INVESTIGATION OF THE STRESSED NONLINEAR ELASTIC PLANE WITH CRACKS

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ABSTRACT

The elastic field is represented by means of two analytical functions and the explicit solution of the problem is obtained. It is shown that the strains at the end-points of cut are bounded.

KEYWORDS

Harmonic Type Material, Crack Problems, Plane nonlinear Problems.

There are investigated some plane problems for the nonlinear elastic harmonic type plane (cf. John (1960)) with the straight-forward cuts, which are disposed along the straight fixed-line. On the cut's borders acts well-balanced system of the outward efforts and there is the homogeneous field of the strains on the infinity.

1. Let us consider the physical domain S which is the $z=x+iy$ plane with cuts along of finite number segments $L_k=[a_k,b_k]$ $(k=1,2,\dots,n)$ of the real axis L.

Let $\Gamma=\sum_{k=1}^{n} L_k$ and $\Gamma'=L\setminus\Gamma$. On the Γ prescribed the known system of outward efforts, and there is the homogeneous field of the strains at the infinity. Rotation at the infinity and tangent effort on the line L are absent. The boundary conditions have the form (cf. Muskhelishvili (1966))

$$Y_y^+=N_1(x),\quad Y_y^-=N_2(x)\ \text{on}\ \Gamma,\qquad X_y^{\pm}=0\ \text{on}\ L, \tag{1}$$

where Y_y, X_y are the components of Cauchy's strains tensor, signs $(\cdot)^+$ and $(\cdot)^-$ denote the limiting values from the upper and the lower borders of the cuts, respectively. Here

$N_1(x)$ and $N_2(x)$ are real functions given on the Γ satisfying Holder condition. Besides

$$X_x^{(\infty)}=P_1, \quad Y_y^{(\infty)}=P_2, \quad X_y^{(\infty)}=0.$$

For solving the above boundary-value problem the complex representation by two analytic functions $\phi(z)$ and $\psi(z)$ defined in physical domain S (cf. Lourie (1980) and Doborjginidze (1989)) are used

$$X_x+Y_y+4\mu = (\lambda+2\mu)J^{-1/2}q\Omega(q),$$

$$Y_y-X_x-2iX_y = -4(\lambda+2\mu)J^{-1/2}\Omega(q)q^{-1}(\partial z^*/\partial z)(\partial z^*/\partial\bar{z}),$$

$$\partial z^*/\partial z = \mu(\lambda+2\mu)^{-1}\phi'^2(z)+(\lambda+\mu)(\lambda+2\mu)^{-1}\phi'(z)\overline{\phi'(z)}^{-1}, \qquad (2)$$

$$\partial z^*/\partial\bar{z} = (\lambda+\mu)(\lambda+2\mu)^{-1}\left[\phi(z)\overline{\phi''(z)}/\overline{\phi'^2(z)}-\overline{\psi'(z)}\right],$$

$$u+iv=\mu/(\lambda+2\mu)\int\phi'^2(z)dz+(\lambda+\mu)/(\lambda+2\mu)\left[\phi(z)/\overline{\phi'(z)}+\overline{\psi(z)}\right]-z, \qquad (3)$$

where

$$J^{1/2}=(\partial z^*/\partial z)(\partial\bar{z}^*/\partial\bar{z})-(\partial z^*/\partial\bar{z})(\partial\bar{z}^*/\partial z), \quad q=2|\partial z^*/\partial z|,$$

$$\Omega(q)=q-2(\lambda+\mu)/(\lambda+2\mu) \qquad (4)$$

It is known that if $|z|$ is sufficiently large

$$\phi(z)=a_o z+\phi_o(z), \quad \psi(z)=b_o z+\psi_o(z), \qquad (5)$$

holds (cf. Doborjginidze (1989)), where

$$a_o^2=[(\lambda+\mu)/\mu]\left[(2\mu(P_1+P_2)+P_1P_2+4\mu^2)/(\lambda(P_1+P_2)-P_1P_2+4\mu(\lambda+\mu))\right],$$

$$b_o=\left[(\lambda+2\mu)(P_1-P_2)\right]/\left[\lambda(P_1+P_2)-P_1P_2+4\mu(\lambda+\mu)\right]. \qquad (6)$$

Besides it is proven, that $\psi'(z)\neq 0$ for each $z\in S+\Gamma$. By (1)-(4) the formulated above problem can be reduced to the next problem

with $$|\phi'^2(x)| = F^{\pm}(x) \quad \text{on } \Gamma, \qquad (7)$$

$$F(x) = \frac{\lambda+\mu}{\mu}\cdot\frac{(Y_y+2\mu)(Y_y+2\mu+\gamma)}{(\lambda+2\mu)(2Y_y+\gamma+4\mu)-(Y_y+2\mu)(Y_y+2\mu+\gamma)}, \qquad (8)$$

and γ is the given real constant

$$\gamma = \frac{4\mu(\lambda+\mu)(\lambda+2\mu)a_o^2b_o}{[\mu a_o^2+(\lambda+\mu)(1-b_o)][\mu a_o^2+(\lambda+\mu)(1+b_o)]}. \qquad (9)$$

The most general class of solutions of the problem (7),(5) has the following form

$$\phi'(z) = \exp\left[(4\pi i X(x))^{-1}\int_\Gamma(F_o^+ + F_o^-)X(x)(x-z)^{-1}dx + (4\pi i)^{-1}\int_\Gamma(F_o^+ + F_o^-)(x-z)^{-1}dx + P_n(z)(2X(z))^{-1}\right], \qquad (10)$$

where

$$X(z)=\left[(z-a_i)(z-b_i)\dots(z-a_n)(z-b_n)\right]^{1/2}, \quad (\lim z^{-n}X(z)=1), \qquad (11)$$

and $P_n(z)=c_o z^n+c_1 z^{n-1}+\dots+c_n$ is any polynom of degree less then n with real coefficients. The constants c_o and c_1 are defined as follows

$$c_o=\ln a_o^2, \quad c_1=-(a_1+b_1+\dots+a_n+b_n)\ln a_o^2/2, \qquad (12)$$

while other coefficients are to be find from the condition of single-valueness of displacement vector.

After $\phi(z)$ has been defined, potential $\psi(z)$ can be found from the relation

$$\psi(z) = \overline{\phi(z)}\phi''(z)(\phi'(z))^{-2} - \left[\mu/(\lambda+\mu)+|\phi'^2(z)|^{-1}\right](Y_y-X_x)\phi'^2(z)/(X_x+Y_y+4\mu) \qquad (13)$$

applying the well known way.

2. Let us consider the particular case of one cut (n=1) where on the interval [-b,b] of the cut [-a,a] (b<a) the uniformly distributed normal load acts with intensity N_o, while the remainder part of the cut is free of outward efforts. It is assumed that rotation and strains are absent at the infinity. Then from (10), according to (8), after some calculations we get

$$\phi'(z) = \exp\left\{-F_o(2\pi i)^{-1}\left[\ln\frac{(z+b)(d(z)+bz)}{(z-b)(d(z)-bz)} - \frac{z}{(z^2-a^2)^{1/2}}\ln\frac{(b^2-a^2)^{1/2}+b}{(b^2-a^2)^{1/2}-b}\right]\right\}, \qquad (14)$$

where

$$F_o = \ln\left[(\lambda+\mu)\mu^{-1}(2\mu+N_o)(2(\lambda+\mu)-N_o)^{-1}\right],$$

$$d(z) = \left[(b^2-a^2)(z^2-a^2)\right]^{1/2} - a^2. \qquad (15)$$

Using this formulae and (2), we obtain $Y_y^- \equiv N(x)$ where

$$N(x) = 2\mu\left[A^{-\beta(x)+1}-1\right]/\left[1+\mu(\lambda+\mu)^{-1}A^{-\beta(x)+1}\right], \qquad (16)$$

for $x \geq 0$ $(\theta = \mathrm{arctg}((a^2-b^2)^{1/2}/b))$, and

$$A= (\lambda+\mu)\mu^{-1}(2\mu+N_o)(2(\lambda+\mu)-N_o)^{-1},$$

$$\beta(x) = \frac{1}{\pi}\,\mathrm{arctg}\,\frac{2b|x|(a^2-b^2)^{1/2}(x^2-a^2)^{1/2}}{2b^2x^2-a^2b^2-a^2x^2} + \frac{(1-2\theta/\pi)|x|}{(x^2-a^2)^{1/2}}. \qquad (17)$$

As we can see from the formulae obtained values of N(x) essentially depends of the elastic properties of a material. Besides

$$\lim_{|x|\to a} N(x) = 2(\lambda+\mu),$$

which means that in the neighbourhoods of the end points of the cut the normal strains receive sufficiently large finite values. Note, that according to the classic linear theory this strains on the pointed out neighbourhoods have singularity of order 1/2. Certainly it does not correspond to reality.

3. Let us consider the case of one cut where on the middle point of the each border of the cut act opposite directed equal strengths with intensity P_o (that is $N(x)=P_o\delta(x)$, where $\delta(x)$ is Dirak's function).

The solution of this problem can be obtained from preceding problem by passing to limit:

$$\lim_{\substack{a\to 0\\ N_o\to\infty}} (N_o \cdot 2a) = P_o.$$

After some calculation and using (15),(16) we will get

$$\phi'(z) = \exp\left[(\lambda+2\mu)aP_o/4\pi\mu(\lambda+\mu)z(z^2-a^2)^{1/2}\right], \quad \text{for } z\in S, \qquad (18)$$

$$N(x) = 2\mu(\lambda+\mu)\left[\exp h(x)-1\right]/\left[\lambda+\mu+\mu\exp h(x)\right], \quad \text{for } |x|\geq a, \qquad (19)$$

where

$$h(x) = \left[(\lambda+\mu)aP_o\right]/\left[4\pi\mu(\lambda+\mu)|x|(x^2-a^2)^{1/2}\right]. \qquad (20)$$

Therefore $\lim_{|\lambda|\to a+} N(x) = 2(\lambda+\mu)$.

In this case the normal strains on the ends of the cut turned out bounded values whilst in the classic linear theory we have singularity there. Viz, recall the classical analogy of formula (19)

$$N(x) = \left[aP_o\right]/\left[\pi|x|(x^2-a^2)^{1/2}\right], \quad \text{when } |x|>a. \qquad (21)$$

4. Suppose that we have the one-axis homogeneous stretching along the y axis at the infinity. That is, $P_1=0$, $P_2=P_o$ and borders of the cut are free of outward efforts.

Then according to (20) we have

$$\phi'(z) = \left[\frac{2(\lambda+\mu)(2\mu-N_o)}{4\mu(\lambda+\mu)-\lambda N_o}\right]^{1/2} \times$$

$$\times \exp\left[\frac{z}{2(z^2-a^2)^{1/2}} \ln \frac{(2\mu+N_o)(4\mu(\lambda+\mu)-\lambda N_o)}{(2\mu-N_o)(4\mu(\lambda+\mu)+\lambda N_o)}\right], \qquad (22)$$

and using (2) we get

$$N(x) = N_o/2+2\mu\left[\alpha_o\alpha(x)^{\beta(x)}-1\right]/\left[1+2\mu\alpha_o\alpha(x)^{\beta(x)}\right]+$$
$$+\left(\left[N_o^2/4+4\mu^2(\lambda+2\mu)^2\alpha_o^2\alpha(x)^{2\beta(x)}\right]/\left[1+2\mu\alpha_o\alpha(x)^{\beta(x)}\right]^2\right)^{1/2}, \qquad (23)$$

where

$$\alpha_o=\frac{2\mu-N_o}{4\mu(\lambda+\mu)-\lambda N_o}, \quad \alpha(x)=\frac{2\mu+N_o}{2\mu-N_o}\cdot\frac{4\mu(\lambda+\mu)-\lambda N_o}{4\mu(\lambda+\mu)+\lambda N_o}, \quad \beta(x)=|x|(x^2-a^2)^{1/2}.$$

The last formula gives distribution of the normal strains for $|x|\geq a$ without peculiarities.

It can be considered, in the same way, the boundary value problems with displacements, when on the borders of a cut the normal elastic displacements are given. The typical peculiarity of this problems is that the normal strains are bounded at the ends of the cut.

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