ON A GENERAL THREE-DIMENSIONAL WEIGHT FUNCTION METHOD

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ABSTRACT

In this investigation, a general method for three-dimensional weight function determination is developed. It is shown that a mode I weight function for a flat 3-D crack may be obtained from any known solution for such crack if conformal mapping of the crack domain into the unit circle is available. As an example, an exact three-dimensional weight function for a straight through "tunnel" crack is established.

KEYWORDS

Linear elastic fracture mechanics, stress intensity factor, three-dimensional weight function, circular crack, straight through "tunnel" crack, exact solution.

INTRODUCTION

Weight function concept is a powerful tool in linear elastic fracture mechanics analysis. When the weight function is known, the problem of stress intensity factor determination for a given applied stress field reduces simply to calculation of some definite integrals.

For mode I crack problem the weight function by definition is the stress intensity factor caused by two normal opposite directed concentrated unit forces applied to the crack surfaces, and the straightforward method for its determination consists in solving the elasticity boundary value problem with concentrated loads. However, dealing with two-dimensional crack problems, such as plane and axisymmetric problems, there is no need to solve boundary value problem with concentrated loads, since two-dimensional weight function is available if solution for given crack geometry is known for any boundary conditions on crack faces (Rice, 1972; Banks-Sills, 1988).

In this paper a new general method for three-dimensional weight function determination is proposed which bases on the variational formula for crack problems (Rice, 1985; N. M. Borodachev, 1986, 1989) and on the theory of harmonic functions. This method allows us to compute the 3-D weight function for a given crack problem if we know conformal mapping of the crack domain into the unit circle and some trial solution of the crack problem under consideration corresponding to more simple boundary conditions on crack faces. For example, it is shown that three-dimensional weight function for circular crack may be deduced from a simple axisymmetric trial solution, and a new 3-D weight function for a straight through "tunnel" crack is deduced from classic two-dimensional Griffith crack solution.

VARIATIONAL FORMULA FOR CRACK PROBLEMS

With the use of a Cartesian coordinate system x_1, x_2, x_3, consider a three-dimensional linear elasticity problem for a solid with an inner flat crack laying in the plane $x_3=0$ and occupying the domain S with regular boundary Γ. Hereafter the points of the crack domain S will be denoted by $Q=(x_1,x_2,0)$ and the points of the crack front Γ will be denoted by $M=(y_1,y_2,0)$.

It is known that the stress intensity factor at any point of the crack front may be estimated according to the formula

$$K_I(M) = \iint_S W(Q;M)p(Q)dS(Q) \tag{1}$$

where p(Q) is the given normal pressure on the crack surfaces, W(Q;M) is the appropriate weight function, and dS is the area element in rectangular coordinates. Thus the stress intensity factor determination reduces simply to the calculation of definite integrals when the weight function is available.

A general 3-D weight function method, which we are about to propose, bases on a variational formula for crack problems. Using the approach developed by N. M. Borodachev (1989), this variational formula for mode I crack problem may be written in the following general form

$$\delta u_3(Q) = C\int_\Gamma W(Q;M)K_I(M)\delta n(M)ds(M) \tag{2}$$

where δu_3 is the variation of the crack surface normal displacements due to a small variation $\delta n(M)$ of the crack front Γ in the direction of the exterior normal, and ds is arc length element on Γ. The crack surface normal displacement $u_3(Q)$ and the stress intensity factor $K_I(M)$ corresponds to some trial solution. The constant C depends only on the elastic material properties and is determined by the first terms of normal stress and displacement asymptotic expansions, which are valid in the vicinity of crack front. So, if

$$\sigma_{33} = K_I(2r)^{-1/2}, \quad u_3 = cK_I(2r)^{1/2}, \quad r \to 0 \tag{3}$$

where r is the distance in the crack plane from the crack front, then $C=\pi c/2$. For an isotopic and homogeneous elastic material this constant is $C=\pi(1-\nu)/(2\mu)$, where ν is the Poisson's ratio and μ is the shear modulus.

GENERAL 3-D WEIGHT FUNCTION REPRESENTATION

Now consider a Dirichlet boundary value problem for two-dimensional Laplace's equation in the crack domain S

$$\Delta f(Q) = 0, \quad f(M) = f_0(M) \tag{4}$$

where Δ is the Laplacian operator and f_0 is a specified function. From the theory of harmonic functions it is known that the solution f(Q) may be written in the form

$$f(Q) = -\int_\Gamma \partial G(Q;M)/\partial n(M) f_0(M)ds(M) \tag{5}$$

where G(Q;M) is the Green's function for the domain S, and $\partial/\partial n$ is a derivative in the direction of the exterior normal to the boundary.

Another representation of the harmonic function f(Q), which is deduced formally from the variational formula (2), is

$$f(Q) = C\int_\Gamma f(Q)W(Q;M)K_I(M)\delta n(M)(\delta u_3(Q))^{-1}ds(M) \tag{6}$$

Now by comparing the equations (5) and (6) the following general representation for a 3-D weight function is established

$$W(Q;M) = -F(Q;M)U(Q;M)\partial G(Q;M)/\partial n(M) \tag{7}$$

where

$$F(Q;M) = f_0(M)/f(Q), \qquad U(Q;M) = \delta u_3(Q)/(CK_I(M)\delta n(M)) \tag{8}$$

According to the formulae (7) and (8), in order to compute a 3-D weight function we must know only the Green's function for the crack domain S and some trial solution for the crack problem under consideration. The trial solution is needed for the determination of the function U(Q;M) from the equation (8).

In many cases the Green's function may be obtained by using the theory of functions of a complex variable. It is known that if w(z) is the conformal mapping of the domain S into the unit circle $|w|<1$ and $z=x_1+ix_2$, then the Green's function for the Dirichlet problem (4) has the form

$$G(Q;M) = -\ln|w(z;\zeta)|/(2\pi) \tag{9}$$

where $w(z;\zeta)=(w(z)-w(\zeta))/(1-\overline{w}(\zeta)w(z))$ and $\zeta=y_1+iy_2$. Thus the main restriction on the use of the general 3-D weight function representation formula (7) is the availability of a trial solution for the crack problem.

At this point consider a three-dimensional problem for an infinite isotropic homogeneous elastic solid containing an inner circular crack with radius a.

With the use of polar coordinates r and φ, a simple axisymmetric trial solution, corresponding to constant normal pressure p applied to the crack surfaces, may be written in the form

$$u_3(r,0) = 2p(1-\nu)(a^2-r^2)^{1/2}/(\pi\mu), \quad r<a; \quad K_I = 2pa^{1/2}/\pi \tag{10}$$

In the case of an axisymmetric variation of the crack front, when $\delta n(M)=\delta a$, the variation of the crack surface displacement may be obtained as

$$\delta u_3(Q) = (\partial u_3(r,0)/\partial a)\delta a = 2ap(1-\nu)(a^2-r^2)^{-1/2}\delta a/(\pi\mu) \tag{11}$$

and the function U(Q;M) from the equation (7) takes the form

$$U(Q;M) = 2a^{1/2}(a^2-r^2)^{-1/2}/\pi \tag{12}$$

For a circular crack domain S the conformal mapping into the unit circle is $w(z)=z/a$, and somewhat cumbersome but straightforward calculations result in the following formula

$$\partial G(Q;M)/\partial n(M) = -(a^2-r^2)(a^2+r^2-2ar\cos(\varphi-\psi))^{-1}/(2\pi a) \tag{13}$$

where ψ is the value of polar angle φ at the point M of the crack front.

It is clear that function $f(Q)=1$ is harmonic in the crack domain $r<1$ and $f_0(M)=1$. Hence, $F(Q;M)=1$ and by virtue of the equation (7) we obtain the following representation for the 3-D weight function for an inner circular crack in an infinite, isotropic homogeneous elastic solid

$$W(Q;M) = (a^2-r^2)^{1/2}(a^2+r^2-2ar\cos(\varphi-\psi))^{-1}/(\pi^2a^{1/2}) \tag{14}$$

which is in agreement with the known result (Kassir and Sih, 1975).

STRAIGHT THROUGH "TUNNEL" CRACK PROBLEM

Suppose that an infinite isotropic homogeneous elastic solid contains a straight through "tunnel" crack. In this case the crack domain S is the strip $-a<x_1<a$, $-\infty<x_2<\infty$, and the crack front Γ consists of the two parts $\Gamma^\pm$ (where $y_1=\pm a$).

Using the 3-D weight functions $W^\pm(Q;M)$, the stress intensity factors $K_I^\pm$ for the crack front parts $\Gamma^\pm$ may be written in the form

$$K_I^\pm(M) = \iint_S W^\pm(Q;M)p(Q)dS(Q) \tag{15}$$

where $p(Q)$ is the known normal pressure on the crack surfaces. If the pressure does not depend on the coordinate x_2, i. e. $p(Q)=p(x_1)$, the problem in question reduces to classic plane strain Griffith crack problem. The simplest solution of the Griffith crack problem, corresponding to uniform normal pressure $p(x_1)=p=const$, is

$$u_3(x_1,0) = (1-\nu)p(a^2-x_1^2)^{1/2}/\mu, \quad K_I = pa^{1/2} \tag{16}$$

which may be utilized as a trial solution.

It is clear that $W^-(x_1,x_2;Q)=W^+(-x_1,x_2;Q)$, and it is sufficient to compute only the weight function $W^+(Q;M)$ corresponding to the crack front part Γ^+. In order to do this, choose a such variation of the crack front parts $\Gamma^\pm$, for which $\delta n^+(M)=\delta a=const$ and $\delta n^-(M)=0$. The variation of the crack face displacement due to the variation of the crack front may be obtained by differentiating $u_3(x_1,0)$ with respect to a and assuming that a and $-a$ are independent variables. This results in the formula

$$\delta u_3(Q) = p(1-\nu)(x_1+a)^{1/2}(a-x_1)^{-1/2}\delta a/(2\mu) \tag{17}$$

and the corresponding function $U^+(Q;M)$ takes the form

$$U^+(Q;M) = (x_1+a)^{1/2}(a-x_1)^{-1/2}/(\pi a^{1/2}) \tag{18}$$

Function $f(Q)=(x_1+a)/(2a)$ is harmonic in the crack domain S, equals to zero on the boundary Γ^- and equals to unity on the boundary Γ^+, i. e. $f_0^+(M)=1$. Hence, the appropriate function $F^+(Q;M)$ is

$$F^+(Q;M) = 2a/(x_1+a) \tag{19}$$

It is known that the function $w(z)=tg(\alpha z/2)$, where $\alpha=\pi/(2a)$, is the conformal mapping of the strip S into the unit circle. Substituting this function into the equation (9) gives

$$G(Q;M) = -\ln\{[ch\alpha(x_2-y_2)-\cos\alpha(x_1-y_1)]/[ch\alpha(x_2-y_2)+\cos\alpha(x_1+y_1)]\}/(4\pi) \tag{20}$$

and the normal derivative of the Green's function on the boundary Γ^+ may be written after some calculations in the form

$$\partial G(Q;M)/\partial n^+(M) = -\cos\alpha x_1(ch\alpha(x_2-y_2)-\sin\alpha x_1)^{-1}/(4a) \tag{21}$$

Substituting the relations (18), (19) and (21) into the general equation (7) and using the property that $W^-(x_1,x_2;Q)=W^+(-x_1,x_2;Q)$, gives the following new analytical representation of 3-D weight functions for a straight through "tunnel" crack:

$$W^\pm(Q;M) = \cos\alpha x_1(a^2-x_1^2)^{-1/2}(ch\alpha(x_2-y_2)\mp\sin\alpha x_1)^{-1}/(2\pi a^{1/2}) \tag{22}$$

This is an exact representation of the 3-D weight function under consideration, an approximate one was proposed by Oore and Burns (1980).

NUMERICAL RESULTS

Using the established weight functions it is possible to estimate the stress intensity factors for arbitrary 3-D stress fields. For example, consider the problem for an infinite solid with a "tunnel" crack when the uniform normal pressure $p=const$ is applied only inside a rectangular area $-a<x_1<a$, $-b<x_2<b$.

Substituting the weight functions (22) into the equation (15) results in the explicit formulae for the stress intensity factors corresponding to the given stress field

$$K_I^{\pm}(y_2) = (p/2\pi a^{1/2}) \int_{-a}^{a} (a^2-x_1^2)^{-1/2}\cos\alpha x_1 dx_1 \int_{-b}^{b} (\mathrm{ch}\alpha(x_2-y_2)\mp\sin\alpha x_1)^{-1}dx_2 \qquad (23)$$

It is not difficult to evaluate the integral with respect to the variable x_2 in the closed form, and the equations (23) may be written as

$$K_I^{\pm}(y_2) = K_I^0 H(h,y) \qquad (24)$$

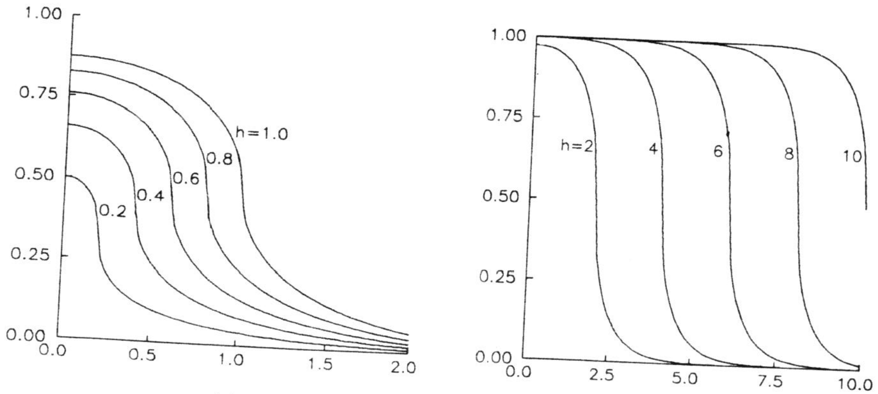
where

$$H(h,y) = (2/\pi^2) \int_{-1}^{1} (1-x^2)^{-1/2}\chi(x,y)dx \qquad (25)$$

$$\chi = \mathrm{arctg}\{[\exp(\pi(h-y)/2)+\sin(\pi x/2)]/\cos(\pi x/2)\}-\mathrm{arctg}\{[\exp(-\pi(h+y)/2)+\sin(\pi x/2)]/\cos(\pi x/2)\}$$

$$h = b/a, \quad y = y_2/a, \quad K_I^0 = pa^{1/2}$$

One can verify that the function $\chi(x,y)$ tends to $\pi(1-x)/2$ and the function $H(h,y)$ tends to unity as $h \to \infty$. This limit corresponds to the plane strain solution and the equation (24) gives for this case the following stress intensity factor values: $K_I^{\pm}(y_2)=K_I^0$, which are in agreement with the classic Griffith crack solution.



(a) (b)

Fig. 1a,b. Variation of H with y for different values of h

For finite values of the parameter h (when the problem in question is essentially three-dimensional) the values of the integral from the equation (25) are not available in the closed form. In order to estimate the stress intensity factors, the integral in this case is replaced by a quadrature formula.

Variation of function $H(h,y)$ with y is shown in Fig. 1a for $h=0.2$, 0.4, 0.6, 0.8, and 1.0 and in Fig. 1b for $h=2$, 4, 6, 8, and 10.

CONCLUSION

Proposed three-dimensional weight function method is a generalization of a technique by Borodachev A. N. (1990) for a circular crack 3-D weight function determination. This method allows us to obtain a variety of new three-dimensional weight functions by using known solutions of crack problems as trial ones. Also it is worth to note that one may utilize approximate trial solutions in order to develop approximate 3-D weight functions.

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