INVESTIGATION ON THE SINGULARITIES OF ASYMPTOTIC SOLUTIONS IN STATIONARY CRACK AND STEADY CRACK GROWTH WITH ELASTIC-PLASTIC MATERIAL

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Abstract: In the present paper the functional forms of asymptotic stress and deformation fields of mode I stationary crack and steady state crack extension in an elastic plastic material are investigated. The material is characterized by J_2-flow theory with linear hardening and power-law hardening. All stresses and strains of the asymptotic crack tip field are separable functional forms of r and θ which represent to the polar coordinate system centred at the actual crack tip. The results of stress and deformation fields for both materials are comparable.

1. Introduction

The investigation of the near tip stress and deformation fields with an elastic plastic material is necessary for the development of fracture mechanics and has become one of its central problems. The asymptotic solution for a stationary crack with a power-law hardening material using deformation theory of plasticity, the so-called HRR-solution /1,2/, supplies a theoretical basis for elastic plastic fracture mechanics, but the elastic deformation is neglected in this solution. The asymptotic solution of stationary crack with linear hardening material was also investigated in /2/, where the important nonlinear term was neglected. Therefore, the solutions for both materials are not comparable. This problem is intensified for steady state crack extension. For a power-law hardening material a solution with logarithmic functions for stresses and plastic strains in the active plastic loading zone has been found in /3/ under some assumptions, but for a linear hardening material a stress field has been found in /4/ using a power-law function and in /5,6/ a complete solution for this material was investigated. The different functional forms for both materials lead to mathematical difficulties and the solutions are farther not comparable.

The issues are now, what functional form an asymptotic solution near the crack tip must possess and why the solutions for both materials are different only for various approximations. In order to answer these questions we will try to present in this paper uniform functions for asymptotic solutions with an elastic plastic material for stationary crack and steady state crack growth. The investigation assumes mode I plane stress and plane strain, small deformations, quasi static case, isotropic material and J_2-flow plasticity theory.

2. Constitutive Equations

We can first view both elastic plastic materials with linear hardening and power-law hardening in Fig.1 and Fig.2.

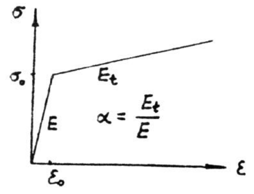

Fig.1 linear hardening material

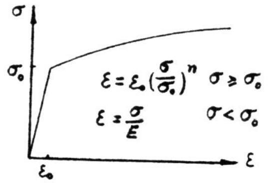

Fig.2 power-law hardening material

σ_0 denotes the yield stress, ε_0 the yield strain, E is Young's modulus and E_t is the tangent modulus. The incremental plasticity theory with the J_2 yield condition is used and the constitutive equation can be written as follows:

$$\dot{\varepsilon}_{ij} = \frac{1}{E}[(1+\nu)\dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\,\delta_{ij}] + \dot{\lambda}\, s_{ij} \qquad (1)$$

where

$$\dot{\lambda} = \begin{cases} \dfrac{3}{2}\dfrac{\omega}{E}\dfrac{\dot{\sigma}_e}{\sigma_e} & \dot{\sigma}_e \geq 0 \quad \textit{linear hardening} \\[2ex] \dfrac{3}{2E}[n(\dfrac{\sigma_e}{\sigma_0})^{n-1} - 1]\dfrac{\dot{\sigma}_e}{\sigma_e} & \dot{\sigma}_e \geq 0 \quad \textit{power-law hardening} \\[2ex] 0 & \dot{\sigma}_e < 0 \quad \textit{elastic behaviour} \end{cases} \qquad (2)$$

with $\omega = \alpha^{-1} - 1$.

Basing on the assumption that a proportional loading condition is considered we can integrate the above constitutive equations (1) and (2), so that they become the following form:

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}] + \lambda\, s_{ij} \qquad (3)$$

with

$$\lambda = \begin{cases} \dfrac{3}{2}\dfrac{\omega}{E}(1 - \dfrac{\sigma_0}{\sigma_e}) & \sigma_e \geq \sigma_0 \quad \textit{linear hardening} \\[2ex] \dfrac{3}{2E}[(\dfrac{\sigma_e}{\sigma_0})^{n-1} - 1] & \sigma_e \geq \sigma_0 \quad \textit{power-law hardening} \\[2ex] 0 & \sigma_e < \sigma_0 \quad \textit{elastic behaviour} \end{cases} \qquad (4)$$

where in (1), (2), (3) and (4) σ_{ij} is the stress tensor, ε_{ij} the strain tensor, σ_e the effective stress, s_{ij} the stress deviator, δ_{ij} the Kronecker delta and ν Poisson's ratio. $(\dot{\ })$ is the material time derivative. The above constitutive equations (1), (2), (3) and (4) together with the equilibrium equations

$$\sigma_{ij,\,j} = 0 \qquad (5)$$

and the geometric conditions

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,\,j} + u_{j,\,i}) \qquad (6)$$

supply the investigation basis for the stationary crack and the steady state crack growth problem.

3. Asymptotic Solution Form

In the following we investigate the crack problem for both plane stress and plane strain mode I cases. Let x, y be a Cartesian coordinate system of fixed orientation travelling with the crack tip such that the x-axis is in the direction of the crack extension. Similarly, let r and θ be polar coordinates corresponding to x and y in Fig.3.

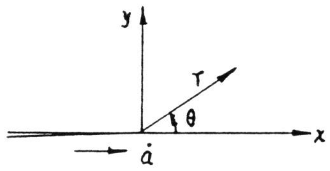

Fig.3 coordinate system

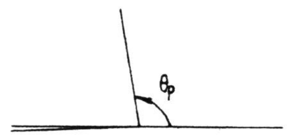

Fig.4 growing crack tip field

In the Fig.3 $\dot{a}$ is the crack growing velocity. For the stationary crack $\dot{a}$ is zero. We consider first the asymptotic solution forms for linear elastic material. They are

$$\begin{aligned} \sigma_{ij}(r,\theta) &= A\sigma_0 r^{-s}\tilde{\sigma}_{ij}(\theta) \\ \varepsilon_{ij}(r,\theta) &= A\varepsilon_0 r^{-s}\tilde{\varepsilon}_{ij}(\theta) \\ u_i(r,\theta) - u^*_i &= A\varepsilon_0 r^{-s+1}\tilde{u}_i(\theta) \end{aligned} \tag{7}$$

where $s = 0.5$ and $A\sigma_0 = K / \sqrt{2\pi}$. K is the so-called stress intensity factor and u^*_i the rigid body displacement.

Further we can still show the asymptotic solution for steady state crack growth with a linear hardening material. Basing on the assumption that the crack tip field can be divided into a plastic loading zone and an elastic unloading zone, Fig.4, the asymptotic stress field is given in /4/, while a complete solution is presented in /5,6/ similar to the equations (7). The asymptotic solutions for a linear elastic material and for steady state crack growth with a linear hardening material can satisfy the above equations (1), (2), (5) and (6) and are exact solutions.

We will now investigate the asymptotic solution for stationary crack with a linear hardening material keeping in mind the structure of eqs. (7), but there is a problem for the field formulation in (4) with the nonlinear term σ_0 / σ_e. The question is, how the solution is influenced by this nonlinear term for linear hardening material. This nonlinear term σ_0 / σ_e is neglected in /2/ and the solutions become those of the linear elastic case, therefore these solution are not interesting. For this reason we will investigate the asymptotic solution for a stationary crack with a linear hardening material considering this nonlinear term. Let us first consider the nonlinear term σ_0 / σ_e. Regarding the field equation formulation in (4) und considering the solution structure (7) this term depends then on the angle θ and the radius r, which injures the solution structure (7). The question is, whether it is possible to select a radius r, so that this term is only a function of the angle θ. So we have investigated in /7/ the crack flank and get some results. Using these results and considering the in /8/ suggested crack tip opening displacement it is possible, that the nonlinear term σ_0 / σ_e becomes then $\varepsilon_0[\tilde{u}_x(\pi)+\tilde{u}_y(\pi)]/\tilde{\sigma}_e(\theta)$. Here, the yield strain and the displacements at the crack flank are used. They are very important values. The yield strain or the yield stress characterizes whether the material particle is elastic plastic or elastic and the displacements at the crack flank determine a crack. Finally, considering the non-stressed crack surface the asymptotic solution leads to an eigenvalue problem.

We have extended this result for the stationary crack with a power-law hardening material and steady state crack extension with a power-law hardening material based on the assumption that the steady state growing crack tip field may be divided into a plastic loading zone and an elastic unloading zone. For the solution of these eigenvalue problems we used the same solution procedure given in /5,6/.

4. Results

In this paper the stress and strain distributions are presented first for plane stress around the crack tip for a stationary crack with a linear hardening material with α = 0.05 and a power-law hardening material with n = 2 in Figs. 5,6,9 and 10. For steady state crack extension the stress and strain distributions are given for a power-law hardening material with n = 2 in Figs 8,12 and for a linear hardening material with α = 0.05 in Figs. 7,11 which are given in /4,5,6/. It is remarkable, that the stresses and strains for both materials for a stationary crack as well as for a steady state crack extension are approximately comparable, only the stress σ_{xx} near the crack flank for both materials in the case of the stationary crack is different. Further the stress distribution for a power-law hardening material for steady state crack extension is compared with the results for the same material assuming the small scale yield condition from a finite element calculation from /9/. They supply also comparable results in Fig. 13. The values for singularities and unloading angles are as follows:

stationary crack:	s = 0.515		for linear hardening $\alpha = 0.05$
	s = 0.521		for po.-law hardening n = 2
steady growing crack:	s = 0.178	$\theta_p = 1.214$	for linear hardening $\alpha = 0.05$
	s = 0.105	$\theta_p = 1.102$	for po.-law hardening n = 2

5. Conclusion

In this paper the asymptotic solution structures for both a linear hardening material and a power-law hardening material has been investigated in both limiting cases, namely stationary crack and steady state crack extension. The solution structures have the form of (7) and the stresses and strains have the same singularities. The angular functions for stresses and strains depend on the yield strain ε_0. The displacements at the crack flank are used, so that the nonlinear term σ_0 / σ_e in the constitutive equation becomes $\varepsilon_0[\tilde{u}_x(\pi)+\tilde{u}_y(\pi)] / \sigma_e$ by the field equation formualtion and the asymptotic solution leads to an eigenvalue problem.

We have presented the complete solutions for the asymptotic crack tip stresses and strains with a linear hardening material and a power-law hardening material for the stationary crack and steady state crack extension. The stresses and strains for both materials are comparable. We will try to explain the approximate comparability for linear hardening material with α = 0.05 and power-law hardening material with n = 2. Let us consider the asymptotic stress behaviour $\sigma_{ij} = A\sigma_0 r^{-s}\tilde{\sigma}_{ij}(\theta)$ which has a singular form at the crack tip. The linear hardening material has an approximation of the σ-εcurve with a constant value E_t, but the power-law hardening material is approximated with its actual stress which has a singular form at the crack tip, so that the strain for the power-law hardening material at the crack tip is higher than that for the linear hardening material. Only the amplitude factor for the asymptotic solution remains undetermined which should be dependent on loading and geometry conditions.

Acknowlegements

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Reference

/1/ Rice,J.R. & Rosengren,G.F.: "Plane strain deformation near a crack tip in a power-law hardening material"; J. Mech. Phys. Solids, Vol. 16, 1968, pp 1-12

/2/ Hutchinson,J.W.: "Singular behavior at the end of a tensile crack in a hardening material"; J. Mech. Phys. Solids, Vol. 16, 1968, pp 13-31

/3/ Gao,Y.C. & Hwang,K.C.: "Elastic-plastic fields in steady crack growth in a strain hardening material"; Proceedings of Fifth International Conference on Fracture, Francois,D., Ed., Pergamon Press, New York, Vol. 2, 1981, pp 669-682

/4/ Amazigo,J.C. & Hutchinson,J.W.: "Crack tip fields in steady crack growth with strain hardening"; J. Mech. Phys. Solids, Vol. 25, 1977, pp 81-97

/5/ Brocks,W.; Guo,T.M. & Fricke, St.: "Asymptotic and Numerical Analysis of Crack Tip Fields of Mode I in Steady Crack Growth with Linear Hardening Material", Proceeding of 9th European Conference on Fracture, S.Sedmak et al, Ed., Volume II, 1992, pp 821-828

/6/ Guo,T.M. & Brocks,W.: "Asymptotische Analyse der stationär wachsenden Rißspitzenfelder von Modus I in linear verfestigendem Material", to be printed in Proceeding of 25th DVM-Arbeitskreise Bruchvorgänge, 1993

/7/ Guo,T.M.: "Untersuchung des singulären Rißspitzenfeldes bei stationärem Rißwachstum in verfestigendem Material", Dissertation to be printed, Ruhr-University Bochum, 1993

/8/ Tracey,D.M.: Trans. ASME, Ser. H, J. Engng Materials and Technol. 98, 146.

/9/ Deng,X.M. & Rosakis,A.: "A finite element investigation of quasi-static and dynamic asymptotic crack-tip fields in hardening elastic-plastic solids under plane stress", International Journal of Fracture 58, 137-156, 1992

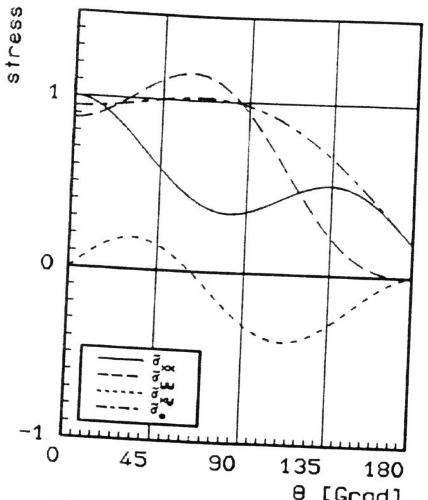

Fig.5: linear hardening

St. crack for pl. stress
$\alpha = 0.05$, $\nu = 0.3$, $\varepsilon_0 = 0.005$
Normalized such that $\tilde{\sigma}_{xx}(0)=1$

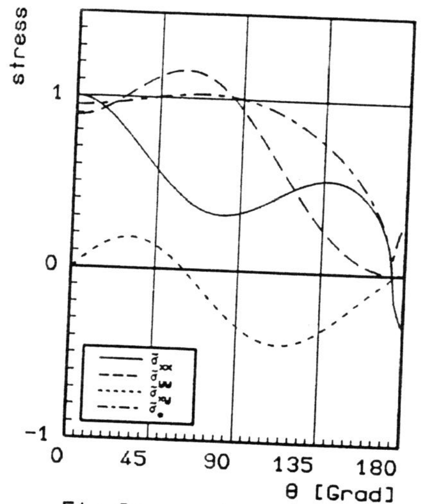

Fig.6: po.-law hardening

St. crack for pl. stress
$n = 2.0$, $\nu = 0.3$, $\varepsilon_0 = 0.005$
Normalized such that $\tilde{\sigma}_{xx}(0)=1$

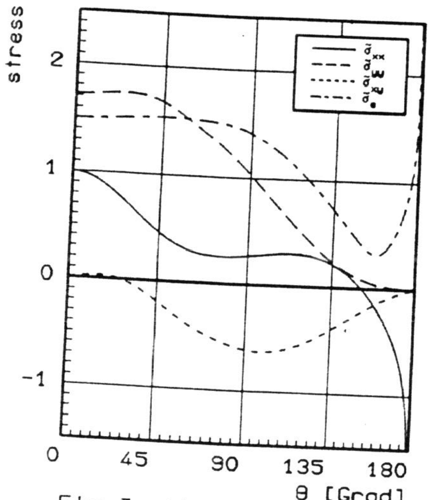

Fig.7: linear hardening

St. gr. crack for pl. stress
$\alpha = 0.05$, $\nu = 0.3$
Normalized such that $\tilde{\sigma}_{xx}(0)=1$

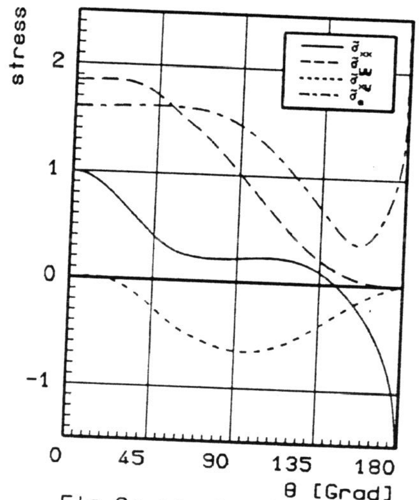

Fig.8: po.-law hardening

St. gr. crack for pl. stress
$n = 2.0$, $\nu=0.3$, $\varepsilon_0 = 0.005$
Normalized such that $\tilde{\sigma}_{xx}(0)=1$

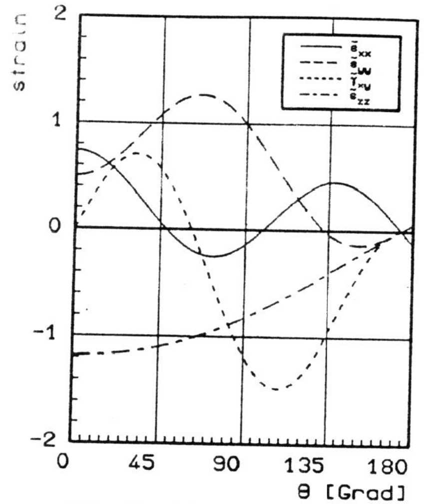

Fig.9: linear hardening

St. crack for pl. stress
$\alpha = 0.05$, $\nu = 0.3$, $\varepsilon_0 = 0.005$
Normalized such that $\tilde{\varepsilon}_{\psi\psi}(90^\circ)=1$

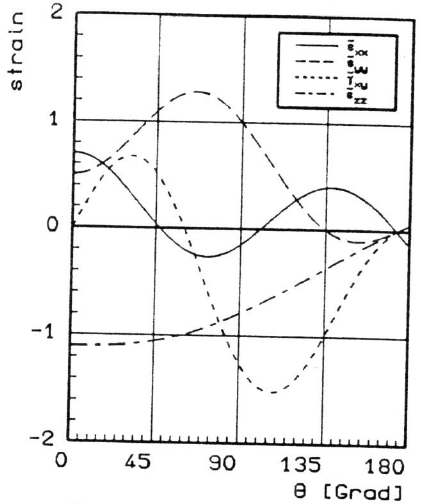

Fig.10: po.-law hardening

St. crack for pl. stress
$n = 2.0$, $\nu = 0.3$, $\varepsilon_0 = 0.005$
Normalized such that $\tilde{\varepsilon}_{\psi\psi}(90^\circ)=1$

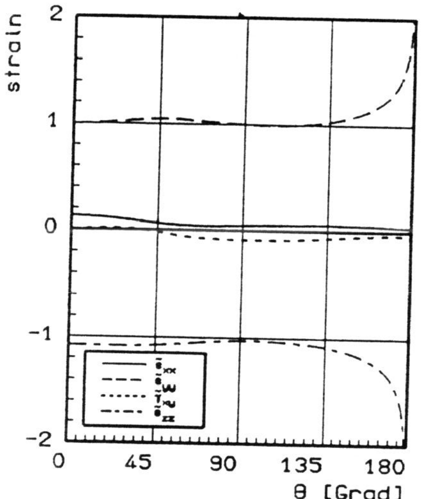

Fig.11: linear hardening

St. gr. crack for pl. stress
$\alpha = 0.05$, $\nu = 0.3$
Normalized such that $\tilde{\varepsilon}_{\psi\psi}(90^\circ)=1$

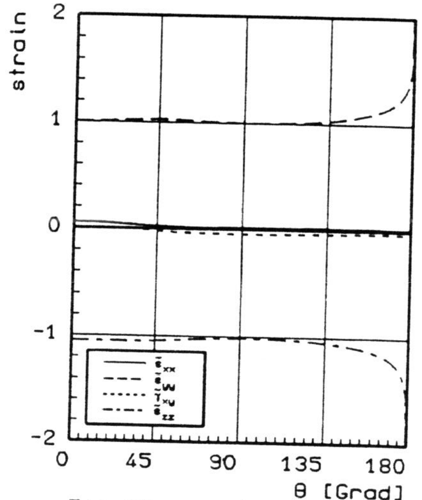

Fig.12: po.-law hardening

St. gr. crack for pl. stress
$n = 2.0$, $\nu=0.3$, $\varepsilon_0 = 0.005$
Normalized such that $\tilde{\varepsilon}_{\psi\psi}(90^\circ)=1$

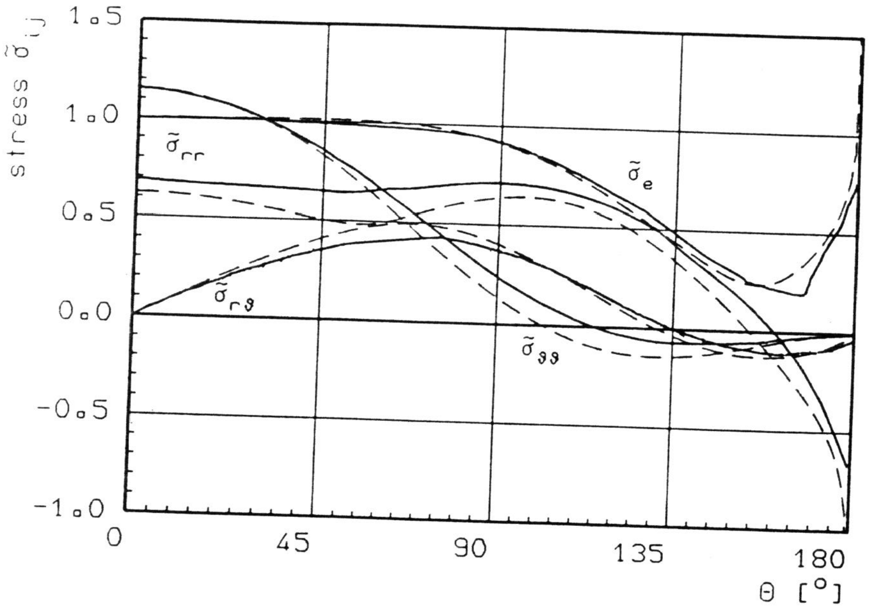

Fig.13: Comparison with results from fem

Power-law hardening material (plane stress)
$n = 2.0$, $\nu = 0.3$, $\varepsilon_0 = 0.005$
Normalized such that $\tilde{\sigma}_e(0°) = 1$

– – – current study
——— Deng & Rosakis