ON A NUMERICAL INVESTIGATION OF BOUNDARY COLLOCATION PROCEDURE AND WILLIAMS SERIES CONVERGENCY

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ABSTRACT

An exact solution to an elastic plate with a crack has been studied. The solution is a series expression within the convergent domain. This series is found to be identical to the Williams Series. Thus, convergency of the Williams Series for mode I and mode II is discussed. Results indicate that the convergent domain of the Williams Series is related to the crack length. A series of calculations at convergent and divergent domains are analysed.

KEYWORDS

Williams series, convergency, fracture mechanicas, stress intensity factor, boundary collocation procedure.

INTRODUCTION

Boundary collocation procedure is one of efficient methods to compute stress intensity factors K_1 and K_2. However, it is still vague whether there are such problems as convergency of the Williams Series and stability of the calculated values. This paper provides an exact solution to the elastic plate with a crack. After finding the Taylor Series expansion of the solution and transforming the subscripts it is found that the expansion is identical to the Williams Series, so the convergent domain of the Williams Series and the exact coefficients of the Williams Series are obtained. Analysis and evaluation show that the solution of the Williams Series is not suited to the whole region of the finite elastic plate.

SERIES SOLUTION FOR MODE I CRACK

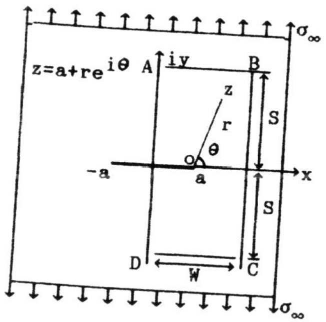
Consider a crack in a elastic isotropic body subjected to mode I loading. Stresses can be obtained from the elastic theory as follows:

$$\sigma_y+\sigma_x=\sigma_\infty\left[2\mathrm{Re}\frac{z}{\sqrt{z^2-a^2}}-1\right] \quad (1) \qquad \sigma_y-\sigma_x+2i\sigma_{xy}=\sigma_\infty\left[\frac{2ia^2y}{\sqrt{(z^2-a^2)^{3/2}}}+1\right] \quad (2)$$

where $z=a+re^{i\theta}$. Rectangular coordinates and polar coordinates are shown in Fig.1. The stresses at any point near the crack ($r<2a$) can be derived using the Taylar Series expansion. The elastic stresses are as follows:

$$\begin{cases}\sigma_y=\sigma_\infty\sum_{n=1}^{\infty}A_n\left[\frac{2n+3}{8}\left(\frac{r}{2a}\right)^{n-\frac{3}{2}}\cos(n-\frac{3}{2})\theta+\left(\frac{r}{2a}\right)^{n-\frac{1}{2}}\cos(n-\frac{1}{2})\theta-\frac{2n-1}{8}\left(\frac{r}{2a}\right)^{n-\frac{3}{2}}\cos(n-\frac{7}{2})\theta\right]\\ \sigma_x=\sigma_\infty\sum_{n=1}^{\infty}A_n\left[-\frac{2n-5}{8}\left(\frac{r}{2a}\right)^{n-\frac{3}{2}}\cos(n-\frac{3}{2})\theta+\left(\frac{r}{2a}\right)^{n-\frac{1}{2}}\cos(n-\frac{1}{2})\theta+\frac{2n-1}{8}\left(\frac{r}{2a}\right)^{n-\frac{3}{2}}\cos(n-\frac{7}{2})\theta\right]-\sigma_\infty\\ \sigma_{xy}=\sigma_\infty\sum_{n=1}^{\infty}A_n\frac{2n-1}{8}\left(\frac{r}{2a}\right)^{n-\frac{3}{2}}\left[\sin(n-\frac{3}{2})\theta-\sin(n-\frac{7}{2})\theta\right]\end{cases} \quad (r<2a) \quad (3)$$

where $A_n=(-1)^{n+1}\frac{(2(n-1)-1)!!}{(2(n-1))!!}$ $\quad (A_1=\frac{(-1)!!}{0!!}=-1)$



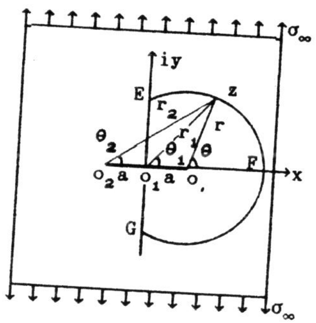


Fig.1 The rectangular domain model. Fig.2 The circular domain model.

Notice that the convergent domain of equation (3) is a circle, the centre of the circle is the crack tip and the convergent radius is equal to 2a. This equation can be arranged as:

$$\begin{cases}\sigma_y=\sigma_\infty\sum_{n=1}^{\infty}A_n\frac{(2n-1)}{8(2n-3)}\left(\frac{r}{2a}\right)^{n-\frac{3}{2}}\left[(2n-7)\cos(n-\frac{3}{2})\theta-(2n-3)\cos(n-\frac{7}{2})\theta\right]\\ \sigma_x=\sigma_\infty\sum_{n=1}^{\infty}A_n\frac{(2n-1)}{8(2n-3)}\left(\frac{r}{2a}\right)^{n-\frac{3}{2}}\left[-(2n+1)\cos(n-\frac{3}{2})\theta+(2n-3)\cos(n-\frac{7}{2})\theta\right]-\sigma_\infty\\ \sigma_{xy}=\sigma_\infty\sum_{n=1}^{\infty}A_n\frac{(2n-1)}{8}\left(\frac{r}{2a}\right)^{n-\frac{3}{2}}\left[\sin(n-\frac{3}{2})\theta-\sin(n-\frac{7}{2})\theta\right]\end{cases} \quad (r<2a) \quad (4)$$

Equation (4) is the solution of the Williams Series, It is using equation (4) that convergency of the Williams Series is analysed.

Moreover, at the domain of $r>2a$, the stresses can also be derived from equation (1) and equation (2)

$$\begin{cases}\sigma_y=\sigma_\infty\sum_{n=1}^{\infty}A_n\Big[\left(\frac{2a}{r}\right)^{n-1}\cos(n-1)\theta+\frac{1}{2}\left(\frac{2a}{r}\right)^{n}\cos n\theta+\frac{2n-1}{8}\left(\frac{2a}{r}\right)^{n+1}\cos(n+1)\theta\\ \qquad -\frac{2n-1}{8}\left(\frac{2a}{r}\right)^{n+1}\cos(n+3)\theta\Big]\\ \sigma_x=\sigma_\infty\sum_{n=1}^{\infty}A_n\Big[\left(\frac{2a}{r}\right)^{n-1}\cos(n-1)\theta+\frac{1}{2}\left(\frac{2a}{r}\right)^{n}\cos n\theta-\frac{2n-1}{8}\left(\frac{2a}{r}\right)^{n+1}\cos(n+1)\theta\\ \qquad +\frac{2n-1}{8}\left(\frac{2a}{r}\right)^{n+1}\cos(n+3)\theta\Big]-\sigma_\infty\\ \sigma_{xy}=\sigma_\infty\sum_{n=1}^{\infty}A_n\frac{2n-1}{8}\left(\frac{2a}{r}\right)^{n+1}\left[\sin(n+3)\theta-\sin(n+1)\theta\right]\end{cases} \quad (r>2a) \quad (5)$$

As a result, the stresses at any point ($0<r<\infty$) can be obtained by equation (4) and equation (5).

TWO MODELS OF EXACT SOLUTION FOR MODE I

Consider the two models shown in Fig.1 and Fig.2, their boundary lines are ABCDA and EFGE,respectively. The stresses in the two models can be shown as equation (4) ($r<2a$) and equation (5) ($r>2a$), respectively. Stress boundary conditions are: (F.Tianyou,1978)

$$\begin{cases}\sigma_y^0=\sigma_\infty\frac{r_1}{a}\left(\frac{a^2}{r\,r_2}\right)^{\frac{3}{2}}\sin\theta_1\sin\frac{3}{2}(\theta+\theta_2)+\sigma_\infty\frac{r_1}{(r\,r_2)^{1/2}}\cos(\theta_1-\frac{1}{2}\theta-\frac{1}{2}\theta_2)\\ \sigma_x^0=-\sigma_\infty\frac{r_1}{a}\left(\frac{a^2}{r\,r_2}\right)^{\frac{3}{2}}\sin\theta_1\sin\frac{3}{2}(\theta+\theta_2)-\sigma_\infty\left[1-\frac{r_1}{(rr_2)^{1/2}}\cos(\theta_1-\frac{1}{2}\theta-\frac{1}{2}\theta_2)\right]\\ \sigma_{xy}^0=\sigma_\infty\frac{r_1}{a}\left(\frac{a^2}{r\,r_2}\right)^{\frac{3}{2}}\sin\theta_1\cos\frac{3}{2}(\theta+\theta_2)\end{cases} \quad (6)$$

Thus, the two models have exact stress boundary conditions and exact series solutions. The two models are called a rectangular domain model and a circular domain model, respectively.

WILLIAMS STRESS SERIES FOR MODE I

The stresses of a finite plate with an edge crack subjected to mode I case can be shown as follows(M.L.Williams,1957):

$$\begin{cases} \sigma_y=\sum_{n=1}^{\infty} C'_n r^{n/2-1}\frac{n}{2}\left[(\frac{n}{2}-2+(-1)^n)\cos(\frac{n}{2}-1)\theta-(\frac{n}{2}-1)\cos(\frac{n}{2}-3)\theta\right] \\ \sigma_x=\sum_{n=1}^{\infty} C'_n r^{n/2-1}\frac{n}{2}\left[-(\frac{n}{2}+2+(-1)^n)\cos(\frac{n}{2}-1)\theta+(\frac{n}{2}-1)\cos(\frac{n}{2}-3)\theta\right] \\ \sigma_{xy}=\sum_{n=1}^{\infty} C'_n r^{n/2-1}\frac{n}{2}\left[(\frac{n}{2}+(-1)^n)\sin(\frac{n}{2}-1)\theta-(\frac{n}{2}-1)\sin(\frac{n}{2}-3)\theta\right] \end{cases} \qquad (7)$$

This equation is the solution of the Williams Series. The coefficients C'_n can be split into even C'_{2n} and odd C'_{2n-1} parts, we get:

$$\begin{cases} \sigma_y=\sum_{n=1}^{\infty} C'_{2n-1} r^{n-\frac{3}{2}}\frac{(2n-1)}{4}\left[(2n-7)\cos(n-\frac{3}{2})\theta-(2n-3)\cos(n-\frac{7}{2})\theta\right] \\ \quad +\sum_{n=1}^{\infty} C'_{2n} r^{n-1} n(n-1)\left[\cos(n-1)\theta-\cos(n-3)\theta\right] \\ \sigma_x=\sum_{n=1}^{\infty} C'_{2n-1} r^{n-\frac{3}{2}}\frac{(2n-1)}{4}\left[-(2n+1)\cos(n-\frac{3}{2})\theta+(2n-3)\cos(n-\frac{7}{2})\theta\right] \\ \quad +\sum_{n=1}^{\infty} C'_{2n} r^{n-1} n\left[-(n+3)\cos(n-1)\theta+(n-1)\cos(n-3)\theta\right] \\ \sigma_{xy}=\sum_{n=1}^{\infty} C'_{2n-1} r^{n-\frac{3}{2}}\frac{(2n-1)(2n-3)}{4}\left[\sin(n-\frac{3}{2})\theta-\sin(n-\frac{7}{2})\theta\right] \\ \quad +\sum_{n=1}^{\infty} C'_{2n} r^{n-1} n\left[(n+1)\sin(n-1)\theta-(n-1)\sin(n-3)\theta\right] \end{cases} \qquad (8)$$

When equation (8) is applied to the two models, its convergent domain is identical to the domain of equation (4). Comparing equation (8) with equation (4), we can find the coefficients in equation (8)

$$\begin{cases} C'_2=\frac{\sigma_\infty}{4}, \quad C'_{2n}=0 & (n=2,3,\ldots) \\ C'_{2n-1}=\frac{1}{2(2n-3)}\frac{A_n\sigma_\infty}{(2a)^{n-3/2}} & (n=1,2,\ldots) \end{cases} \qquad (r<2a) \quad (9)$$

The first coefficient C'_1 is:

$$C'_1=\frac{K_I}{\sqrt{2\pi}} \qquad (K_I=\sigma_\infty\sqrt{\pi a}) \qquad (10)$$

Because the convergent domain of equation (4) is r<2a under this condition, it is near the crack (r<2a) that equation (8) is convergent. This indicates that the convergent domain of the Williams Series is related to the crack length. Comparison of equation (5) with equation (8) shows that the exact solution at the domain of r>2a do not conform to the Williams Series.

SERIES SOLUTION FOR MODE II

For an elastic body subjected to mode II loading, stresses are obtained as follows:

$$\begin{cases} \sigma_y=\tau_\infty\sum_{n=1}^{\infty} A_n\frac{(2n-1)}{8}(\frac{r}{2a})^{n-\frac{3}{2}}\left[\sin(n-\frac{3}{2})\theta-\sin(n-\frac{7}{2})\theta\right] \qquad (r<2a) \quad (11) \\ \sigma_x=\tau_\infty\sum_{n=1}^{\infty} A_n\frac{(2n-1)}{8(2n-3)}(\frac{r}{2a})^{n-\frac{3}{2}}\left[-(2n+5)\sin(n-\frac{3}{2})\theta+(2n-3)\sin(n-\frac{7}{2})\theta\right] \\ \sigma_{xy}=\tau_\infty\sum_{n=1}^{\infty} A_n\frac{(2n-1)}{8(2n-3)}(\frac{r}{2a})^{n-\frac{3}{2}}\left[-(2n+1)\cos(n-\frac{3}{2})\theta+(2n-3)\cos(n-\frac{7}{2})\theta\right] \end{cases}$$

The stresses of the Williams Series with an edge crack subjected to mode II loading can also become:

$$\begin{cases} \sigma_y=\sum_{n=1}^{\infty} D'_{2n-1} r^{n-\frac{3}{2}}\frac{(2n-1)(2n-3)}{4}\left[\sin(n-\frac{3}{2})\theta-\sin(n-\frac{7}{2})\theta\right] \\ \quad +\sum_{n=1}^{\infty} D'_{2n} r^{n-1} n\left[(n-3)\sin(n-1)\theta-(n-1)\sin(n-3)\theta\right] \\ \sigma_x=\sum_{n=1}^{\infty} D'_{2n-1} r^{n-\frac{3}{2}}\frac{(2n-1)}{4}\left[-(2n+5)\sin(n-\frac{3}{2})\theta+(2n-3)\sin(n-\frac{7}{2})\theta\right] \\ \quad +\sum_{n=1}^{\infty} D'_{2n} r^{n-1} n\left[-(n+1)\sin(n-1)\theta+(n-1)\sin(n-3)\theta\right] \\ \sigma_{xy}=\sum_{n=1}^{\infty} D'_{2n-1} r^{n-\frac{3}{2}}\frac{(2n-1)}{4}\left[-(2n+1)\cos(n-\frac{3}{2})\theta+(2n-3)\cos(n-\frac{7}{2})\theta\right] \\ \quad +\sum_{n=1}^{\infty} D'_{2n} r^{n-1} n(n-1)\left[-\cos(n-1)\theta+\cos(n-3)\theta\right] \end{cases} \qquad (12)$$

Comparing equation(11) with equation(12), we get:

$$\begin{cases} D'_{2n}=0 & (n=1,2,3\ldots) \\ D'_{2n-1}=\frac{1}{2(2n-3)}\frac{A_n\tau_\infty}{(2a)^{n-3/2}} & (n=1,2,3\ldots) \end{cases} \qquad (r<2a) \quad (13)$$

EVALUATION OF BOUNDARY COLLOCATION PROCEDURE FOR MODE I

For the two models with mode I loading, boundary collocation procedures are carried out using equation (8). The stress boundary conditions are found in equation (6).The calculated results at the convergent and divergent domains are listed in Table 1 and Table 2.

Table 1. Stability of the values C_1 and C_9 with the number of collocation points m

	The rectangular domain model				The circular domain model			
	S=W=2a		S=2W=4a		r/a=1.8		r/a=3.5	
m	C_1 [a)]	C_9	C_1	C_9	C_1	C_9	C_1	C_9
15	.505938	.009422	.429753	.009987	.500052	.020795	.473858	.057765
19	.498736	.018815	.479131	.015679	.499761	.019441	.491942	.071085
20	.500255	.018815	.486966	.034955	.500032	.019465	.587154	.001757
21	.501209	019084	.486543	.042408	.499914	.019505	.498472	.027746
25	.500354	.019769	.492696	.019992	.499998	.019538	.497665	.061044
30	.498921	.019894	.496149	.027657	.500005	.019531	.498126	.062747
35	.499641	.019554	.500343	-.000836	.499998	.019532	.507997	.022865
40	.500263	.020379	.498351	.006508	.499999	.019530	.496336	.064605
45	.499995	.016367	.500128	-.010690	.500002	.019535	.496861	.060108
50	.503616	.017669	.503162	.027335	.500011	.019539	.498225	.063659

a) The sign C_1 denotes a dimensionless conficient $C_1 = \frac{C_1'}{\sqrt{2a}\,\sigma_\infty} = 0.5$

Table 2. Convergency for the first 11 terms of the Williams Series at the convergent domain and divergent domain. (m=35)

Cn	Theoretic	S=W=2a	RE(%)	r=1.8a	RE(%)	S=2W=4a	RE(%)	r=3.5a	RE(%)
C_1	.500000	.499641	0.07	.499998	0.0	.500343	0.00	.507997	1.60
C_2	.250000	.250009	0.00	.249999	0.0	.240531	0.95	.254120	-1.64
C_3	-.250000	-.250428	-0.17	-.249999	0.0	-.198379	20.65	-.246445	1.42
C_5	.062500	.062086	0.66	.062500	0.0	.130936	-109.50	.058012	7.18
C_7	-.031250	-.031559	-0.99	-.031250	0.0	.001278	104.09	-.032612	4.30
C_9	.019531	.019554	-0.12	.019532	0.0	-.000836	104.28	.022865	-17.07
C_{11}	-.013672	-.013679	-0.05	-.013672	0.0	-.016318	-19.36	-.015066	-10.20
C_{13}	.010254	.010178	0.74	.010253	0.0	-.009488	192.53	.009163	10.64
C_{15}	-.008056	-.008360	-3.77	-.008055	0.0	-.044127	-447.75	-.006928	14.00
C_{17}	.006546	.006235	4.74	.006545	0.0	.041628	-535.93	.005735	12.39
C_{19}	-.005455	-.005657	-3.70	-.005454	0.0	.015754	388.79	-.004732	13.25

For the two models the calculated results, as shown in Table 1 and Table 2, indicate that very exact coefficients of the Williams Series are obtained by taking a wide scope of collocation points within the convergent domain. A fairly good value K_1 can also be obtained at the divergent domain near the convergent domain such as the case of the rectangular domain model where S=2W=4a. The calculated value C_1 is still exact and stable, but the other coefficients do not conform to the theoretical values.

DISCUSSION

The boundary collocation method is a procedure for approximating functions, which replaces exact solutions by the Williams approximating polynomial. The boundary condition is approximated by summing this polynomial. As we have seen above, the Williams Series is a Taylor Series near the crack. The boundary collocation procedure within the convergent domain uses the partial sum of a convergent infinite Williams Series as an approximation to the sum of the infinite Williams Series. For this case, the type of the approximating function is the partial sum of equation (8) and the type of the exact solution is the Williams Series equation (4),which indicates that the type of the approximating function is identical to that of the exact solution. So the calculated results of the *nth* terms of the Williams Series are exact and stable (G.M.Phillips and P.J.Taylor,1973; G.Polya and G.Szego 1976). On the other hand, for the case of the divergent domain, the type of the approximating function is still the partial sum of the Williams Series equation (8). But the type of the exact solution for equation (5) is different from the one of the Williams Series equation (8). Calculated results of some terms of the Williams polynomial thus appear as inexact and unstable. For the case of limited divergent domain, the coefficients of the Williams polynomial are of weight character (G.M. Phllips and P.J.Taylor 1973) and the maximum weight of the Williams polynomal is the first term C_1. Calculated results indicate that a fairly good value K_1 can be obtained, but the other coefficients do not conform to the exact values. Thus, we can say there exists *calculated domain with the stress intensity factor* K_1 . The calculated value K_1 is only exact and stable within this domain. For the circular domain model, the domain range is r=3.5a.

CONCLUSIONS

1) Theoretical analysis shows that the convergent domain of the Williams Series for mode I and mode II is a circular domain.The centre of the circle is the crack tip. The convergent radius r is related to the crack length.

2) The calculated results indicate that very exact coefficients of the Williams Series can be obtained by taking a wide scope of collocation

points within the convergent domain.

3) There exists calculated domain with the stress intensity factor K_1 within a limited divergent domain. The calculated value K_1 is only exact and stable, but the other coefficients do not conform to the terms of the Williams polynomial.

NOMENCLATURE

a=half length of the crack
m=the number of the boundary collocation points
K_1=mode I sterss intensity factor
$\sigma_\infty, \tau_\infty$=the boundary stresses of the infinite plate
r, r_1, r_2=polar coordinates
$\theta, \theta_1, \theta_2$=polar coordinates
$\sigma_x, \sigma_y, \sigma_{xy}$=stress components in the rectangular coordinate system
$\sigma_x^0, \sigma_y^0, \sigma_{xy}^0$=stress boundary conditions for the two models
A_n, C_n', D_n'=constants
RE(%)=relative error

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