FRACTAL APPROACHES IN FRACTURE MECHANICS OF HETEROGENEOUS SOLIDS

V.V. SILBERSCHMIDT
Mining Institute, Russian Academy of Sciences,
78A Karl Marx St. 614007 Perm, Russia

ABSTRACT

The problem of the macroscopic crack propagation is examined for a case of a randomly heterogeneous brittle media. The main structural mechanisms are accounted by means of continuum damage mechanics and stochastic values of K_I. The 2D problem of a crack propagation initiated by a V-shaped notch specimen was numerically simulated. The proposed approach allowed to examine the spatial and temporal propagation of the crack accompanied by the stress redistribution. It is shown that the fractal dimension of a crack front can be used as an invariant of brittle fracture process for stochastic materials.

KEYWORDS

Fracture mechanics, heterogeneous solids, fractals, brittle fracture

INTRODUCTION

Analysis of crack propagation in various materials under different conditions is one of the central problems in fracture mechanics (Tada et al., 1973; Knott, 1973). Achievements in material sciences and production of composites caused the transition from linear fracture mechanics to non-linear one. The standardized methods for experimental evaluation of fracture critical parameters together with specific numerical procedures (including special types of finite elements, etc.) formed a basement for a precise life prediction for various constructions. The crack propagation process is sufficiently complicated in the cases of so called "randomly heterogeneous" materials (Beale and Srolovitz, 1988; Hassold and Srolovitz, 1988). Macrocrack development in such media (brittle rocks and stochastic composites being examples) interacts with a non-uniform evolution of microdefects at different scale levels. A stochastic distribution of mechanical properties and elements' orientations causes a stress localization and a multiple growth of fracture nucleus. Some ideas for crack-damage or crack-microcrack interactions were proposed by Chaboche (1988) and Kachanov and Laures (1989). The present paper is aimed on the elaboration of an adequate description for a stochastic fracture process development in randomly heterogeneous media with non-uniform damage accumulation.

MODELS OF STOCHASTIC BRITTLE MATERIALS

Analysis of the processes of damage accumulation and fracture evolution in stochastic media is usually complicated by many factors: heterogeneity and anisotropy of mechanical properties, non-uniform redistribution of stress under a crack propagation, etc. All these factors have a different effect on the crack-damage interaction, which determines the specificity of a spatial fracture development. A correct description of these processes is possible on the basis of adequate continuum and quasi-continuum models of a randomly heterogeneous solid with structural defects, also taking into the account the stochastic nature of mechanical properties.

It is recognized that the stochastic crack growth could be represented as a random walk mechanism, described with the Markov-chain formalism. But such an approach presupposes the full independence of elementary fracture acts, so one could ignore the spatial parameters. Recently new methods of the fracture analysis were proposed, mainly based on the percolation theory. The specimen is approximated by the lattice model with a random distribution of strength parameters. So the fracture of the structural element is spatially 'frozen' and influences the behaviour of neighboring links. The macroscopic failure corresponds to the critical (infinite) cluster formation and could be characterized by the scalar parameter - a percolation threshold. The continuum fracture models became more and more complicated: a Swiss- and blue-cheese models were proposed for a media with cracks by Sornette (1988). The interest to a percolation approach was mainly stimulated by a respectively simple procedure for obtaining a threshold parameter (a concentration one, as a rule) directly from a model or by analogy with electrical conductivity. But recently a difference between percolation thresholds for electrical conductivity and brittle fracture was proved by Sornette (1988). It could be naturally explained by more complicated hierarchical system of defects carriers if compared with electricity.

These methods, which were utilized for the investigations of fracture scaling laws, made an impetus to the use of the fractal theory (Mandelbrot 1977, 1982). This process was promoted by two factors. The first was the deep understanding of the self-similar character of the fracture evolution; the second - a proximity of fractal and percolation approaches: usually fractal and percolational clusters coincide. But in spite of geometrical realizations' similarity, the percolation

theory doesn't deal with a hierarchy of levels for media description. Besides, the fractal theory allows to study more precisely (and to obtain qualitative characteristics) the spatial and/or temporal evolution of quasi-brittle fracture process.

THEORY OF FRACTALS AND FRACTURE MODELS

As it seems, the possibility for the use of fractals in brittle fracture analysis was proposed by G.I.Barenblatt (1983). The interest to fractals was to a certain extent stimulated by experimental data which proved the fractal character of porous media and fracture surfaces. The main part of theoretical fractal approaches, which investigate fracture initiation, introduce additional mechanisms (analogous to the structural reconstructions) or a hierarchy of different scales. Some approaches exploit the external geometrical similarity of crack morphology and fractal objects (Lung, 1986; Lung and Zhang, 1989; Heping, 1989) but usually it doesn't allow the whole fracture process to be described. Some approaches deal with models for solids of a spring or rod type (Solla, 1986: Hermann et al., 1989; Hinrichsen et al., 1989; Louis et al., 1986; Arcangelis, 1989). Elastic constitutive equation is supplemented either by a local fracture condition or by a fracture initiation (cut of the spring). Other properties of real structure could be introduced by different procedures. For example, two kinds of elements are exploited (Solla, 1986), representing various fractions of a composite.

Thus, we could point out that traditional fractal approaches of brittle fracture are based on the relatively simple models and do exploit mainly the linear elasticity as a constitutive equation. They, as a rule, don't take into consideration the evolution of a microstructure (damage accumulation and the like) which is one of the main reasons of the fracture.

FRACTAL ANALYSIS OF DAMAGE - CRACK INTERACTION

<u>Model of Damage Accumulation.</u> Approaches being developed within the framework of linear fracture mechanics involve study of the conditions leading to instability of a solid containing cracks of particular geometry. These methods as was shown by Chaboche (1988) and Kachanov and Laures (1989) should account the time-evolution of the microdefects for correct prediction of the fracture development in the specimen or construction. Besides, the conventional approaches, as a rule, deal with the averaged structural and mechanical features. A more adequate approximation to modeling a heterogeneous random medium gives continuum fracture mechanics whose fundamentals were laid down in the works by L.Kachanov and Yu.Rabotnov, and at present have reached a broad audience (Chaboche, 1988; Krajcinovic and Fonseka, 1981). The present work uses a continuum approach to

description of a medium with microdefects (Naimark and Silberschmidt, 1991; Betekhtin et al., 1989).

The second-order tensor p_{ik} is used as a damage parameter. It characterizes the volume concentration and the preferential orientation of the penny-shaped microcracks. The constitutive equations which account the interaction of change of the stress-strain state and damage accumulation were obtained on the basis of the statistical-thermodynamic description of a medium with microcracks. In the general form these equations can be written as (Silberschmidt, 1990, 1991)

$$\dot{\sigma}_{ik} = f_1(\sigma_{ik}, p_{ik}, \dot{p}_{ik}, \varepsilon_{ik}, \dot{\varepsilon}_{ik}, \delta, \dots),$$

$$\dot{p}_{ik} = f_2(p_{ik}, \sigma_{ik}, \dot{\sigma}_{ik}, \varepsilon_{ik}, \dot{\varepsilon}_{ik}, \delta, \dots). \qquad (1)$$

where σ_{ik}, ε_{ik} are macroscopic stress and strain tensors, respectively, δ - structural parameter, "·" means differentiation with respect to time.

The Stochastic Fracture of a V-notched Specimen. The analysis of the brittle fracture interacting with stochastic damage evolution should include a description of the spatial scatter of strength and the possibility of a strong fragmentation (formation of plurality of intersecting fracture surfaces). However, creating computational procedures can be complicated in this case due to the increasing connectedness of the region under study as a result of formation of new free surfaces. When applied to the conventional space discretization schemes,- finite elements - this requires reformulation of the boundary conditions particularly on each time step. This shortcoming can be overcome by using the fractal theory to describe the stochastic nature of fracture.

To study the brittle fracture of specimens without sharp notches the fractal tree was used as an analog of the discretization scheme of the region of space under study. But in a traditional case of the V-shape notch the crack-damage interaction becomes the leading factor in crack propagation and in the evolution of ensemble of defects (Silberschmidt, 1990; Silberschmidt and Silberschmidt, 1990a, b). So, besides the state equations in form (1) one should account the exact crack in terms of the stress intensity factors (Silberschmidt, 1991). The stress redistribution in this case could be described on the basis of known results, for example by formulas from (Cherepanov, 1979).

Let us study the application of the fractal approach for propagating crack characterization by an example of the rectangular region ABCD, which contains the symmetry plane (and an apex) of the V-shape notch. We shall consider this region to be the cross-section of the rectangular beam loaded

with a constant force $\hat{S}$ on its end far from the cross-section under study. The direction of the force is perpendicular to the region ABCD. We shall divide this cross-section into the elements which dimensions correspond to the requirements of the elementary volume (all macroscopic parameters could be considered to be constant within such an element). All the non-uniformity of the mechanical properties and a stochastic character of the damage accumulation will be taken into account by setting the distribution of these parameters along the elementary volumes in a random way.

For the case under study the system of the constitutive equations (1) can be reduced to the load redistribution law for structural elements due to the crack propagation and to the kinetic quasi-linear equation of damage accumulation. The latter could be written for the $p = Sp\, p_{lm}$ in the form

$$\dot{p}^{ij} = A\,\sigma^{ij} + B\,p^{ij}, \qquad (2)$$

where a pair (i, j) corresponds to the element of the i-th row and j-th column, rows are perpendicular to the crack front, $\sigma = \sigma_{zz}$ is a parameter of a macroscopic stress tensor, z being the load direction; A and B are parameters of material (in a common case the random values, too). In order to fulfill automatically the equilibrium law let's consider

$$\bar{G}^{i} = \sum_{j=1}^{N} G^{ij} = const \qquad (3)$$

where G^{ij} is the stiffness of the (i, j)-element, N - the number of columns of the elements. Let's note that we distribute G^{ij} over all elements, even in the region of crack. The model distribution of G^{ij} can be defined by the formula

$$G^{ij} = G\left[1 + \sin^2(\pi k/M)\right] \qquad (i, j = \overline{1, N}) \qquad (4)$$

where k is a random integer, 1<k<M; $M = N^2$ is a total number of elements for a square lattice of N rows by N columns; the ratio of maximum to minimum stiffness equals 2. So G^{ij} is a random magnitude set in a numerical simulation by the random number generator. The load redistribution law must reflect the stress field disturbance caused by a crack propagation. Because of the spatial non-uniformity of this process, linked with the random character of mechanical properties, one should take into account the difference of the crack length along its front. So, exploiting the well known relation $\sigma_{zz} = K_I/(2\pi y)^{1/2}$ where y is a coordinate perpendicular to a crack front; one may obtain the stress redistribution relation by integrating this equation along the y axis in the rows :

$$\sigma^{ij} = K_{ij}\,\sigma^{ij}_{in} \qquad (5)$$

where the initial stress value (depending on the random stiffness of elements) is

$$\sigma^{ij}_{in} = \hat{S}\, G^{ij} / \bar{\bar{G}} l_x l_y \qquad (6)$$

Here $\bar{\bar{G}} = \sum_{i=1}^{N} \sum_{j=1}^{N} G^{ij}$ is the total stiffness: l_x and l_y are the dimensions of the structural element along x and y axes, respectively. The reloading coefficients K_{ij} reflect two interacting fracture mechanisms: the crack propagation under a constant load and the local failure of the elements in the crack front. The structural element fracture condition is defined either by the normal (natural) kinetic criterion (asymptotic $p \xrightarrow[t \to t_f]{} \infty$, t_f is the fracture time) or by the fulfillment of the postulated local fracture criterion (force, concentration). We'll assume that for the failed element its stiffness tends to zero. The second mechanism is accounted by eq.(2): the increase caused by the stress redistribution accelerates the damage accumulation. The coefficients K_{ij} could be obtained from the formula

$$K_{ij} = \hat{K}_{ij}\, \bar{G}^i / \hat{G}^i \qquad (7)$$

where $\hat{K}_{ij}$ are linked with the stress intensity factor of the i-th row of the crack front:

$$\hat{K}_{ij} = K^i_I \sqrt{\pi l_y}\, (\sqrt{j-l_i} - \sqrt{j+1-l_i}) \;, \qquad (8)$$

l_i being the number of columns in the i-th row occupied by the crack. The second multiplier in eq.(7) accounts the decrease of the total stiffness for the i-th row under the crack propagation and failure of structural elements: $\hat{G}^i = \sum_{j=l_i}^{N} (G^{ij} \hat{K}_{ij})$. The value of K^i_I could be approximated by the known relations, for example, by (Cherepanov, 1979) :

$$K^i_I = \hat{S}\, \bar{G}^i \frac{1.11 - 5(l_i/H)^4}{\hat{G}(1 - l_i/H)} \;. \qquad (9)$$

where H is a specimen dimension along y.
The numerical simulation was fulfilled for the case of N = 50 and 100 (i.e. the square lattice of 2500 or 10000 elements, respectively), and the V-shaped notch, occupying 4 per cent of

specimen's width. The process of fracture propagation is sufficiently non-uniform, the crack-front movement being of the step-like character.

In order to prove that the crack front is a fractal object one should calculate its fractal dimension. It could be obtained, for example, as the slope of the curve $N^* - R$ in double logarithmic coordinates, R being the distance from the L_{cr} and N^* - the number of failed elements in crack front within R. Here L_{cr} is the column (farthest from the notch apex) which contain only failed elements. It is a traditional procedure for a fractal, developing from the straight line (Mandelbrot, 1977, 1982). It's obvious that the fractal dimension insufficiently changes with the crack propagation though the crack front's shape varies largely. Calculations, carried out for various statistical realizations (various random distributions of material's properties over the region under study), give the value of the fractal (Hausdorff- Besikovitch) dimension for a propagating crack $D_{HB} = 0.68 \pm 0.03$.

The resulting magnitude of D_{HB} does not contradict the data of the numerical simulation (Hermann et al., 1989) and the experimental definition of the fractal dimension for the real porous solids (brown coal, sandstone, others). With the crack approaching the specimen's opposite side the fractal dimension increases with an asymptote, $D_{HB} \to 1$: the last rows in this case contains mainly failed elements, so the 'mass' of fractal increases proportionally to its radius.

REFERENCES

ARCANGELIS, L. de, HANSEN, A., HERMANN, H.J. and ROUX, S. (1989) Model of fracture growth. *Phys. Rev. B*, 40, 1, 877-880.
BARENBLATT, G.I. (1983) Similarity, Self-Similarity and Intermediate Asymptotes, 2d ed., Gidrometeoizdat, Leningrad (in Russian).
BEALE, P. D., and SROLOVITZ, D. J. (1988) Elastic fracture in random materials. *Phys. Rev. B*, 37, 10, 5500-5507.
BETEKHTIN, V.I., NAIMARK, O.B. and SILBERSCHMIDT, V.V. (1989) Fracture of solids with microcracks: experiments, statistical thermodynamics and constitutive equations. In: Adv. Res. Fract. Proc. 7th Int. Conf. Fract. Vol.6, Pergamon Press, New York, p.38-43.
CHABOCHE, J.-L. (1988) Continuum damage mechanics. Part 1. General concepts. Part 2. Damage growth, crack initiation and crack growth. *Trans. ASME, J. Appl. Mech.* 55, 1, 59-72.
CHEREPANOV, G.P. (1979) Mechanics of Brittle Fracture, McGraw - Hill, New York.
HASSOLD, G.N. and SROLOVITZ, D.J. (1988) Brittle fracture in materials with random defects. *Phys. Rev. B*, 39, 13, 9273-9281.
HEPING, X. (1989) The fractal effect of irregularity of crack

branching on the fracture toughness of brittle materials. *Int. J. Fracture*, 41, 4, 267-274.
HERMANN, H.J., HANSEN, A. and ROUX, S. (1989) Fracture of disordered, elastic lattices in two dimensions. *Phys. Rev. B*, 39, 1, 637-648.
HINRICHSEN, E.L., HANSEN, A. and ROUX, S. (1989) A fracture growth model. *Europhysics Letters*, 8, 1, 1-7.
KACHANOV, M. and LAURES, J.-P. (1989) Three-dimensional problems of strongly interacting arbitrarily located penny-shaped cracks. *Int. J. Fracture*, 41, 4, 289-313.
KNOTT, J.F. (1973) Fundamentals in Fracture Mechanics, Butterworth, London, 1973.
KRAJCINOVIC, D. and FONSEKA, G. V. (1981) The continuous damage theory of brittle materials. *Trans. ASME. J. Appl. Mech.* 48, 809-824.
KRAUSZ, A.S. and KRAUSZ, K. (1988) Fracture Kinetics of Crack Growth, Kluwer Academic, Dordrecht.
LOUIS, E., GUINEA, F. and FLORES, F. (1986) Fractal nature of cracks. In: Fractals in Physics, Elsevier Science Publishers, North Holland, Amsterdam, p.177-181.
LUNG, C.W. (1986) Fractals and fracture of metals with cracks. In: Fractals in Physics, Elsevier Science Publishers, North Holland, Amsterdam, p.189-193.
LUNG, C.W. and ZHANG, S.Z. (1989) Fractal Dimension of the Fractured Surface of Materials, Preprint 76, International Centre for Theoretical Physics.
MANDELBROT, B.B. (1977) Fractals, Form and Chance, W.H.Freeman and Company, San-Francisco.
MANDELBROT, B.B. (1982) The Fractal Geometry of Nature, W.H.Freeman and Company, San-Francisco.
NAIMARK, O.B. and SILBERSCHMIDT, V.V. (1991) On the fracture of solids with microcracks. *Eur.J.Mech.A/Solids*, 10, 6, 607-612.
SILBERSCHMIDT, V.V. (1990) Fractal Models in Fracture Analysis, Sverdlovsk, (in Russian).
SILBERSCHMIDT, V.V. (1991) Fractal Analysis of Crack Propagation, Sverdlovsk, (in Russian).
SILBERSCHMIDT, V.V, and SILBERSCHMIDT, V.G. (1990a) Fractal models in rock fracture analysis. *Terra Nova*, 2, 4, p.483-487.
SILBERSCHMIDT, V.V. and SILBERSCHMIDT, V.G. (1990b) Fractal Approaches in Mechanics of Jointed Rocks. In: Mechanics of Jointed and Faulted Rocks. (Ed. H.P.Rossmanith), Balkema, Rotterdam, p.83-87.
SOLLA, S.A. (1986) Fracture of loaded fractal trees. In: Fractals in Physics, Elsevier Science Publishers, North Holland, Amsterdam, p.185-189.
SORNETTE, D. (1988) Critical transport and failure in continuum crack percolation. *Journal de Physique*, 49, p.1365-1377.
TADA, H., PARIS, P. and IRWIN, G. (1973) The Stress Analysis of Cracks Handbook, Del Research Corporation, Hellertown, Pennsylvania.