INVESTIGATION OF A KINKED CRACK SYSTEM WITH GENERALIZED STRESS SINGULARITIES BY MEANS OF THE INTEGRAL EQUATION METHOD

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Abstract

The problem of a crack running in a homogeneous matrix material towards an inclusion with different material properties and kinking at the material interface into an interface crack is discussed, using a modified CT - specimen as an example. For the sake of the solution of this boundary value problem a singular integral equation for the density $\omega(t)$ of the corresponding complex potentials $\varphi_j(z)$ is formulated. The analytical treatment of this equation shows the existence of complicated eigenvalues at the kinking point.

Keywords

Crack kinking, generalized stress singularities, singular integral equation

1 Introduction

The modified CT - specimen (cf. Fig. 2) divides the complex plane in two simply connected domains S_0 and S_1 which are related to the matrix and the inclusion, respectivly. The matrix is loaded by two known, opposite equal forces F_n acting at the points z_n $(n = 1, 2)$. The stress and displacement state is given in terms of Kolosov's complex potentials φ and ψ.

The complex potentials $\varphi_j(z)$ $(z \in S_j,\ j = 0, 1)$ decompose into a sectionally holomorphic function $\varphi(z)$ and a function $\varphi^*(z)$ due to the forces $F_n = X_n + iY_n$ in the following way:

$$\varphi_j(z) = \varphi(z) + \varphi^*(z)\,, \quad z \in S_j \tag{1}$$

where

$$\varphi^*(z) = \sum_{n=1}^{2} -\frac{X_n + iY_n}{2\pi(1+\kappa_0)} \ln(z - z_n) \qquad (2)$$

For the matrix ($z \in S_0$) this decomposition is obvious. In case of the inclusion it must be kept in mind, that $\varphi_1(z)$ and $\varphi^*(z)$ are both holomorphic in S_1 hence $\varphi(z)$ is holomorphic in S_1 too. This kind of decomposition is not really necessary, but simplifies the resulting equations. In the same way the potential $\psi_j(z)$ ($z \in S_j$) is decomposed as

$$\psi_j(z) = \psi(z) + \psi^*(z), \quad z \in S_j \qquad (3)$$

where

$$\psi^*(z) = \sum_{n=1}^{2} \left[\kappa_0 \frac{X_n - iY_n}{2\pi(1+\kappa_0)} \ln(z - z_n) + \frac{\bar{z}_n(X_n + iY_n)}{2\pi(1+\kappa_0)} \frac{1}{z - z_n}\right] \qquad (4)$$

For the sectionally holomorphic functions $\varphi(z)$ and $\psi(z)$ the following system of boundary equations is valid:

$$\begin{aligned} \varphi^{\pm}(t) + t\overline{\varphi'^{\pm}(t)} + \overline{\psi^{\pm}(t)} &= f^{\pm}(t) + c_k; \\ \kappa^{\pm}(t)\varphi^{\pm}(t) - t\overline{\varphi'^{\pm}(t)} - \overline{\psi^{\pm}(t)} &= 2\mu^{\pm} g^{\pm}(t); \end{aligned} \quad t \in L_k \qquad (5)$$

with $f^{\pm}(t)$ and $g^{\pm}(t)$ defined on L_k by

$$f^{\pm}(t) = \int_{t_k}^{t} (\sigma_n^{\pm}(\tau) + i\sigma_t^{\pm}(\tau))\, d\tau - f^{*\pm}(t) \qquad (6)$$

$$g^{\pm}(t) = u(t) + iv(t) - g^{*\pm}(t) \qquad (7)$$

Here, the functions $f^{*\pm}(t)$ and $g^{*\pm}(t)$ are due to the known functions $\varphi^*(z)$ and $\psi^*(z)$ [1]. The integration constants c_k are calculated during the solution process while $\kappa^{\pm}$ and $\mu^{\pm}$ are the elastic parameters of the matrix and the inclusion.

In order to solve the equation system (5), the boundary conditions have to be specified. Therefore, the latter are given by

$$\sigma_n^{\pm}(t) + i\sigma_t^{\pm}(t) = 0; \quad t \in L_2, L_1'' \qquad (8)$$

$$\begin{aligned} \sigma_n^{+}(t) + i\sigma_t^{+}(t) &= \sigma_n^{-}(t) + i\sigma_t^{-}(t); \\ u^{+}(t) + iv^{+}(t) &= u^{-}(t) + iv^{-}(t); \end{aligned} \quad t \in L_1' \qquad (9)$$

$$\sigma_{xx}^{\infty} = \sigma_{yy}^{\infty} = \sigma_{xy}^{\infty} = 0 \qquad (10)$$

which leads to

$$f^{\pm}(t) = -f^{*\pm}(t); \quad t \in L_2, L_1'' \qquad (11)$$

$$\begin{aligned} f^{+}(t) &= f^{-}(t); \\ g^{+}(t) &= g^{-}(t); \end{aligned} \quad t \in L_1' \qquad (12)$$

2 The singular integral equation

The sectionally holomorphic function $\varphi(z)$ can be written in terms of a Cauchy integral with a density $\omega(t)$, $t \in L$ as follows

$$\varphi(z) = \frac{1}{2\pi i} \int_L \frac{\omega(\tau)\, d\tau}{\tau - z}, \qquad \varphi^{\pm}(t) = \pm\frac{1}{2}\omega^{\pm}(t) + \frac{1}{2\pi i} \int_L \frac{\omega(\tau)\, d\tau}{\tau - t}, \quad t \in L \qquad (13)$$

The last equation is known as the Sokhotskii–Plemelj formula for the boundary values of a Cauchy integral. The integral in this equation is singular and must be understood in the Cauchy principle value sense. In an analogous way as the function $\varphi(z)$ to $\omega(t)$ the sectionally holomorphic function $\psi(z)$ is related to a density $\Omega(t)$ according to

$$\psi(z) = \frac{1}{2\pi i} \int_L \frac{\Omega(\tau)\, d\tau}{\tau - z}, \qquad \psi^{\pm}(t) = \pm\frac{1}{2}\Omega^{\pm}(t) + \frac{1}{2\pi i} \int_L \frac{\Omega(\tau)\, d\tau}{\tau - t}, \quad t \in L \qquad (14)$$

The introduction of the boundary conditions (11, 12) and the Cauchy integral representation of $\varphi^{\pm}(t)$ and $\psi^{\pm}(t)$ into the boundary equations (5) leads to a system of equations

$$\omega(t) + t\overline{\omega'(t)} + \overline{\Omega(t)} = 0; \quad t \in L \qquad (15)$$

$$(K\omega)(t) + t\overline{(K\omega')(t)} + \overline{(K\Omega)(t)} = f^{+}(t) + f^{-}(t) + 2c_k; \quad t \in L_1'', L_2 \qquad (16)$$

$$\begin{aligned} &\frac{\kappa^+}{4\mu^+}((K\omega) + \omega)(t) - \frac{1}{4\mu^+}\left[t\overline{((K\omega') + \omega')(t)} + \overline{((K\Omega) + \Omega)(t)}\right] - g^{*+}(t) = \\ &\frac{\kappa^-}{4\mu^-}((K\omega) - \omega)(t) - \frac{1}{4\mu^-}\left[t\overline{((K\omega') - \omega')(t)} + \overline{((K\Omega) - \Omega)(t)}\right] - g^{*-}(t); \end{aligned} \quad t \in L_1' \qquad (17)$$

where K denotes the operator of singular integration.

$$(K\omega)(t) = \frac{1}{\pi i} \int_L \frac{\omega(\tau)\, d\tau}{\tau - t} \qquad (18)$$

By using the first equation (15) to eliminate the density $\Omega(t)$ from the second set of equations (16, 17) the desired boundary integral equation is obtained in terms of $\omega(t)$.

$$(M\omega)(t) \equiv A(t)\omega(t) + (k_1\omega)(t) + \overline{(k_2\omega)(t)} = h(t); \quad t \in L \qquad (19)$$

The integral operators k_1 and k_2 are given by

$$(k_1\omega)(t) = \frac{1}{\pi i}\int_L \left\{B(t) + C(t)\left[\frac{d\bar{\tau}}{d\tau}\frac{\tau - t}{\bar{\tau} - \bar{t}} - 1\right]\right\}\frac{\omega(\tau)\,d\tau}{\tau - t} \tag{20}$$

$$(k_2\omega)(t) = \frac{C(t)}{\pi i}\int_L \left[\frac{d\bar{\tau}}{d\tau} - \frac{\bar{\tau} - \bar{t}}{\tau - t}\right]\frac{\omega(\tau)\,d\tau}{\tau - t} = -\frac{C(t)}{\pi i}\int_L \frac{\bar{\tau} - \bar{t}}{\tau - t}\omega'(\tau)\,d\tau \tag{21}$$

The right hand side $h(t)$ of eq. (19) is

$$h(t) = \begin{cases} f^+(t) + f^-(t) + 2c_k\,; & t \in L_1'', L_2 \\ -g^+(t) + g^-(t)\,; & t \in L_1' \end{cases} \tag{22}$$

The piecewise constant coefficients $A(t)$, $B(t)$ and $C(t)$ in (19) are

$$A(t) = \begin{cases} 0\,; \\ \frac{1+\kappa^+}{4\mu^+} + \frac{1+\kappa^-}{4\mu^-}\,; \end{cases} \quad B(t) = \begin{cases} 2\,; \\ \frac{1-\kappa^+}{4\mu^+} - \frac{1-\kappa^-}{4\mu^-}\,; \end{cases} \quad C(t) = \begin{cases} 1\,; & t \in L_1'', L_2 \\ \frac{\kappa^-}{4\mu^-} - \frac{\kappa^+}{4\mu^+}\,; & t \in L_1' \end{cases}$$

3 The behaviour of M in the nodes of L

In general it is impossible to solve the singular integral equation (19) using the characteristic equation concept (cf. [2]) if there exist nodes in which the curve system is not smooth. In the present case, the nodes of the curve system L are the crack tip of the interface crack and the kinking point, respectivly. Let t_0 be one of these nodes and denote with $\Gamma_1, \ldots, \Gamma_N$ the curves with t_0 as a common point. It is noted, that t_0 is either a starting-point or a terminal point of each curve Γ_n $(n = 1, \ldots, N)$. The density $\omega(t)$ belongs to the class $H_0(t_0)$ with a derivative of the class $H^*(t_0)$. Such a function can be defined as follows

$$\omega(t) = \omega_{\lambda n}(t - t_0)^{\lambda} + \omega_{\bar{\lambda} n}(t - t_0)^{\bar{\lambda}}\,; \qquad t \in \Gamma_n \tag{23}$$

with a generally complex power λ whose real part is restricted to $0 < \Re(\lambda) < 1$. Further, in the coefficients $\omega_{\lambda n}$ and $\omega_{\bar{\lambda} n}$, λ and $\bar{\lambda}$ are understood as indices. The power λ is an eigenvalue of M, if the dominant part of M is nullified by ω, $(M\omega)(t) \sim 0$. For the sake of the determination of the dominant parts of k_1 and k_2 the following singular approximation formula is used, which holds true in the case $\Re(\lambda) > -1$

$$\frac{1}{\pi i}\int_{\Gamma_n} \frac{(\tau - t_0)^{\lambda}\,d\tau}{\tau - t} \sim a_{mn}(\lambda)(t - t_0)^{\lambda}\,, t \in \Gamma_m\,; \qquad a_{mn}(\lambda) = \mp\frac{e^{\mp i\pi\lambda}}{i\sin\pi\lambda} - \delta_{mn}, \tag{24}$$

where the upper sign is used if t_0 is a starting-point of Γ_n and the lower sign, if t_0 is a terminal point of Γ_n. Using this formula, the dominant part of k_1 is given by

$$\begin{aligned}(k_1\omega)(t) &\sim \sum_n \frac{k_{1\,n}(t)}{\pi i}\int_{\Gamma_n} \frac{\omega(\tau)\,d\tau}{\tau - \tau} \\ &\sim \sum_n k_{1\,mn}\left(a_{mn}(\lambda)\omega_{\lambda n}(t - t_0)^{\lambda} + a_{mn}(\bar{\lambda})\bar{\omega}_{\bar{\lambda} n}(t - t_0)^{\bar{\lambda}}\right)\,; \qquad t \in \Gamma_m\end{aligned} \tag{25}$$

with the definitions

$$k_{1\,n}(t) = \lim_{\tau \to t_0}\left\{B(t) + C(t)\left[\frac{d\bar{\tau}}{d\tau}\frac{\tau - t}{\bar{\tau} - \bar{t}} - 1\right]\right\}, \qquad \tau \in \Gamma_n \tag{26}$$

$$k_{1\,mn} = \lim_{t \to t_0} k_{1\,n}(t) \tag{27}$$

In a similar way, the dominant part of k_2 is given by the following expression

$$\begin{aligned}(k_2\omega)(t) &\sim -\frac{C(t)}{\pi i}\int_L \frac{\bar{\tau} - \bar{t}_0}{\tau - t_0}\frac{(\tau - t_0)\omega'(\tau)\,d\tau}{\tau - t} + \frac{C(t)(\bar{t} - \bar{t}_0)}{\pi i}\int_L \frac{\omega'(\tau)\,d\tau}{\tau - t} \\ &\sim \sum_n C_m(k_{2\,m} - k_{2\,n})\left[\lambda a_{mn}(\lambda)(t - t_0)^{\lambda} + \bar{\lambda} a_{mn}(\bar{\lambda})(t - t_0)^{\bar{\lambda}}\right]\end{aligned} \tag{28}$$

with the definition

$$k_{2\,n} = \lim_{t \to t_0}\frac{\bar{t} - \bar{t}_0}{t - t_0}, \qquad t \in L_n \tag{29}$$

By using these results, the dominant part of M is found to be

$$\begin{aligned}(M\omega)(t) &\sim \sum_n \left(r_{mn}(\lambda)\omega_{\lambda n} + \overline{s_{mn}(\bar{\lambda})}\bar{\omega}_{\bar{\lambda} n}\right)(t - t_0)^{\lambda} \\ &+ \sum_n \left(\overline{s_{mn}(\lambda)}\bar{\omega}_{\lambda n} + r_{mn}(\bar{\lambda})\omega_{\bar{\lambda} n}\right)(t - t_0)^{\bar{\lambda}}\,; \qquad t \in \Gamma_m\end{aligned} \tag{30}$$

with the definitions

$$r_{mn}(\lambda) = A_m\delta_{mn} + \left\{B_m + C_m\left[e^{2i(\alpha_n - \alpha_m)} - 1\right]\right\}a_{mn}(\lambda) \tag{31}$$

$$s_{mn}(\lambda) = \lambda C_m\left[e^{-2i\alpha_m} - e^{-2i\alpha_n}\right]a_{mn}(\lambda)e^{2i\alpha_m\lambda} \tag{32}$$

If λ is an eigenvalue, then the following system of equations is obtained

$$\sum_n \left[r_{mn}(\lambda)\omega_{\lambda n} + \overline{s_{mn}(\bar{\lambda})}\bar{\omega}_{\bar{\lambda} n}\right] = 0 \tag{33}$$

$$\sum_n \left[s_{mn}(\lambda)\omega_{\lambda n} + \overline{r_{mn}(\bar{\lambda})}\bar{\omega}_{\bar{\lambda} n}\right] = 0 \tag{34}$$

which leads to the characteristic equation

$$\det(T) = 0 \tag{35}$$

where T is the matrix of the coefficients of the system of equations (33, 34) and can be written as a system of 2×2–submatrices $T_{mn}(\lambda)$ defined by

$$T_{mn}(\lambda) = \begin{pmatrix} r_{mn}(\lambda) & \overline{s_{mn}(\bar{\lambda})} \\ s_{mn}(\lambda) & \overline{r_{mn}(\bar{\lambda})} \end{pmatrix} \tag{36}$$

3.1 The tip of the interface crack

At the tip of the interface crack the curves $\Gamma_1 = L_1''$ and $\Gamma_2 = L_1'$ meet in a smooth manner, hence $s_{mn} = 0, \forall (m,n)$ is valid. The eigenvalue equations (33, 34) decompose in two uncoupled equivalent systems of equations for the unknown quantities ω_λ and $\bar{\omega}_{\bar{\lambda}}$. Hence, the eigenvalue equation reduces to

$$\sum_{n=1}^{2} r_{mn}(\lambda)\omega_{\lambda n} = 0\,; \qquad m = 1,2 \tag{37}$$

At the interface crack and the bonding line, respectivly, the coefficients of the integral equation are specified as

$$A_1 = 0; \qquad B_1 = 2;$$
$$A_2 = \frac{\kappa_0 + 1}{4\mu_0} + \frac{\kappa_1 + 1}{4\mu_1}; \quad B_2 = \frac{\kappa_0 - 1}{4\mu_0} - \frac{\kappa_1 - 1}{4\mu_1}; \tag{38}$$

The crack tip is a terminal point of Γ_1 and at the same time a starting-point of Γ_2. Therefore, the matrix T can be described as follows

$$T = (r_{mn}) = \begin{pmatrix} 2\frac{\cos\pi\lambda}{i\sin\pi\lambda} & 2\left(-\frac{\cos\pi\lambda}{i\sin\pi\lambda} + 1\right) \\ B_2\left(\frac{\cos\pi\lambda}{i\sin\pi\lambda} + 1\right) & A_2 - B_2\frac{\cos\pi\lambda}{i\sin\pi\lambda} \end{pmatrix} \tag{39}$$

The characteristic equation is given by

$$\det(T) = 2A_2\frac{\cos\pi\lambda}{i\sin\pi\lambda} - 2B_2 = 0 \tag{40}$$

The solution of this equation leads to the following eigenvalue

$$\lambda = \frac{1}{2} + i\frac{1}{2\pi}\ln\frac{A_2 - B_2}{A_2 + B_2} = \frac{1}{2} + i\beta \tag{41}$$

where the quantity

$$\frac{A_2 - B_2}{A_2 + B_2} = \frac{\kappa_1\mu_0 + \mu l}{\kappa_0\mu_1 + \mu_0} \tag{42}$$

is known as the bimaterial constant which is related to the oscillating stress singularity $\gamma = -1/2 + i\beta$.

3.2 The kinking point

At the kinking point the three curves $\Gamma_1 = L_1'$, $\Gamma_2 = L_1''$ and $\Gamma_3 = L_2$ join in a non-smooth manner, thus leading to complicated eigenvalues. The eigenvalue equations form a linear system of order 6. The characateristic equation can be factorized in two parts. The first eigenvalue belongs to the problem of the homogeneous quarter-plane formed by the curves L_2^+ and L_1''. The characteristic equation leading to this eigenvalue is known to be (cf. [3])

$$\sin(\pi\lambda/2) = \lambda\sin(\pi/2) \tag{43}$$

The second eigenvalue belongs to the problem of a quarter-plane bonded to a half-plane and is defined by the equation (cf. [4])

$$A\beta^2 + 2B\alpha\beta + C\alpha^2 + 2D\beta + 2E\alpha + F = 0, \tag{44}$$

with
$A = 4\sin^2(\pi\lambda)[\sin^2(\pi\lambda) - \lambda^2]$, $B = 2\lambda^2\sin^2(\pi\lambda)$, $C = \sin^2(\pi\lambda) - \lambda^2$, $D = -B$,
$E = (2\lambda^2 - 1)\sin^2(\pi\lambda) + \sin^2(\pi\lambda/2) - \lambda^2$, $F = \sin^2(3\pi\lambda/2) - \lambda^2$
and the material parameters α and β

$$\alpha = \frac{\mu_1 m_0 - \mu_0 m_1}{\mu_1 m_0 + \mu_0 m_1} \qquad \beta = \frac{\mu_1(m_0 - 2) - \mu_0(m_1 - 2)}{\mu_0 m_1 + \mu_1 m_0} \tag{45}$$

where the constants m_j depend on the associated stress state.

$$m_j = \begin{cases} 4(1 - \nu_j) & \text{plane strain;} \\ \frac{4}{1+\nu_j} & \text{plane stress;} \end{cases} \qquad j = 0,1 \tag{46}$$

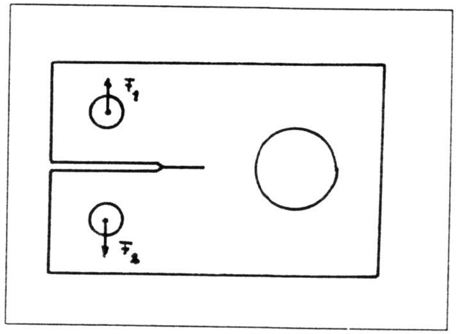

Fig. 1: *Modified CT– specimen*

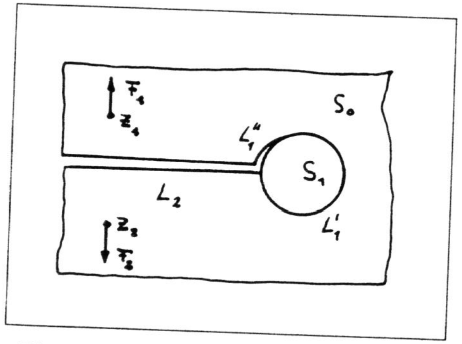

Fig. 2: *Kinked crack system*

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