# HIGH-ORDER ASYMPTOTIC FIELDS AT THE APEX OF A NOTCH IN A POWER-LAW HARDENING BIMATERIAL

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## ABSTRACT

In the present paper, the basic formulas of the singular field near a notch tip in a power–law hardening bimaterial are derived and the zero–order asymptotic solution with the HRR singularity are obtained. In the light of different notch open angles and hardening exponents the relation between the singularity of stresses and open angle of a sharp notch as well as the hardening exponent is given. Using the method of asymptotic analysis and interficial displacement–match technique, the singularity of the stresses corresponding to the first–order solutions and some typical groups of figures of zero–order and first–order stress distribution for bimaterial with different open angles of notch are also given.

**Key words:** notch problem, power–law hardening bimaterial, high–order asymptotic solution, interfacial crack

## INTRODUCTION

In composite materials and micro–electronic packages, some geometric figures with different open angles of notch between bi–, or multi– materials often exist as shown in Fig 1. It is a notch problem of bimaterial. The interface crack problem is a special case ($\beta_1=\beta_2=0$). The investigation on the fields near a notch tip of bimaterial in multilayered film and micro–electronic packages have begun recently. In the situation of notch in a power–law hardening bimaterial with special open angle $\beta_1=\frac{\pi}{2}$, $\pi$ *or* $\frac{3\pi}{2}$, the relation between the singularity of stresses corresponding to zero–order main singular field $s_0$ and the hardening exponent of power–law hardening material on a rigid basis was deduced by Duva [1], the relation between $\beta$ and $s_0$ under a general situation and the solution of angular distribution of stresses had not been investigated. The interface crack problem was studied by shih and Asaro [2,3,4]. Gao and Lou [5] obtained the solution with HRR singularity. But an approximated model of rigid substrate with $\tilde{U}_\theta(0^+)=0$ was adopted. Wang [6] investigated the singular field with the HRR singularity of interface crack, but the material below the interface was assumed to be linearly isotropic, i.e. $n_2=1$. Zhou and Yu [7,8] used the asymptotic analysis with interficial displacement–match technique to deduce the close form

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solution of near tip fields of antiplane mode–Ⅲ interfacial crack problem, Using the solution obtained by Li and Wang, [10] Xia [9] investigated the high–order asymptotic solution of bimaterial interface crack problem and deduced the zero–order and high–order asymptotic fields in a certain range of material constants. This problem had also considered by Aravas and sharma [11] and Champion and Atkinson [12].

Based on the Ref.[7–9], the high–order asymptotic solution of near tip fields of notch in a power–law hardening bimaterial is studied in this paper. The solutions in this paper satisfy asymptotically the continuity conditions of stresses and displacements on the interface.

## FORMULATION OF THE PROBLEM

Consider the problem of a bimaterial notch with arbitrary open angles, as shown in Fig.1. The material 1 and 2 are both power–law hardening materials. Let $E_\beta, \nu_\beta, \sigma_{0\beta}$, $n_\beta$ and $\alpha_\beta$ ($\beta = 1,2$ corresponding to the material 1 and 2, respectively) denote the Young's modulus, Possion ratios, yielding stresses, hardening exponents and hardening coefficients of bimaterial, respectively. Here, we assume that $n_1 > n_2$ and let $n = \max(n_1, n_2)$.

The constitutive relations can be expressed as the following dimensionless form

$$\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \gamma\sigma_{kk}\delta_{ij} + \frac{3}{2}\alpha\sigma_e^{n-1}S_{ij} \tag{2.1}$$

where $\varepsilon = \dfrac{\bar{\varepsilon}}{\varepsilon_0}, \sigma = \dfrac{\bar{\sigma}}{\sigma_0}$, n is the hardening exponent.

The equilibrium equation is satisfied automatically by introducing a dimensionless stress function $\Phi_{(r,\theta)}$. In polar coordinates the relation between the stresses and the stress function is

$$\begin{cases} \sigma_r = r^{-1}\Phi' + r^{-2}\Phi^{\cdot\cdot} \\ \sigma_\theta = \Phi'' \\ \tau_{r\theta} = -(r^{-1}\Phi^{\cdot})' \end{cases} \tag{2.2}$$

where $\Phi = \Phi/(\sigma_0 L^2)$, $r = \dfrac{r}{L}$, L denotes the characteristic length of crack,

$( )' = \partial/\partial r \quad ,( )^{\cdot} = \dfrac{\partial}{\partial\theta}$

For a plane strain problem, the compatibility equation is

$$r^{-1}(r\varepsilon_\theta)'' + r^{-2}(\varepsilon_r)^{\cdot\cdot} - r^{-1}\varepsilon'_r - 2r^{-2}(r\varepsilon_{r\theta})^{\cdot\prime} = 0 \tag{2.3}$$

The above equations from (2.1) to (2.3) consist of the basic equations of the plane strain problem for power–law hardening materials.

Assume that there is a separated form of HRR singular fiels at the near tip of notch. The two beginning terms of the asymptotic expansion of the dimensionless stress function $\Phi(r,\theta)$ are expressed as

$$\Phi(r,\theta) = K_1 r^{s_0}\tilde{\Phi}(\theta) + K_2 r^{s_1}\tilde{\Phi}_1(\theta) \tag{2.4}$$

where the ripple denotes the function of the angle $\theta$, i.e. the angle distribution function.



The compatibility equation can be written as

$$K_1 r^{s_0-4}\Omega_0^e + K_1\Gamma r^{s_0+\Delta s_1-4}\Omega_1^e + \alpha K_1^n r^{n(s_0-2)-2}\Omega_0^p + \alpha K_1^n \Gamma r^{n(s_0-2)+\Delta s_1-2}\Omega_1^p = 0 \tag{2.5}$$

in which

$$\begin{cases} \Omega_0^e = \tilde{\varepsilon}_{r_0}^{e\,\cdot\cdot} - (s_0-2)\tilde{\varepsilon}_{r_0}^{e} + (s_0-2)(s_0-1)\tilde{\varepsilon}_{\theta_0}^{e} - 2(s_0-1)\tilde{\varepsilon}_{r\theta_0}^{e\,\cdot} \\ \Omega_1^e = \tilde{\varepsilon}_{r_1}^{e\,\cdot\cdot} - (s_0+\Delta s_1-2)\tilde{\varepsilon}_{r_1}^{e} + (s_0+\Delta s_1-2)(s_0+\Delta s_1-1)\tilde{\varepsilon}_{\theta_1}^{e} \\ \quad - 2(s_0+\Delta s_1-1)\tilde{\varepsilon}_{r\theta_1}^{e\,\cdot} \\ \Omega_0^p = \tilde{\varepsilon}_{r_0}^{p\,\cdot\cdot} - n(s_0-2)[n(s_0-2)+2]\tilde{\varepsilon}_{r_0}^{p} - 2[n(s_0-2)+1]\tilde{\varepsilon}_{r\theta_0}^{p\,\cdot} \\ \Omega_1^p = \tilde{\varepsilon}_{r_1}^{p\,\cdot\cdot} - [n(s_0-2)+\Delta s_1][n(s_0-2)+\Delta s_1+2)\tilde{\varepsilon}_{r_1}^{p} - \\ \quad 2[n(s_0-2)+\Delta s_1+1)]\tilde{\varepsilon}_{r\theta_1}^{p\,\cdot} \qquad \Gamma = K_2/K_1 \end{cases} \tag{2.6}$$

The governing equations of zero–order and first–order of singular field can be deduced from Eq. (2.5). The following three group conditions are required to determine the solution of the problem

1. The traction–free boundary conditions of notch surfaces are

$$\begin{cases} \sigma_\theta|_{\theta=\pi-\beta_1} = 0, & \sigma_\theta|_{\theta=-(\pi-\beta_2)} = 0 \\ \tau_{r\theta}|_{\theta=\pi-\beta_1} = 0, & \tau_{r\theta}|_{\theta=-(\pi-\beta_2)} = 0 \end{cases} \tag{2.7}$$

which can be recast as the following form expressed by stress function

$$\tilde{\Phi}(\pi-\beta_1) = \tilde{\Phi}^{\cdot}_k(\pi-\beta_1) = \tilde{\Phi}_k(-\pi+\beta_2) = \tilde{\Phi}^{\cdot}_k(-\pi+\beta_2) = 0 \quad (k=0,1) \tag{2.7$'$}$$

2. The stress continuity conditions on the interface are

$$\sigma_\theta|_{\theta=0^+} = \sigma_\theta|_{\theta=0^-}, \quad \tau_{r\theta}|_{\theta=0^+} = \tau_{r\theta}|_{\theta=0^-} \tag{2.8}$$

or $$\tilde{\Phi}_k(0^+) = \tilde{\Phi}_k(0^-), \quad \tilde{\Phi}^{\cdot}_k(0^+) = \tilde{\Phi}^{\cdot}_k(0^-) \quad (k=0,1) \tag{2.8$'$}$$

3. The continuity conditions of displacements on the interface are

$$u_a|_{\theta=0^+} = u_a|_{\theta=0^-} \tag{2.9}$$

It can be written in concrete form as

$$r^{n(s_0-2)+1}\{a_1 K_1^{n1}\tilde{u}_{a_0}^p(0^+) + [a_1\Gamma K_1^{n1} r^{\Delta s_1}\tilde{u}_{a_1}^p(0^+) - a_2 K_1^{n2} r^{-(n_1-n_2)(s_0-2)}\tilde{u}_{a_0}^p(0^-)] - a_2\Gamma K_1^{n_2} r^{-(n_1-n_2)(s_0-2)+\Delta s_1}\tilde{u}_{a_1}^p(0^-) + K_1 r^{-(n_1-1)(s_0-2)}[\tilde{u}_{a_0}^e(0^+) - \tilde{u}_{a_0}^e(0^-)] + \Gamma K_1 r^{-(n_1-1)(s_0-2)+\Delta s_1}[\tilde{u}_{a_1}^e(0^+) - \tilde{u}_{a_1}^e(0^-)]\} = 0 \quad (a = r,\theta) \tag{2.10}$$

The displacement continuity conditions are different for the zero–order and first–order asymptotic fields, as will be discussed in the following sections.

Eq. (2.5) and the conditions (2.7), (2.8) and (2.9), which describe the asymptotic solutions of the notch problem of power–law hardening bimaterial, consist of a boundary–value problem of a nonhomogeneous, nonlinear ordinary differential coupled equation. The fourth-order Runge–Kutta method with constant steps in used to calculate the integral of equations. The minimum step is about 0.5 °, as is controlled by the Newton–Raphson automatic control program. The accuracy of solutions in the present paper is controlled to be better than $10^{-5}$.

## THE ZERO-ORDER ASYMPTOTIC SOLUTION OF THE NOTCH PROBLEM

From (2.5) and (2.6), the governing equation of the zero-order asymptotic fields expressed by stress function can be deduced as

$$\{\frac{d^2}{d\theta^2} - n_k(s_0-2)[n_k(s_0-2)+2]\}\{\tilde{\sigma}_{e_0}^{n_k-1}[s_0(2-s_\sigma)\tilde{\Phi}_0+\tilde{\Phi}_0]\}+$$
$$4(s_0-1)[n_k(s_0-2)+1](\tilde{\sigma}_{e0}^{n_k-1}\tilde{\Phi}_0)^{\cdot}=0$$
$$(k=1, 0\leqslant\theta\leqslant\pi-\beta_1;\quad k=2,\quad -\pi+\beta_2\leqslant\theta\leqslant 0) \tag{3.1}$$

The three groups of definite conditions are as follows

1. The traction-free boundary conditions on the notch surface are

$$\begin{cases}\tilde{\Phi}_0(\pi-\beta_1)\tilde{\Phi}_0(\pi-\beta_1)=0\\ \tilde{\Phi}_0(-\pi+\beta_2)=\tilde{\Phi}_0(-\pi+\beta_2)=0\end{cases} \tag{3.2}$$

2. The continuity conditions of stresses on the interface are

$$\tilde{\Phi}_0(0^+)=\tilde{\Phi}_0(0^-),\qquad \tilde{\Phi}_0(0^+)=\tilde{\Phi}_0(0^-) \tag{3.3}$$

3. When $r\rightarrow 0$, the first term in the braces in (2.10) is far bigger than others. So the continuity conditions of displacements can be rewritten as

$$\tilde{u}_{a_0}^{p}(0^+)=0\qquad (a=r,\theta) \tag{3.4}$$

This is equivalent to that the material 2 is considered to be rigid.

which can also be expressed in form of stress function

$$\begin{cases}\tilde{\Phi}_0(0^+)-s_0(s_0-2)\tilde{\Phi}_0(0^+)=0\\ \tilde{\Phi}_0(0^+)+[4(s_0-1)(n_1(s_0-2)+1)-s_0(s_0-2)]\tilde{\Phi}_0(0^+)=0\end{cases} \tag{3.5}$$

The zero-order asymptotic field and the singularity of stresses $s_0$ in thé upper material can be obtained from (3.1), (3.2) and (3.5). So is related with the hardening exponent $n_1$ and the open angle $\beta_1$. $s_0=(2n_1+1)/(n_1+1)$ for $\beta_1=0$, i.e. the interface crack. The solution of the problem can be obtained from (3.1)–(3.4).

In this paper, the notch problem of bimaterial with $n_1=3$ is calculated. The singularity of zero-order stresses $s_0$ and the stress angle distribution function $\tilde{\sigma}_{\alpha\beta}(\theta)$ are shown in figures numbered from 2 to 9.

The singularity of stresses $s_0$ decreases with the increase of open angle $\beta_1$. In the case of $\beta_1=90$ °, the singularity exponent of stress $s_0-2$ is equal to be $-0.2126$, which is consistent with be result of Duva [1]. The change of $s_0$ with the hardening exponents $n_1$:

The results show that the singularity of stresses decreases with the increase of $n_1$.



## THE FIRST-ORDER ASYMPTOTIC SOLUTIONS

As shown from (2.5), the governing equation corresponding to the first-order asymptotic fields has three different forms for different value of $n_k$ $(k=1,2)$.

The traction-free boundary conditions of the notch surfaces and the stress continuity conditions on the interface are the same as eq. (3.2) and (3.3), respectively.

From the displacement continuity conditions in (2.10), the first term in the braces has been used in the zero-order fields. The two term in the first square brackets are the first-order fields of displacement in the upper material and the zero-order fields of displacement in the lower material, respectively. From the exponents of r of the two terms, we know that the following two different cases exist.

(1) For $0<\Delta s_1<-(n_1-n_2)(s_0-2)$ and $n_2\geqslant 1$, we have

$$\tilde{u}_{a1}^{p}(0^+)=0\qquad (\alpha=r,\theta) \tag{4.1}$$

which shows that the displacement fields corresponding to the zero-order solutions in the lower half-space will be matched by that corresponding to the second-order solutions (of higher order solutions) in the upper half-plane. This case is not discussed here.

(2) For $\Delta s_1=-(n_1-n_2)(s_0-2)$ and $n_2>1$, we have

$$\alpha_1\Gamma K_1^{n1}\tilde{u}_{\alpha_1}^{p}(0^+)-\alpha_2K_1^{n2}\tilde{u}_{\alpha_0}^{p}(0^-)=0\qquad(\alpha=r,\theta) \tag{4.2}$$

Let $\Gamma=\alpha_2K_1^{n_2-n_1}/\alpha_1$. Then, we have

$$\tilde{u}_{a_1}^{p}(0^+)-\tilde{u}_{a_0}^{p}(0^-)=0 \tag{4.3}$$

which show that the displacements continuity conditions on the interface corresponding to the first-order solutions are obtained by matching the zero-order asymptotic displacements in the lower half-plane with the first-order asymptotic displacements in the upper half-plane.

(4.3) can also be rewritten in the form of stress function

$$\begin{cases}\tilde{\Phi}_1(0^+)-s_1(s_1-2)\tilde{\Phi}_1(0^+)=C_1\\ \tilde{\Phi}_1(0^+)+[4n_1(s_1-1)(1+n_2(s_1-2))-s_1(s_1-2)]\tilde{\Phi}_1(0^+)=C_2\end{cases} \tag{4.4}$$

From above discussion, and assumed $n_1=5$, $n_2=3$, then we have

$$\Delta s_1=-(n_1-n_2)(s_0-2)=-2(s_0-2) \tag{4.5}$$

Then the singularity exponent of the first-order asymptotic solutions of stresses $s_1-2$ is

$$s_1-2=s_0+\Delta s_1-2=-s_0+2=-(s_0-2) \tag{4.6}$$

And the governing equation of the first-order fields corresponding to two cases discussed above are

$$\Omega_1^p = 0 \qquad 0 \leqslant \theta \leqslant \pi - \beta_1, \qquad \Omega_1^p = -\frac{\alpha_1}{\alpha_2}\Omega_0^e \qquad -\pi + \beta_2 \leqslant \theta \leqslant 0 \tag{4.7}$$

respectively. Here $\alpha_1$ is assumed to be unity, and the ratio of $\alpha_1$ and $\alpha_2$ is chosen to be 50. The possion ration in $\Omega_0^e$ is related only with the material 2, and let $\gamma = 0.3$. Using the shooting method and the fourth–order Lunge–Kutta method with constant steps, the following examples are solved.

a) $\beta_1 = 0°$ $\beta_2 = 0°$

For the interface crack in Fig.6, the magnitude of the first–order stress field is much bigger than that of the zero–order stress field. The singularity exponent of stress is $s_1 - 2 = 0.166667$.

b) $\beta_1 = 45°$, $\beta_2 = 45°$.

For the notch problem in Fig. 7 the first–order stresses are much bigger than the above zero–order stresses. The stresses in the near tip of the notch with open angles $\beta_1 + \beta_2 = 90°$ distribute more smoothly than that in the interface crack. The stress singularity exponent $s_1 - 2$ is equal to 0.165085.

For several cases, the angular distribution function of the first–order asymptotic solutions are also shown in the Fig. 8 to 10.

## CONCLUSIONS

After investigating the plane–strain problem of interface notch of a power–law hardening bimaterial, the following conclusions can be shown.

1. For different open angles ($\beta_1$) of notch and hardening exponents ($n_1$), the related curves between the singularity exponent of stress and these two parameters ($\beta_1$, $n_1$) are given. The singularity exponent decreases with the increase of open angles, and reaches the maximum value for the interface crack. The singularity exponent reduces also with the increase of hardening exponent, and then disappears for the perfectly–plastic materials ($n \to \infty$), as holds for any open angle of $\beta_1$.

2. The zero–order asymptotic fields with the HRR singularity at the tip of a notch with several open angles ($\beta_1$) are obtained. These solutions satisfy asymptotically the continuity conditions of stresses and displacements on the interface.

3. Based on the zero–order asymptotic solution, using the asymptotic analysis as displacement–match technique the first–order asymptotic fields at the tip of notch are calculated corresponding to several open angles. The results show that no singularity of stresses exists and the angle distributions of stresses exist and the angle distributions of stresses are much bigger than that of the zero–order asymptotic fields. These solutions also asymptotically satisfy the continuity conditions of stresses and the displacements on the interface.



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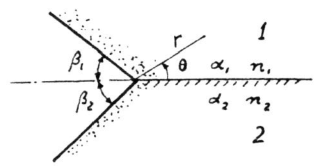


Fig.1

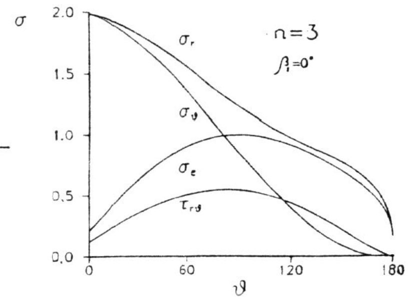


Fig.2

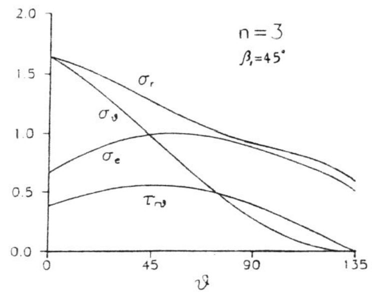


Fig.3

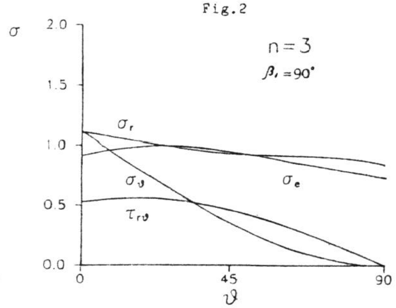


Fig.4

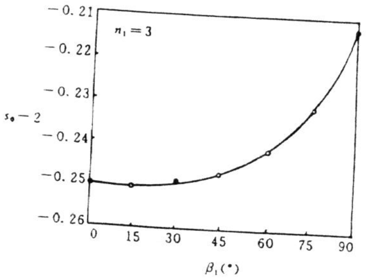


Fig.5 $S_0-2 \sim \beta_1$

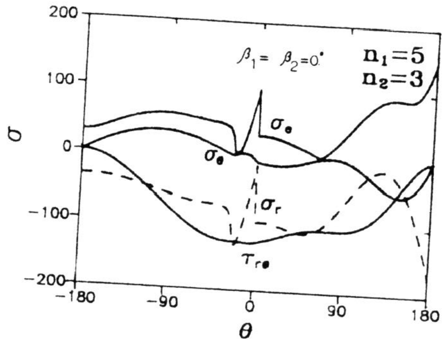


Fig.6

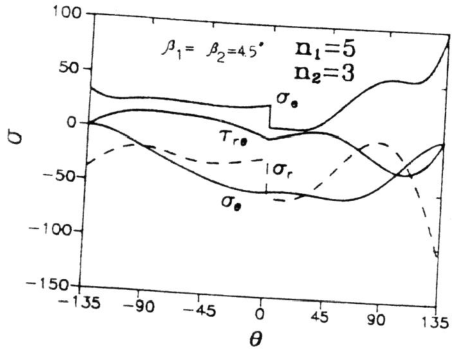


Fig.7

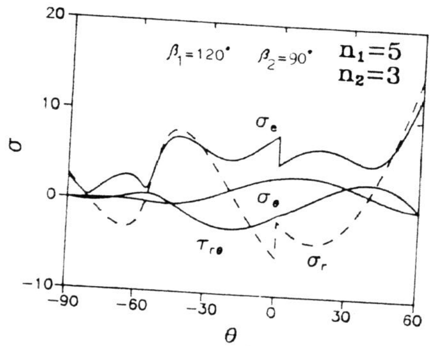


Fig.8

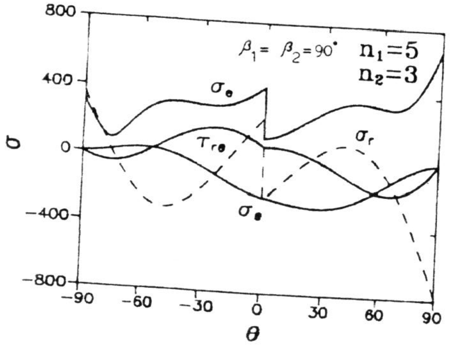


Fig.9

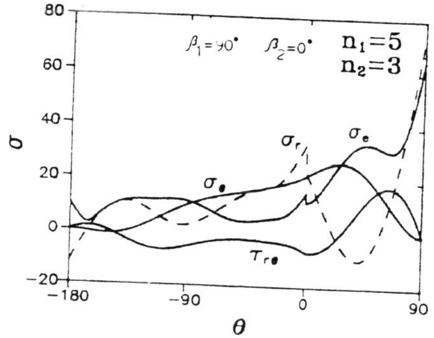


Fig.10