GRADIENT CRITERIA OF STRENGTH AND THEIR RELATION WITH LINEAR ELASTIC FRACTURE MECHANICS

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ABSTRACT

The paper considers a gradient approach for the estimation of local strength of brittle materials in a stress concentration zone. The specific forms of gradient strength criteria for nonuniform stressed state and their combined variant are formulated. The relation of the proposed gradient criteria with linear elastic fracture mechanics is demonstrated.

KEYWORDS

Gradient strength criteria, crack-type concentrators.

GRADIENT APPROACH

The material characteristics such as the strength limit σ_b determined under uniform stressed state are used to calculate the strength. The value σ_b is often assumed to be also valid for nonuniform stressed state of the material of real structures. However, validity of such an assumption for elements of the structures with stress concentrators is doubtful. This is especially so for large values of the concentration factor $\alpha = (max\ \sigma_1)/p$, where $(max\ \sigma_1)$ is the first main stress in the vertex of a concentrator; p is the nominal stress. The effective stress concentration factor $\alpha_* = \sigma_b/p_*$ defined as the ratio of the strength limit and the nominal fracture stress is usually less than α. Hence, at the moment of crack initiation $(max\ \sigma_1) > \sigma_b$. Therefore, in case of brittle fracture, i.e. in the absence of any considerable equalization of stress peaks at the expense of plastic deformations, we may speak about a local increase of material strength in the mostly stressed point. This effect is pronounced in case of brittle and structurally-inhomogeneous materials, such as iron, marble, graphite, phenolic and epoxy glasses.

At the moment of crack initiation the value of $(max\ \sigma_1)$ is called the local strength limit σ_* which is not a constant value and depends on the degree of the stressed state nonuniformity in the vicinity of the tip of the most dominant crack. The nonuniformity can be characterized by the relative gradient of the first main stress

$$g_1 = |grad\ \sigma_1| \ / \ (max\ \sigma_1) \qquad (1)$$

which is calculated in the concentrator vertex from an elastic solution of the corresponding problem.

FORMULATION OF THE CRITERION.

The experimental results on determination of strength of phenolic and epoxy glasses under stress concentration conditions can be found in literature (Serensen and Strelyayev, 1962; Zaitsev and Strelyayev, 1968). According to these authors, the increase of local strength in the most dangerous point can be described by the functional dependence

$$\sigma_* = \sigma_b \cdot f(g_1) \qquad (2)$$

The function $f(g\ )$ is determined on the basis of special experimental data. It is noted (Zaitsev and Strelyayev, 1968) that a dependence of the form

$$\sigma_* = \sigma_b \left[1 + B\ g_1^n \right] \qquad (3)$$

describes satisfactorily the experimental results obtained for the AG-4s and 33-18s phenolic and epoxy glasses.

It is shown (Novopashin and Suknev, 1987) that for the structure element with a crack-type strain concentrator, only with n = 1/2 , we'll obtain the finite values of the limiting nominal stresses p_*. Otherwise, it either will be impossible to break the body having the crack, or it'll fail under zero loading. Hence, the gradient strength criterion satisfying the requirement of finiteness of the fracture load in case of the crack-type concentrator can be written as

$$\sigma_* = \sigma_b \left[1 + \sqrt{L_1 g_1} \right] , \qquad (4)$$

where L_1 is the parameter which has the length dimension and depends on the properties of material, i.e. it is the characteristic size.

RELATION OF GRADIENT CRITERION WITH LINEAR ELASTIC FRACTURE MECHANICS

It has been shown (Legan, 1990) that the parameter L_1 should be connected with the characteristic of fracture tougness of the material, i.e. the critical stress intensity factor K_{Ic}, as follows

$$L_1 = \frac{2}{\pi} K_{Ic}^2 \ / \ \sigma_b^2 \qquad (5)$$

In this case, for the Griffith crack concentrator, gradient criterion (4) gives the well-known (Sedov, 1970) equation

$$p_* = K_{Ic} \sqrt{\frac{2}{\pi} / d} \ , \qquad (6)$$

where d is the crack length.

To prove this, it is necessary to use the solution of the problem of uniaxial tension of a plate with an elliptic hole (Fig.1) and not the asymptotic expression for stresses in the vicinity of the crack vertex. For $b \rightarrow 0$ this solution gives a more precise history of stress distribution in the vicinity of the vertex of the crack-type concentrator. The mentioned solution is presented in some papers (Muskhelishvili, 1966; Sedov, 1970), where it is written as complex stress functions in special complex coordinates. Using Kolosov's formulae we can obtain the expressions for the stressed state components. It is then necessary to turn to real coordinates of the problem. After transition to real coordinates we can write the distribution of the first main stress $\sigma_1(x)$ over the dangerous section. By virtue of the problem symmetry, $|grad\ \sigma_1| = |\partial\sigma_1/\partial x|$

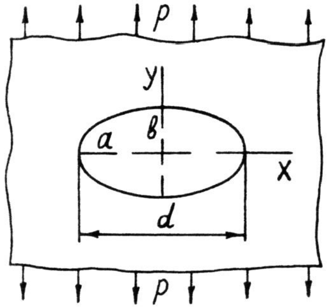

Fig.1. The problem of uniaxial tension of a plate with an elliptic hole.

Differentiating the function $\sigma_1(x)$ in the x-axis, determining the values of $|grad\ \sigma_1|$ and $(max\ \sigma_1)$ at x = a and substituting them into (1) yields the formula for the relative gradient g_1 in the concentrator vertex as

$$g_1 = (\alpha - 1)^2 (1 + \frac{1}{2\alpha}) \ / \ d \ , \qquad (7)$$

where $\alpha = 1 + 2 \frac{a}{b}$ is the stress concentration factor;
d = 2a is the hole size in the dangerous region.
Substituting (7) into (4), the expression for σ_* becomes

$$\sigma_* = \sigma_b \left[1 + (\alpha - 1) \sqrt{(1 + \frac{1}{2\alpha}) L_1 \ / \ d} \right]$$

The nominal stress at the crack initiation p_* is equal to the maximum stress σ_* divided by the concentration factor α , i.e.

$$p_* = \sigma_b \left[\frac{1}{\alpha} + \frac{\alpha-1}{\alpha} \sqrt{(1 + \frac{1}{2\alpha}) L_1 \ / \ d} \right] \qquad (8)$$

In the case of the crack-type concentrator, for $\alpha \rightarrow \infty$ we have

$$p_* = \sigma_b \sqrt{L_1 \ / \ d} \qquad (9)$$

Substitution of (5) into (9) gives

$$p_* = K_{Ic} \sqrt{\frac{2}{\pi} / d} \qquad (10)$$

Hence, the well-known equation of determination of the nominal fracture stress p_* is obtained for the case of the Griffith crack. Thus, there is interrelation between the gradient criterion (4) and the linear elastic fracture mechanics.

Formulation of a Combined Gradient Criterion. After the interrelation between the gradient criterion (4) and fracture mechanics has been established, it is worth to note that the mentioned form of the criterion is not the only one case in which this interrelation exists. For example, taking into consideration the gradient approaches and experimental data in the fatigue region (Afanasyev, 1953), the gradient strength criterion can be written as

$$\sigma_* = \sigma_b \sqrt{1 + L_1 g_1} \qquad (11)$$

Since in the case of the crack-type concentrators $g_1 \to \infty$, then both criterion (11) and criterion (4) will give the same results. However, for stress concentrators, other than cracks, criteria (11) and (4) will give different results. Thus, at the same material characteristics σ_b and L_1, there are two different gradient strength criteria. In this case the advantages of one criterion over the other are not evident. A combined variant of the gradient strength criterion would be

$$\sigma_* = \sigma_b \left[1 - \beta + \sqrt{\beta^2 + L_1 g_1} \right], \qquad (12)$$

where β is a variable parameter ($\beta \geqslant 0$). When $\beta = 0$, the combined criterion transforms into (4), and when $\beta = 1$, this criterion changes into (11).

For experimental results obtained on a particular material in a nonuniform stressed state we can find such a value of the parameter β at which combined criterion (12) will describe these results better than the criteria (4) and (11). At the same time, for the crack-type concentrators, criterion (12) will give the same equations as those of criteria (4), (11).

FURTHER CORROBORATION OF RELATION BETWEEN GRADIENT CRITERIA AND FRACTURE MECHANICS

The results obtained allow a hypothesis that the gradient strength criteria lead to equations of linear fracture mechanics in a particular case of the crack-type stress concentrators. For further corroboration the well-known solutions of problems of stress concentration, which permit the limiting transition to the crack-type stress concentrators, can be used. These corroborations are divided into proved for the particular cases of plane (2-dimensional) and space (3-dimensional) problems.

Plane Problem. Let us consider the plane problem of biaxial extension of a plate with an elliptic hole (Fig.2) which solution is known (Muskhelishvili, 1966; Sedov, 1970). The monograph (Sedov, 1970) presents the formulae which give the distribution of the first main stress σ_1 in the critical region, i.e. in the direction of x-axis as

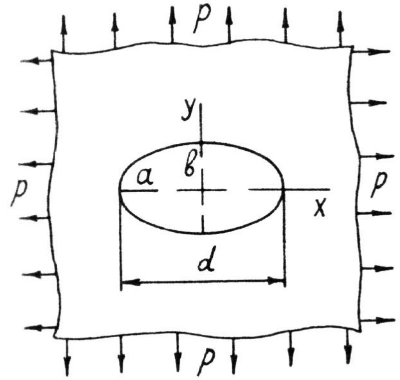

Fig.2. The problem of biaxial extension of a plate with an elliptic hole.

$$\sigma_1 = p \frac{x(x^2 - a^2 + 2b^2)}{(x^2 - a^2 + b^2)^{3/2}} \qquad (13)$$

The maximum value of σ_1 is attained at $x = a$. Hence, $\alpha = 2\frac{a}{b}$. It should be noted that $\alpha \geqslant 2$ since we always have that $a \geqslant b$. By virtue of the problem symmetry,

$$|grad\ \sigma_1| = |\partial\sigma_1/\partial x|$$

Taking into account that $a = d/2$ and $\alpha \geqslant 2$, we obtain

$$|grad\ \sigma_1| = \alpha\, p\, (\alpha^2 - 2) / d$$

Since $(max\ \sigma_1) = \alpha p$, then by definition of (1), we find that

$$g_1 = (\alpha^2 - 2) / d \qquad (14)$$

Further, substitution of this expression into combined criterion (12) gives the value of σ_* as

$$\sigma_* = \sigma_b \left[1 - \beta + \sqrt{\beta^2 + (\alpha^2 - 2) L_1 / d} \right]$$

The nominal fracture stress $p_* = \sigma_* / \alpha$, i.e.

$$p_* = \sigma_b \left[\frac{1-\beta}{\alpha} + \frac{1}{\alpha} \sqrt{\beta^2 + (\alpha^2 - 2) L_1 / d} \right] \qquad (15)$$

At limiting transition of an elliptic hole into mathematical cut $\alpha \to \infty$. Hence, in this case at finite values of the parameter β we have

$$p_* = \sigma_b \sqrt{L_1 / d} \qquad (16)$$

Substituting (5) into (16) results in

$$p_* = K_{Ic} \sqrt{\frac{2}{\pi} / d} \qquad (17)$$

This equation is well-known in fracture mechanics for determining the nominal fracture stress at bi-axial extension of a plate with a straightline cut. Thus, the interrelation between the gradient strength criteria and linear elastic fracture mechanics has been confirmed again. In addition, in considering the problem of biaxial extension of a plate with an elliptic hole the testing of results obtained earlier has been performed because equations (17) and (6) had to coincide, and that they have coincided.

<u>Space Problem</u>. To verify the hypothesis for 3-dimensional problems, let us consider stress distribution around an axi-symmetric oblate spheroidal cavity in an infinite body subjected to uniaxial tension along the symmetrical axis (Fig.3).

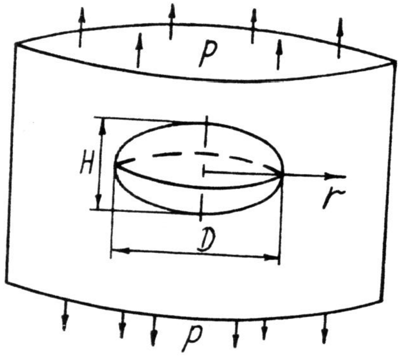

Fig.3. The problem of uniaxial tension of the body with a spheroidal cavity.

D is the cavity diameter in the critical region;

H is the cavity size in the symmetry axis;

r is the current radius from the symmetry axis;

$$V = \sqrt{\frac{H^2}{D^2-H^2}}\ ;$$

$$W = \sqrt{\frac{D^2}{D^2-H^2}}\ ;$$

$$F = W^4 V\, arctg\left(\frac{1}{V}\right) - W^2V^2\ ;$$

$$N = 6\left[8FW^2 - 2(1+\nu)F^2 - 6FW^4 + 4W^6 - 4W^4\right]$$

The solution of this problem is known (Neuber, 1958) and presented in the elliptic coordinates. According to this solution, distrbution of the first main stress σ_1 in the dangerous section i.e., in the critical region at the tip of the hole, is determined by the expression

$$\sigma_1 = p\left[1 + \frac{A+4B}{sh^3(u)} + (12B + 2(1-\nu)C)\left[arctg\left(\frac{1}{sh(u)}\right) - \frac{1}{sh(u)}\right]\right], \quad (18)$$

where use is made of the following notation:

$$A = \frac{V}{N} W^4 \left[(6-8\nu)F - 6FW^2 + 4W^4 - 8(1-\nu)W^2\right]\ ;$$

$$B = \frac{V}{N} W^4\left[2\nu F + 2W^4 - (1+2\nu)W^2\right]; \qquad C = \frac{V}{N} W^4\left[6F - 12W^2\right];$$

$$u = ln\left[2W\frac{r}{D} + \sqrt{\left(2W\frac{r}{D}\right)^2 - 1}\right]\ ;$$

ν is the Poisson coefficient.

The maximum stresses in the dangerous section are attained on the spheroidal cavity surface, where $r = D/2$. It can be proved that in these points $sh(u) = V$. Hence,

$$\alpha = 1 + \frac{A+4B}{V^3} + (12B + 2(1-\nu)C)\left[arctg\left(\frac{1}{V}\right) - \frac{1}{V}\right] \quad (19)$$

Further, by definition of (1), we find that

$$g_1 = \frac{\left|24B+4(1-\nu)C - 6(A+4B)W^2/V^2\right|}{A+4B - (12B+2(1-\nu)C)\left[V^2-V^3 arctg\left(\frac{1}{V}\right)\right] + V^3}\ \frac{1}{D} \quad (20)$$

Let us consider the limiting transition from a spheroidal cavity to a plane circular crack when $H/D \to 0$. In this case $V \to 0$; $W \to 1$. The combinations of coefficients A, B, C met in (19) and (20) are written with allowance for only the values of greater order, i.e.

$$A + 4B = \frac{2}{\pi}V^2\ ; \qquad 12B + 2(1-\nu)C = -\frac{2}{\pi} \quad (21)$$

Substition of (21) into (19) gives the following expression for the concentration factor α at $V \to 0$

$$\alpha = \frac{4}{\pi} / V \quad (22)$$

Analogously, substitution of (21) into (20) and allowance for only the terms of the smallness order V^2 in the denominator give the expression for the relative gradient g_1 in case of the concentrator having the form of a plane circular crack, i.e. for $H/D \to 0$, $V \to 0$, $W \to 1$,

$$g_1 = \frac{4}{V^2D} \quad (23)$$

Further, using the gradient approach, we'll determine the nominal fracture stress $p_* = \sigma_* / \alpha$ for the problem under consideration. With taking into account (22) we obtain

$$p_* = \frac{\pi}{4} V \sigma_* \quad (24)$$

The use of the combined criterion and expression (23) gives

$$p_* = \frac{\pi}{4} V \sigma_b \left[1 - \beta + \sqrt{\beta^2 + \frac{4}{V^2} L_1 / D}\right]$$

Since $V \to 0$, we have

$$p_* = \sigma_b \sqrt{\frac{\pi^2}{4} L_1 / D} \quad (25)$$

Taking into account (5) yields,

$$p_* = K_{Ic} \sqrt{\frac{\pi}{2} / D} \quad (26)$$

Hence, the equation which is well-known (Sack, 1946; Sneddon, 1946) in fracture mechanics for determination of the nominal fracture stresses in the presence of a plane circular crack with a diameter D

CONCLUSION

Thus, it has been confirmed that the application of the gradient strength criteria given as equations (4), (11) and (12) to a particular case of stress concentrators in the form of cracks results in equations of linear elastic fracture mechanics. This peculiarity of gradient criteria is significant and can be very effectively utilized for a variety of strength calculations.

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