}\frac{e^{-\mu\xi}d\xi}{\xi\sqrt{1-\xi^2}}-\mu e^{\mu y}\int\limits_{y}^{1}\frac{C-\xi}{y\sqrt{1-\xi^2}}e^{-\mu\xi}d\xi \qquad (17)$$

From this, tangentional stresses do not meet zero boundary conditions on crack. The discrepancy is taken off by the decision of equation (2) under the following boundary conditions

$$\tau_2=-\tau_1(x=0)\ ,\quad v=0 \qquad (y=0) \qquad (18)$$

All functions appeal to zero at infinity. The decision of the problem (2), (18) is as follows:

$$v(x_2,y)=-\frac{2}{\pi}\frac{\sigma_0\ell}{\sqrt{E_1E_2}}\int\limits_{0}^{\infty}\frac{\omega^{*}(s)}{s}e^{-x_2 s}\sin ys\,ds$$

where

$$\omega^{*}(s)=\int\limits_{0}^{1}\omega(y)\sin sy\,dy\ ,\quad x_2=\sqrt{\frac{E_2}{G}}\,\frac{x_1}{\ell}$$

Now tangential stresses τ meets the zero boundary conditions along line (x_1 0). Coeffisient C (which is determined from equation (16)) is ratio of coefficients of stress intensivity in the crack top for the matrix with fibre and without it. If in cross-section $x=x_1=0$ fibre is torn, then coefficient C is different from (16) (we shall mark this coeffisient C*). As C*$\gg$C, then the advancement of crack in matrix is possible only with broken fibre. If the fibre is not broken, then the question on destruction will be connected with determination of adhesive firm of junction of matrix with fibrous. The influence of neighbouring fibres on stress state around crack can be taken into consideration in the following way. Let us assume, that fibres (they are disposed on the distance b_1 from central fibres) are not displase along axis y_1 with deformation ($v=0, y_1=b_1$) it is problem periodical in direction y_1. That is why we can consider one period and the field $x_1\geqslant 0, 0\leqslant y_1\leqslant b_1$, $b_1>1$ for symmetric problem. The boundary conditions for the equation (1) are $\sigma_1=0(x_1=0, 0\leqslant y_1\leqslant \ell), u=0(x_1=y_1=0; \ell\leqslant y_1\leqslant b_1), u=u_s(y_1=0; y_1=b_1)$,

$\sigma_1 = \sigma_0$ $(x_1 \to \infty)$ The condition of eqilibrium of central fibrous leads to equation (10), and of neighbour fibre leads to the equation: $\varphi_y - \mu\varphi = P_1$ $(y=B)$, $B = B_1/l$

If we introduce function

$$f(z) = \Phi'(z) + i\mu\Phi(z) + i(P+P_1) \tag{19}$$

then boundary problem for it may be formulated as follows:

$$\operatorname{Re} f(z) = 0 \quad (x=0,\ 1<y<b),\quad \operatorname{Im} f(z) = P_1 \ (y=0),\quad \operatorname{Im} f(z) = P \ (y=b),$$
$$\operatorname{Im} f(z) = P + P_1 \quad (x=0,\ y<1),\quad f(z) \sim \mu z \qquad (x \to \infty).$$

The decision of this problem may be found by reflection of considered semy-strip to upper semy-surfase by means of the function

$$\zeta = \xi + i\eta = \operatorname{ch} z_* = \operatorname{ch} x_* \cos y_* + i \operatorname{sh} x_* \sin y_*$$
$$z_* = \pi z/b,\quad x_* = \pi x/b,\quad y_* = \pi y/b.$$

The decision of equation (19) is

$$\Phi(\zeta) = \left(\zeta + \sqrt{\zeta^2-1}\right)^{-i\mu b/\pi} A_1(\zeta),$$
$$A_1(\zeta) = \frac{b}{\pi}\int_{-\infty}^{\zeta} \frac{\left(\tau + \sqrt{\tau^2-1}\right)^{i\mu b/\pi}}{\sqrt{\tau^2-1}} f(\tau)\,d\tau - \left[(P+P_1)/\mu\right]\left(\zeta + \sqrt{\zeta^2-1}\right)^{i\mu b/\pi}$$

P and C are determined from the condition u=0 in points (x=y=0), (x=0, y=1). P_1 and P are connected by ratio $P - P_1 = \mu b$

AXIALSYMMETRIC PROBLEM

Suppose that elastic inclusion (fibre) with circular (radius a) cross-section is placed in the elastic orthotropic solid (matrix) with cylindrical anisotrophy. Axis of fibre coincides with axis z (cylindrical coordinate system). Cross- section z=0 there is disk crack with radius l_1 in matrix. At infinity the solid is subjected to axial stretching by means of stress σ_0 in direction of inclusion. It is necessary to define the distribution law of efforts in fibre, contact efforts between the fibre and matrix, distribution of efforts in solid and influence of fibre on stress state in the crack top. We suppose, that model of unidimensional elastic inclusion continuum in combination with model of contact on cylindrical area for matrix takes place (Pavlenko,1981; Sternberg,1970). On the first stage we came to integration of equation (3) under the following boundary conditions

$$\sigma_3 = 0 \quad (z=0,\ r \le l_1),\quad \sigma_3 = \sigma_0 \quad (|z| \to \infty),$$
$$w = 0 \quad (z=0,\ r=a;\ z=0,\ r \ge l_1),\quad w = w_s \ (r=a)$$

After the introduction of new variables

$$z = x_1 a,\quad r = a e^{y_1},\quad w = w_1 \cdot a,\quad \sigma = e^{2y_1}\sigma_3,\quad S_1 = e^{y_1}\tau_1 \tag{20}$$

equation (3) will be as follows:

$$\partial\sigma/\partial x_1 + \partial S_1/\partial y_1 = 0$$

The last equation is satisfied if function F is determined from (7) and effort P* in

cross-section of fibre $x_1 = 0$ is

$$P^* = (AE/E_3) F_{y_1} - 2AF,\quad y_1 = 0,\quad A = \pi a^2 \tag{21}$$

Utilization of ratioes (9) (after substitution in (9) E_1 to E_3, l to $\ln(l_1/a)$, u to w_1, τ_1 to S_1) shows, that functions φ and ψ satisfy the equation

$$\psi_y = \varphi_x,\quad -\psi_x = e^{-2ly}\varphi_y \tag{22}$$

Equation (21) goes over into equation (10) with $\mu = lE_3/E$, $P = P^* E_3/\sigma_0 EA$ and the conditions $\sigma_3 = 0$ on crack and w=0, when z=0 r=a and z=0, $r \ge l_1$ lead to conditions (11), (12). Hence, the solution of problem consists in finding of functions $\varphi(x,y)$, $\psi(x,y)$, which satisfy the system (22) and the conditions (10)-(12). Out of the solution decision of system (22) $\varphi(x,y)$ and $\psi(x,y)$ we compose the function of the complex variable $\zeta = x+iy$, $\Phi(\zeta) = \varphi + i\psi$. The "derivative" $\Phi'(\zeta)$ is determined in the following way (Lavrentiev et al.,1973)

$$\Phi'(\zeta) = \varphi_x + i e^{ly}\psi_x \tag{23}$$

Function $\Phi(\zeta)$ must sutisfy the condition (at infinity $\sigma_3 = \sigma_0$, $\tau_1 = 0$)

$$\Phi(\zeta) \backsim -i\zeta \quad (\zeta \to \infty)$$

If we introduce function $f(\zeta)$

$$f(\zeta) = \Phi'(\zeta) + i\mu\Phi(\zeta) + iP = (\psi_y - \mu\psi) - i(e^{-ly}\varphi_y - \mu\varphi - P)$$

("derivative" $\Phi'(\zeta)$ is defined from (23)), then boundary conditions for $f(\zeta)$ are as follows:

$$\operatorname{Re} f(\zeta) = 0 \quad (x=0,\ y>1),\quad \operatorname{Im} f(\zeta) = 0 \quad (y=0),$$
$$\operatorname{Im} f(\zeta) = P \quad (x=0,\ y<1),\quad f(\zeta) \backsim \mu\zeta \quad (\zeta \to \infty).$$

Further analysis is analogous with plate problem, as far as the formulas of generalization of Koshy theorem and Koshy formula take place here.

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