ON MAXIMAL STRESSES IN ELASTIC ISOTROPIC PLANE WITH A CRACK OF NONLINEAR DEFORMATIONS

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ABSTRACT

We consider here an elastic isotropic plane S belonging to a plane of a complex variable $z = x + iy$. On a line segment $(-a \leq x \leq a)$ of a real axis there is a rectilinear crack whose length is $2a$. At infinity of this domain the components of the stresses are defined by expressions

$$\sigma_y = P = \text{const}, \qquad \sigma_x = \tau_{xy} = 0 \qquad (I)$$

The load on the edges of the crack is absent. The dependence between the deformations and stresses in some vicinity of the ends A $(|x| = a)$ is nonlinear and in the rest of the domain S is linear. By virtue of this we assume that the stress σ_y at the points A takes the finite value σ_*.

Finding the value σ_* presents the main problem. The solution of this problem is reduced to finding the expression for normal stress $\sigma_y(x)$ on the line L_0 $(|x| > a)$ belonging to the axis x. In this paper we give an approximate expression of this function satisfying the conditions given below. The formulation of the problem begins with the consideration of the analogous problem for a plane with an elliptic hole.

KEYWORDS

Elastic isotropic plane, crack (rectilinear cut), linear and nonlinear deformations.

HEADING

Extension of a Plane with an Elliptic hole. Let the contour of the hole L be an ellipsis with semi-axes a and b coinsiding with the coordinate axes x and y. If $b \to 0$, it transforms into a cut (crack) with the length $2a$. We denote a plane with such a hole by the symbol S and a set of points on a real axis by the symbol L_0 $(|x| > a)$. Let us assume that the components of stresses at infinity are defined by expression (I), the exterior load on L is absent; in the vicinity of the vertices A ($|x|=a$) of the contour L the deformations are nonlinear. We denote the expressions of normal stresses on L for linear and nonlinear deformations by $\sigma_y(x)$ and $\sigma_y^*(x)$ respectively. Consequently, their main vectors in the interval L_0 are defined by the following expressions

$$Y_*(x)=\int_a^x \sigma_y^*(u)\,du, \quad Y(x)=\int_a^x \sigma_y(u)\,du \tag{2}$$

$$\sigma_y^*(x)=dY_*/dx, \quad \sigma_y(x)=dY(x)/dx \tag{3}$$

According to (Muckhelishvili, 1966) we find for linear deformations on L_0

$$Y(\rho)=\frac{PR}{4}\left[3+m+\frac{1-m}{1-m/\rho^2}+\frac{(1-m)(2+m)}{\rho^2-m}\right](\rho-1/\rho) \tag{4}$$

where

$$R=(a+b)/2, \quad m=(a-b)/(a+b), \quad x=R(\rho+m/\rho)$$

$$\rho=|x|/(a+b)+(x^2/(a+b)^2-m)^{1/2} \tag{5}$$

Applying formula (3) we get, (6)

$$\sigma_y(a)=(3+m)P/(1-m), \quad k=(3+m)/(1-m) \tag{7}$$

where k is a coefficient of the concentration of stresses at the point A.

Now let us find the stress $\sigma_y^*(a)$ and the coefficient of the stress concentration k_* at the point A for nonlinear deformation in the above given sense. In this case we consider that $\sigma_y^*(x)$ satisfies the following conditions. I) $\sigma_y^*(x)=\sigma^*(a)$ at $x=a$ and tends to P for $x\to\infty$. 2) Any part of the plane S cut along the line L_0 is in an equilibrium when stresses $\sigma_y^*(x)$ act on L_{c0} and $\sigma_y=P$ at infinity. 3) If the zones of nonlinear deformation tend to zero the expression $\sigma_y^*(x)$ on L_{c0} must coinside with the well-known expression $\sigma_y(x)$ found according to the linear theory. 4) As for nonlinear deformations the strained state of a body depends on its mechanical properties, the expression $\sigma_y^*(x)$ on L_0 must satisfy the condition that for some value $x=\beta\in L_c$, the equality $\sigma_y^*(\beta)=\sigma_0$ must hold where σ_0 is the stress found in an experimental way; is some point in the vicinity of which the deformations are linear. 5) The normal stress at the point A takes the finite value for m = I. In this particular case it is assumed that the opposite edges of the cut (crack) do not interact.

The main problem is reduced to finding the expression $\sigma_y^*(x)$ in the intervals L_{c1} ($a<x<\beta_*$) of nonlinear deformations and L_{c2} ($x>\beta_*$) for linear deformations on the contour L_c, where β_* is the point of division into parts L_{c1} and L_{02}. The position of this point must be defined. But later on we consider that the expression of the function $Y_*(x)$ on L_{01} and L_{c2} must be defined first of all. In is clear that the above given conditions are not sufficient for finding the definite expression of the function $Y_*(x)$. Therefore we consider approximately that $Y_*(\rho)$ is defined by one expression on

$$Y_*(\rho)=Y(\rho)(1-\lambda/(\lambda+\rho^2-m)), \tag{8}$$

where $Y(\rho)$ is a function defined by relation (4); λ is an arbitrare positive parameter.

Appling then formula (3) and taking into consideration (6) we get the explicit expression of the function $\sigma_y^*(\rho)$ on L_0 satisfying all given above conditions. Here λ is defined from the quadratic equation but the normal stress and the coefficient of stress concentrations at the point A (ρ = I) are defined by the following expressions:

$$\sigma_y^*(a)=(3+m)P/(1+\lambda-m), \quad k_*=(3+m)/(1+\lambda-m) \tag{9}$$

The Plane with the Cut along the Line Segment. Assuming that in relation (8) m = I we get

$$Y_*(\rho)=(\rho^2-1)^2\rho a/2\rho(\lambda+\rho^2-1) \tag{I0}$$

$$\sigma_y^*(\rho)=P[(\rho^2+1)+2\lambda\rho^2/(\lambda+\rho^2-1)]/(\lambda+\rho^2-1) \tag{II}$$

From (II) it follows that at the point A corresponding to ρ = I the normal stress and the coefficient are defined by the expressions

$$\sigma_y^*(a)=4P/\lambda, \quad k_*=4/\lambda \tag{I2}$$

In this case according to condition (4) λ is defined from the quadratic equation

$$(\beta^2+1)/(\lambda+\beta^2-1)+2\lambda\beta^2(\lambda+\beta^2-1)^2=n_c \tag{I3}$$

where $n_0=\sigma_0/P$; σ_0 is the value of $\sigma_y^*(x)$ for $\rho=\beta$.

The Transformation of Expressions (I0) and (II). Taking into consideration that there exist dependences between ρ and $x\in L_c$

$$|\dot x|=(\rho+1/\rho)/2, \quad \dot x=x/a$$

$$\rho=|\dot x|+(\dot x^2-1)^{1/2}, \quad \rho-1/\rho=2(\dot x^2-1)^{1/2} \tag{I4}$$

relation (I0) and (II) may be represented in the form

$$Y_*(x)=2PaM^3/D \tag{I5}$$

$$\sigma_y^*(x)=2P[|\dot x|+\lambda/D]/D \tag{I6}$$

where

$$D=\lambda|\dot x|+(2-\lambda)M, \quad M=(\dot x^2-1)^{1/2} \tag{I7}$$

For λ = 0 relation (I6) takes the form

$$\sigma_y^*(x)=P|x|/(x^2-a^2)^{1/2}$$

This result holds for linear deformations everywhere in the domain S.

By virtue of (I6) the parameter λ is defined from the equation

$$2\left[\frac{\beta}{\lambda\gamma+2M}+\frac{\lambda}{(\lambda\gamma+2M)^2}\right]=n_0 \tag{I8}$$

where $M=\sqrt{\beta^2-1}$, $\gamma=\beta-M$, $n_0=\sigma_0/P$, σ_0 is the value of $\sigma_y^*(x)$ at the point $x=\beta\in L_0$.

Consequently, (I6) represents the definite expression of the function $\sigma_y^*(x)$ on L_0. According to this we can find the position of the point $x=\beta_*$ which is the bound of the division into the parts L_{01} and L_{02} of nonlinear and linear deformations along the line L_0. For this it is sufficient to set $x=\beta_*/a=\delta$, $\sigma_y^*(\delta)=\sigma_\pi$ in relation (I6) where σ_π is the maximal stress on the diagram of a simple extension for the sample of the material considered for which the proportional dependence between the stress and the deformation concerned holds. As a result we get the equation from which we can find the value of β_*.

Another Formulation of the Problem. According to (Ponasuk et al., I988) we consider that expression (I6) holds only in the intervals L_{01}, but in L_{02} the function $\sigma_y^*(x)$ must be defined, Under these conditions we assume that the deformations are linear everywhere in the domain S. Consequently, we may consider that the stress components in S are defined by formulae (Muskhelishvili, I988)

$$\sigma_y+\sigma_x=2\left[\Phi(z)+\overline{\Phi(z)}\right] \tag{I9}$$

$$\sigma_y-i\tau_{xy}=\Phi(z)+\Omega(\bar z)+(z-\bar z)\overline{\Phi'(z)} \tag{20}$$

where $\Phi(z)$ and $\Omega(z)$ are holomorphic functions in the domain S satisfying at large $|z|$ the conditions

$$\Phi(z)=P/4+O(1/z^2),\quad \Omega(z)=3P/4+O(1/z^2) \tag{2I}$$

In the intervals L' ($|x|\le a$) and L_{01} these functions must satisfy the boundary conditions

$$\Phi^+(x)+\Omega^-(x)=q(x)$$
$$\Phi^-(x)+\Omega^+(x)=q(x) \tag{22}$$

where $q(x)=0$ is on L' and $q(x)=f(x)$ is on L_{01}; $f(x)$ is the right part of relation (I6).

The solution of the boundary value problem (2I) is defined by relations

$$\Phi(z)=2P(-C_1+f_1(z)/2-z\dot I(z)+iC_2)\quad (z\in S)$$
$$\Omega(z)=2P(C_1+f_1(z)/2+z\dot I(z)-iC_2)\quad (z\in S) \tag{23}$$

where

$$f_1(z)=\left[z/a+\lambda/D_0(z)\right]/D_0(z)$$
$$D_0(z)=\lambda z/a+\gamma M_0(z)$$
$$M_0(z)=((z^2/a^2)-1)^{1/2}$$
$$\dot I(z)=\int_{-a}^{a}\lambda\left[\varphi(\tau)/\tau(\tau-z)\right]d\tau/2\pi i$$
$$\gamma=2-\lambda,\qquad \dot x=x/a \tag{24}$$
$$N(x)=(\lambda^2-\gamma^2)\dot x^2+\gamma^2;\quad C_2=\int_{-a}^{a}(\varphi(\tau)/\tau)\,d\tau/2\pi$$
$$\varphi(x)=\left[\lambda\dot x^2+((\lambda^2+\gamma^2)\dot x^2-\gamma^2)/N(x)\right]/N(x)$$

C_1 is a real constant defined according to (2I) by the expression C = I/8.

Applying then formula (20) we find that $\sigma_y^*(x)$ on L_{02} is also defined by expression (I6). The parameters λ and β_* are defined here as it was said above. The problem considered is equivalent to the problem of finding the functions $\Phi(z)$ and $\Omega(z)$ giving the finite values of stresses on L_b for linearity of deformations everywhere in the domain S. If $\sigma_y^*(x)$ is given on the intervals of nonlinear deformations L_{01} and along the line L_0 we must demand the fulfilment of the above given condition 2). Otherwise the state of the crack will be not in equilibrium. The criteria of its state is the fulfilment of relations

$$\int_{L_0}\left[\sigma_y^*(x)-P\right]dx-2aP=0 \tag{24}$$

$$\sigma_y^*(x)\le\sigma_b\qquad \text{for } x\to a \tag{25}$$

where σ_b is a maximal normal stress when the material breaks. For $\sigma_y^*(a)=\sigma_b$ the limiting equilibrium state of the crack takes place. If relation (24) is not fulfilled the state of the crack will be not in equilibrium. Its length will increase if the left part of equality (24) is less than zero and decrease if the same expression is greater than zero.

The Calculation on Strength. Let us consider that the crack is in an equilibrium state at the given stress at infinity and the stress $\sigma_y^*(x)$ on L is defined by formula (I6). Besides that we consider that the parameter λ depends only on P and that its value for each $n_0=\sigma_0/P$ is defined from equation (I8) where σ_0 is the value of $\sigma_y^*(\beta)$ at the point $x=\beta$ found in the experimental way. For β we can take the coordinate x of any point situated on the right from the point $x=\beta_*$.

The value $P=P_*$ presents the main practical interest for which $\sigma_y^*(a)=[\sigma]$ where $[\sigma]=\sigma_b/n$ is the assumed stress, n - the coefficient of the reserve of strength and also the value $P=\sigma_*$ for $\sigma_y^*(a)=\sigma_b$. As the parameter λ depends on P and does not depend on the coordinates and as $\lambda=0$ the stress $\sigma_y^*(a)=\sigma_y(a)=\infty$ and we consider that for any P in the interval $(0<P<\sigma)$

$$\lambda=A\,(P/\sigma_b)^k \tag{26}$$

where A and k are arbitrary positive coefficients. To find them we shall do the following.

Let for the point $x=\beta>\beta_*$ be found experimentally that when $P=P_1$ and $P=P_2$ $(P_1<P_2<\sigma_b)$ the stress $\sigma_y^*(\beta)$ takes the values σ_1 and σ_2. Then according to (I8) the parameter λ takes the values λ_1 and λ_2 respectively. Taking into consideration this we find

$$\lambda=\lambda_1(P/P_1)^k,\quad k=(\ln(\lambda_1/\lambda_2))/\ln(P_1/P_2) \tag{27}$$

Using then formula (I2) we find

$$\sigma_y^*(a)=4\,(P_1/P)^k\,P/\lambda_1 \tag{28}$$

$$[\sigma] = 4(P_1/P_*)^k P_*/\lambda_1 \tag{29}$$

$$\sigma_b = 4(P_i/\sigma_*)^k \sigma_*/\lambda_1 \tag{30}$$

Considering that $[\sigma]$ and σ_b are known values, we find from (29) and (30)

$$P_* = (4/\lambda_1)^\alpha (P_1/[\sigma])^{\alpha k} [\sigma] \tag{31}$$

$$\sigma_* = (4/\lambda_1)^\alpha (P_1/\sigma_b)^{\alpha k} \sigma_b, \quad \alpha = 1/(k-1) \tag{32}$$

We can use relation (32) for the experimental check of the accuracy of the solution of the problem.

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