FAILURE MODES OF COMPOSITES: A COMPREHENSIVE COMPARISON OF TENSOR POLYNOMIAL CRITERIA

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S U M M A R Y

Several versions of criteria based on failure tensor polynomials (FTP), whereas define identical values for the diagonal terms of the failure fourth-rank tensor, ***H****, as well as for the terms of the strength differential effect second-rank tensor,* ***h****, they present differences between the off- diagonal terms of* ***H****.*

The three most important versions constitute the ***Tsai-Wu criterion*** *with the additional assumption that [1,3] :*

$$H_{ij} = 0 \qquad \text{for } i \neq j, \tag{1}$$

the ***Tsai-Hahn criterion,*** *with [2] :*

$$H_{ij} = - 1/2\,(H_{ii}\,H_{jj})^{1/2} \tag{2}$$

and the ***elliptic paraboloid failure surface criterion*** *(EPFS), with [5-8]:*

$$H_{ij} = 1/2\,(H_{kk} - H_{ii} - H_{jj}) \tag{3}$$

This paper points out their main differences, by comparing their results with existing triaxial tests and it proves that, while (σ_3,σ_1)- and (σ_3,σ_2)-intersections of the failure surface yield doubtful results, plane tests in the (σ_1,σ_2)-isotropic plane is a convenient test for comparing criteria, because the 45 deg off-axis values of failure stresses differ substantially for each particular criterion. Additionally, a biaxial hydrostatic tensile or compressive loading of such anisotropic lamella presents the advantage to be independent, and therefore insensitive, to any off-axis orientation of the specimens, relatively to the loading axes. Therefore, these biaxial tests may complement triaxial ones for the explicit definition of a valid version of criterion. Biaxial tests with different orthotropic materials indicate the superiority of the (EPFS)-criterion and the weaknesses of the other two versions of the (FTP)-criterion.

1. TENSOR POLYNOMIALS FOR FAILURE OF ORTHOTROPIC MATERIALS

The most widely used criteria for isotropic and anisotropic bodies assume complete tensor polynomial functions in a quadratic form, whose linear terms take into account the strength differential effect (SDE) exhibited by all materials.

The anisotropic failure theory is based on the hypothesis, which is largely supported experimentally [4], that isotropic materials can withstand infinitely large amounts of ***hydrostatic compression*** (or tension) without failing. Thus, a ***safe loading path*** exists in the 6D-Euclidean space of symmetric stress tensors which, for isotropic solids, coincides with the direction of the 2nd rank ***spherical tensor,*** $\mu\mathbf{1}(\mu \in R^-)$. This hypothesis, represented in the 3D principal stress space, is evidenced by a paraboloid of revolution with the hydrostatic axis $\sigma_1=\sigma_2=\sigma_3$, as a symmetry axis, whose open end is oriented towards the direction of hydrostatic compression, or tension [5].

A generalization of this failure surface for ***orthotropic solids*** is represented by an ***elliptic paraboloid*** with a symmetry axis, parallel to the hydrostatic axis, and displaced from the origin of the coordinate system by a distance depending on the degree of strength anisotropy of the material. The failure condition expressed in terms of principal stress components, σ_i, was shown to have the general expression of the quadric surface equation in compacted form, that is [7]:

$$H_{ij}\sigma_i\sigma_j + h_i\sigma_i - 1 = 0 \qquad (i, j = 1, \ldots\ldots, 3) \qquad (4)$$

where the 2nd rank tensor H_{ij} and the vector h_i were appropriately defined in terms of the basic strength properties of the material [5-9].

The necessary and sufficient condition for the failure surface (4) to be convex and open-ended is that the tensor **H** must be positive semi-infinite. This condition is expressed by :

$$\sigma \cdot \mathbf{H} \cdot \sigma \geq 0 \qquad (\forall\, \sigma > 0) \qquad (5)$$

Then, the normal components of the failure tensors are expressed by :

$$H_{ii} = \frac{1}{\sigma_{Ti}\sigma_{ci}} \qquad (6)$$

$$(i \leq 3)$$

$$h_i = \frac{1}{\sigma_{Ti}} - \frac{1}{\sigma_{ci}} = (\sigma_{ci} - \sigma_{Ti})H_{ii} \qquad (7)$$

whereas for shear components are given by :

$$H_{ii} = \frac{1}{\sigma^+_{si}\sigma^-_{si}} \qquad (8)$$

$$(i > 3)$$

$$h_i = \frac{1}{\sigma^+_{si}} - \frac{1}{\sigma^-_{si}} = (\sigma^-_{si} - \sigma^+_{si})H_{ii} \qquad (9)$$

In the above relations the repeated index convention does not apply and the σ_{Ti} and σ_{ci}-stresses express the tension (T) and compression (c) failure stress in the i-direction. Furthermore, the σ_{si}^+, σ_{si}^- -stresses express the shear strengths positive or negative in the i-plane (i > 3) and the usual contracted notation of Cartesian indices is used. When the coordinate system defining the failure stresses coincides with the material symmetry directions there is no shear-strength differential effect, that is : $\sigma_{si}^+=\sigma_{si}^-$

Up to this point, no special phenomenological hypothesis was used for the determination of the failure tensor components. The calculation of these components was based upon requirements, common for all anisotropic failure criteria, which are expressed by the general form of eq.(4). However, the off-diagonal components of the failure tensor **H** (H_{ij} , $i\neq j$) should be derived according to the particular assumptions, which are different for the various criteria.

The open end of the failure hypersurface is mathematically assured by imposing the 4th-rank failure tensor **H** to have a ***zero eigenvalue.*** Moreover, the hypothesis that hydrostatic stress is a safe loading path is mathematically formulated by associating the zero eigenvalue of tensor **H** to the 2nd-rank spherical tensor, **1**, which is then the corresponding ***eigentensor*** of **H**. The above implies that the failure hypersurface is a generalized elliptic paraboloid with a symmetry axis parallel to the direction of the spherical tensor, **1**.

These conditions yield the following expressions for the non-diagonal terms of **H** :

$$H_{ij} = \frac{1}{2}(H_{kk} - H_{ii} - H_{jj}), \qquad (i, j, k \leq 3, i\neq j\neq k) \qquad (10)$$

Relations (10) imply that the interaction failure coefficients H_{12}, H_{23} and H_{31} for the elliptic paraboloid failure surface (EPFS) are interrelated with the diagonal components, which are directly defined through relations (6) and (7) with the basic strength data.

This is a significant advantage of the (EPFS)-criterion which is not met with the other similar criteria. Indeed all these criteria are solely based in the experimental evaluation of the off-diagonal coefficients. However, the problem of the optimal experimental determination of the failure tensor components presents unsurmountable difficulties, which were considered by Wu [10]. Complicated and sophisticated experimental techniques were proposed, which were not generally accepted. Thus, several researchers have adopted the ***zeroing*** of the undetermined 4^{th}-rank failure tensor components, which in reality assume very small values. This arbitrary choice altered drastically the concepts of the criterion proposed by Tsai and Wu [1] and the resulting failure condition was shown to yield erroneous predictions, when compared with 3D-experimental failure data [11].

2. THE (EPFS)-CRITERION FOR THE TRANSVERSELY ISOTROPIC BODY

It has been shown that for fiber reinforced composites, the orthotropy of the materials may be satisfactorily approximated by a transverse isotropy. We give therefore in the followings directly the expressions for the transversely isotropic material, which are much simpler than those for the general orthotropic material [7-9]. Indeed, for a transversely isotropic material whose σ_3-axis is the strong axis it is valid that :

$$H_{11} = H_{22} \quad \text{and} \quad h_1 = h_2 \tag{11}$$

The elliptic paraboloid failure surface for the transversely isotropic material is expressed in the $(\sigma_1,\sigma_2,\sigma_3)$-principal stress space, where the σ_3-principal direction corresponds to the strongest one, by [7] :

$$H_{11}(\sigma_1^2+\sigma_2^2) + H_{33}\sigma_3^2 + (H_{33}-2H_{11})\sigma_1\sigma_2 - H_{33}(\sigma_1\sigma_3+\sigma_2\sigma_3) + h_1(\sigma_1+\sigma_2) + h_3\sigma_3=1 \tag{12}$$

The same expression, referred to the Cartesian coordinate system Oxyz, where the Oz-axis, is parallel to the hydrostatic one and the (Oxy)-plane is the deviatoric plane, with the Oy-axis lying on the $O\sigma_1\sigma_3$-plane, is expressed by :

$$(2H_{11} - \tfrac{1}{2}H_{33})x^2 + \tfrac{3}{2}H_{33}y^2 + \sqrt{\tfrac{2}{3}}(h_3-h_1)y + \frac{1}{\sqrt{3}}(2h_1+h_3)z = 1 \tag{13}$$

The principal semi-axes a_1 and a_2 of the elliptic intersection of the (EPFS) by the deviatoric plane are given by [7] :

$$\left.\begin{matrix} a_1 \\ a_2 \end{matrix}\right\} = \begin{cases} \left(\dfrac{2}{3H_{33}}\right)^{1/2}\left\{1+\dfrac{1}{H_{33}}\left(\dfrac{h_1-h_3}{3}\right)^2\right\}^{1/2} \\ \left(\dfrac{2}{(4H_{11}-H_{33})}\right)^{1/2}\left\{1+\dfrac{1}{H_{33}}\left(\dfrac{h_1-h_3}{3}\right)^2\right\}^{1/2} \end{cases} \tag{14}$$

The ratio of the major axis $2a_1$ to the minor axis $2a_2$ of the elliptic intersection expresses the ellipticity of the paraboloid and it is given by :

$$\lambda_0 = \left\{\frac{3H_{33}}{(4H_{11}-H_{33})}\right\}^{1/2} \tag{15}$$

Moreover, the angle θ_0 subtended by the principal axes of the elliptic intersection and the Oxy-system is given by [7] :

$$\theta_0 = 0 \tag{16}$$

The equation expressing the intersection of the elliptic paraboloid failure surface, by the principal plane (σ_3, σ_1) is expressed by :

$$H_{11}\sigma_1^2 + H_{33}\sigma_3^2 + 2H_{31}\sigma_3\sigma_1 + h_1\sigma_1 + h_3\sigma_3 = 1 \tag{17}$$

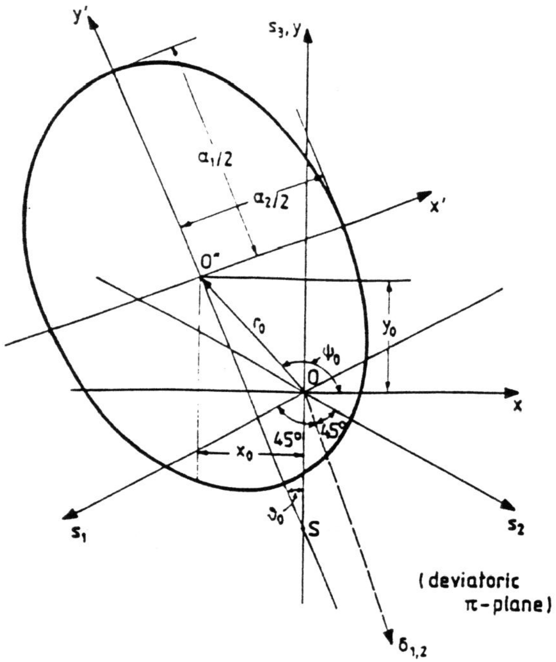

Fig.1 Intersection of the EPFS for the transversely isotropic material by the (σ_1,σ_3)-principal-stress plane for a C-strong transversely isotropic material.

The center of this ellipse is defined by its coordinates $(\sigma_{3M}, \sigma_{1M})$. Figure 1 presents this intersection in the (σ_3, σ_1)-principal stress plane and the coordinates σ_{3M}, σ_{1M} and the angle λ_1 of inclination of the polar radius (OM) are given by [7] :

$$(\sigma_{3M},\sigma_{1M}) = \left(-\frac{(h_1H_{33}+2h_3H_{11})}{H_{33}(4H_{11}-H_{33})} \; , \; -\frac{(2h_1+h_3)}{(4H_{11}-H_{33})} \right) \tag{18}$$

$$\lambda_1 = \tan^{-1}\frac{(h_1H_{33}+2h_3H_{11})}{H_{33}(2h_1+h_3)} \tag{19}$$

The system of Cartesian coordinates $(M\sigma_1,\sigma_3)$, to which this ellipse is central and symmetric, is defined by the angle θ_1, expressed by :

$$\theta_1 = \frac{1}{2}\tan^{-1}\frac{H_{33}}{(H_{11}-H_{33})} \tag{20}$$

On the other hand, the intersection of the (EPFS) by the transverse isotropic principal plane $(\sigma_1\sigma_2)$ is an ellipse whose center lies along the bisector of the angle $(\sigma_1 0 \sigma_2)$ with coordinates:

$$\sigma_{1K}=\sigma_{2K}= -\frac{h_1}{H_{33}} \tag{21}$$

and whose principal semi-axes are given by :

$$\alpha_{1K} = \sqrt{2}\left\{\frac{1}{H_{33}} + \left(\frac{h_1}{H_{33}}\right)^2\right\}^{1/2}$$

$$\alpha_{3K} = \sqrt{2}\left\{\frac{H_{33}+h_1^2}{H_{33}(4H_{11}-H_{33})}\right\}^{1/2} \tag{22}$$

Relations (21) and (22) indicate clearly that the position and the shape of the transverse isotropic intersection of the EPFS is influenced strongly by the anisotropy coefficient H_{33} and this intersection is much different than, the respective intersection of a totally isotropic material presenting the same strength differential effect, as the SDE existing along the transverse isotropic plane (see Fig.2).

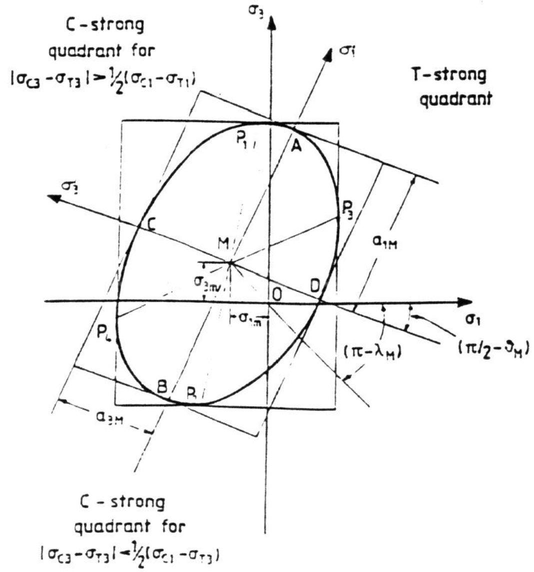

Fig.2 Intersection of the EPFS for the transversely isotropic material by the (σ_1,σ_2) isotropic transverse principal plane for a C-strong transversely isotropic material.

3. THE FAILURE TENSOR POLYNOMIAL CRITERION IN ITS THREE DIFFERENT VERSIONS

The expressions for the transversely isotropic material, which are much simpler than those for the general orthotropic material [12], whose σ_3-axis is the strong axis, are derived from the general orthotropic relationships by introducing the simplifications :

$$H_{11} = H_{22}\ ,\quad h_1 = h_2 \quad \text{and} \quad H_{13} = H_{23} \tag{23}$$

Then, the failure tensor polynomial surface for the transversely isotropic material satisfying all the properties already described is expressed as follows in the $(\sigma_1,\sigma_2,\sigma_3)$-principal stress space, where the σ_3-principal direction corresponds to the strongest one [12] :

$$H_{11}(\sigma_1^2+\sigma_2^2) + H_{33}\sigma_3^2 + 2H_{12}\sigma_1\sigma_2 + 2H_{13}(\sigma_1\sigma_3+\sigma_2\sigma_3) + h_1(\sigma_1+\sigma_2) + h_3\sigma_3 = 1 \tag{24}$$

This expression, referred to the Cartesian coordinate system 0xyz, is expressed by :

$$(H_{11}-H_{12})x^2+\frac{1}{3}(H_{11}+2H_{33}+H_{12}-4H_{13})y^2+\left\{\frac{2\sqrt{2}\,z}{3}(H_{33}-H_{11}+H_{13}-H_{12})-\sqrt{\frac{2}{3}}(h_1-h_3)\right\}y+$$
$$+\frac{z^2}{3}(2H_{11}+H_{33}+2H_{12}+4H_{13})+\frac{z}{\sqrt{3}}(2h_1+h_3)-1=0 \tag{25}$$

(a) Relations (24) and (25) for the **(EPFS)-criterion**, become :

$$H_{11}(\sigma_1^2+\sigma_2^2) + H_{33}\sigma_3^2 + (H_{33}-2H_{11})\sigma_1\sigma_2 - H_{33}(\sigma_1\sigma_3+\sigma_2\sigma_3) + h_1(\sigma_1+\sigma_2) + h_3\sigma_3=1 \tag{26}$$

and :

$$(2H_{11}-\frac{1}{2}H_{33})x^2+\frac{3}{2}H_{33}y^2-\sqrt{\frac{2}{3}}(h_1-h_3)y+\frac{1}{\sqrt{3}}(2h_1+h_3)z=1 \tag{27}$$

(b) For the **Tsai-Hahn criterion**, where relations (1) and (23) hold, take the form :

$$H_{11}(\sigma_1^2+\sigma_2^2) + H_{33}\sigma_3^2 - H_{11}\sigma_1\sigma_2 - (H_{11}H_{33})^{1/2}(\sigma_1\sigma_3+\sigma_2\sigma_3)+h_1(\sigma_1+\sigma_2)+h_3\sigma_3= 1 \tag{28}$$

and :

$$\frac{3}{2}H_{11}x^2+\frac{2}{3}\left(\frac{1}{2}H_{11}^{1/2}+H_{33}^{1/2}\right)^2y^2+\left\{\frac{\sqrt{2}\,z}{3}\left(2H_{33}-H_{11}-(H_{11}H_{33})^{1/2}\right)-\sqrt{\frac{2}{3}}(h_1-h_3)\right\}y+$$
$$+\frac{1}{3}\left[H_{11}+H_{33}-2(H_{11}H_{33})^{1/2}\right]z^2+\frac{1}{\sqrt{3}}(2h_1+h_3)z=1 \tag{29}$$

(c) Finally, for the **Tsai-Wu criterion,** with $H_{ij}=0$, $(i \neq j)$, it is valid that :

$$H_{11}(\sigma_1^2+\sigma_2^2) + H_{33}\sigma_3^2 + h_1(\sigma_1+\sigma_2) + h_3\sigma_3 = 1 \tag{30}$$

and :

$$H_{11}x^2+\frac{1}{3}(H_{11}+2H_{33})y^2-\left[\frac{2\sqrt{2}}{3}(H_{11}-H_{33})z+\sqrt{\frac{2}{3}}(h_1-h_3)\right]y+$$
$$+\frac{1}{3}(2H_{11}+H_{33})z^2+\frac{1}{\sqrt{3}}(2h_1+h_3)z=1 \tag{31}$$

For the principal diagonal $(\sigma_3\delta_{12})$-plane, which presents the symmetric intersection of each of the failure surfaces we have that $\sigma_1=\sigma_2=\delta_{12}/\sqrt{2}$ and $\sigma_3=\sigma_3$, the general expression for the tensor failure polynomial surface for the transversely isotropic body is given by :

$$(H_{11}+H_{12})\delta_{12}^2+H_{33}\sigma_3^2+2\sqrt{2}\,H_{13}\delta_{12}\sigma_3+\sqrt{2}\,h_1\delta_{12}+h_3\sigma_3-1=0 \tag{32}$$

Relation (32) for the three versions of criteria studied take the form :

For the (EPFS)-criterion :

$$\frac{1}{2}H_{33}\delta_{12}^2+H_{33}\sigma_3^2-\sqrt{2}\,H_{33}\delta_{12}\sigma_3+\sqrt{2}\,h_1\delta_{12}+h_3\sigma_3-1=0 \tag{33}$$

for the Tsai-Hahn criterion :

$$\frac{1}{2}H_{11}\delta_{12}^2+H_{33}\sigma_3^2-\sqrt{2}(H_{11}H_{33})^{1/2}\delta_{12}\sigma_3+\sqrt{2}\,h_1\delta_{12}+h_3\sigma_3-1=0 \tag{34}$$

and the Tsai-Wu with $H_{ij}=0$ criterion :

$$H_{11}\delta_{12}^2+H_{33}\sigma_3^2+\sqrt{2}\,h_1\delta_{12}+h_3\sigma_3-1=0 \tag{35}$$

The above relationships were used for tracing the intersections on the principal diagonal symmetry plane $(\sigma_3\delta_{12})$ for a typical transversely isotropic material, which is the oriented polycarbonate (OPC). The failure properties of this material were given by Caddell and coworkers [13-15]. Its failure strengths along the strong σ_3-and the weak σ_1- and σ_2-stress- axes are given as :

$$\sigma_{T3} = 65.2\text{ MPa} \qquad \sigma_{T1} = \sigma_{T2} = 35.2\text{ MPa}$$
$$\sigma_{C3} = 42.7\text{ MPa} \qquad \sigma_{C1} = \sigma_{C2} = 45.2\text{ MPa} \tag{36}$$

Using the values for the oriented polycarbonate [13] given in relations (36) we define the $(\sigma_3\delta_{12})$-intersections of its failure surface, as defined by each of the three versions of the tensor failure polynomial criterion expressed in Eqs. (33-35). Fig. 3 presents these intersections. While the failure loci for the (EPFS)- and the (TH)-criteria are parabolas, the failure locus for the (TW)-criterion with $H_{ij}=0$ is represented by an ellipse.

The equation for the $(\sigma_3\delta_{12})$-intersection of the quadric failure surface in the $(\sigma_1,\sigma_2,\sigma_3)$-principal stress space is expressed by the polynomial [9] :

$$a\delta_{12}^2 + b\sigma_3^2 + 2h\sigma_3\delta_{12} + 2g\delta_{12} + 2f\sigma_3 - 1 = 0 \tag{37}$$

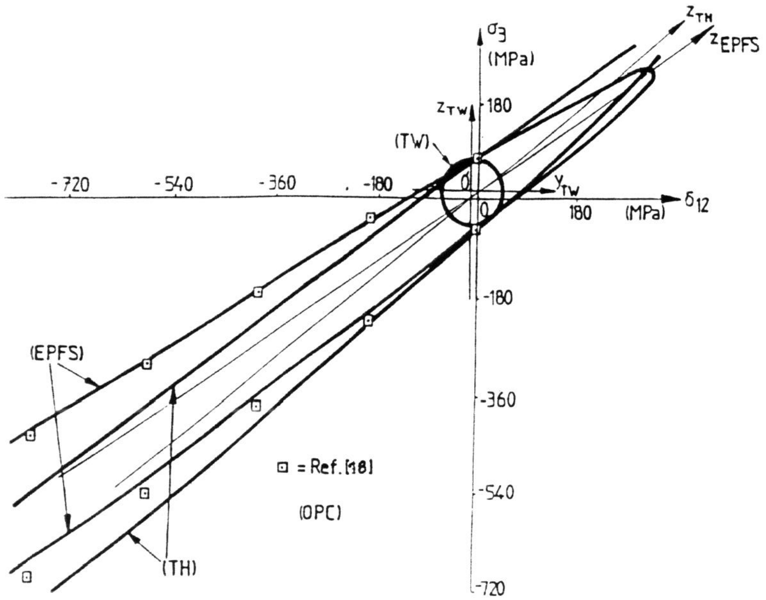

Fig.3 The $(\sigma_3\delta_{12})$-principal diagonal intersections for oriented polycarbonate according to the three versions of failure tensor polynomial criterion (EPFS): The elliptic paraboloid failure surface, (TH) : The Tsai-Hahn criterion and (TW) : The Tsai-Wu with $H_{ij}=0$ $(i\neq j)$ criterion.

The coefficients of this second degree polynomial and the necessary and sufficient conditions so that the curve represented by relation (37) is an open-end curve (parabola), were examined in detail in refs. [6-9] and [16].

It is evident from these plottings that the (EPFS)-criterion and the (TH)-criterion yield open-ended surfaces, whose $(\sigma_3\delta_{12})$-intersections, which are symmetric intersections to the failure surfaces, present the following differences :

(i) The (EPFS)-intersection has its symmetry axis parallel to the hydrostatic axis, whereas the (TH)-intersection is strongly inclined to this axis.

(ii) The (TW)-intersection is aclosed curve (ellipse),whose principal axes are parallel to the σ_3- and δ_{12}- axes and its center is displaced inside the second quadrant of the $(\sigma_3\delta_{12})$-plane.

(iii) The most venerated experiments by Bridgman [4,17] have definitely established that the resistance to failure of all materials (isotropic and orthotropic) is much higher than their resistance to failure in simple compression. This fact implied the principle that the failure strength of materials to a loading mode approaching the hydrostatic pressure should be very high, so that their failure surfaces should be open. Then, the (TW)-criterion with $H_{ij}=0$, $(i\neq J)$ which yields a failure stress under hydrostatic pressure, which is of the same order of magnitude with the failure stress in simple compression is unacceptable, as it is also any other criterion represented in the stress space by a closed surface.

Therefore, the Tsai-Wu criterion stating that all interaction (off-diagonal) terms H_{ij} are independent and should be determined experimentally constitutes, instead of a curve fitting process, not based on any phenomenological trend, failure criterion, and establishing only a failure condition with statistical features.

The difficulty in executing complicated tests to define the failure characteristics of anisotropic materials explains the paucity of such tests in the literature. The only triaxial tests with anisotropic materials existing in the literature are those executed with mildly anisotropic high-polymers, mechanically or thermally deformed, to lend them some amount of orientation as it is the oriented polycarbonate (OPC) [13-15]. There exist also some triaxial tests with foams and porous materials, as well as with sands and rocks, but all these results are not sufficient for studying the failure loci of these materials.

Figure 3 presents the experimental results from ref.[18] referred to the principal $(\sigma_3\delta_{12})$- diagonal plane of the oriented polycarbonate. In the same figure the $(\sigma_3\delta_{12})$-intersections of the three failure loci studied in this paper are also plotted. It is clear from this figure that the (EPFS)-criterion is closer than any other criterion.

4. THE ISOTROPIC (σ_1,σ_2)-INTERSECTIONS OF THE FAILURE LOCI

It has been already stated previously for the (EPFS)-criterion, that the high values of failure stresses in tension and compression along the strongest principal σ_3-stress direction influence considerably the shape of the elliptic intersection on the transverse isotropic $(\sigma_1\sigma_2)$-plane. Furthermore, the centers of these ellipses are displaced along the 45°-diagonal in the first or the third quadrant, so that for compression-strong materials, like oriented polycarbonate, the higher differences in failure stress appear in the compression quadrant, whereas for tension-strong materials, like oriented polypropylene (OPP), the highest differences lie in the first tension-tension quadrant.

While for strong σ_3-stress axis composites the transverse ellipses become much oblonger as the σ_{T3} - and σ_{C3} -stresses become higher, for weak-symmetry axis composites, like the woven-fabric composites [19] and the fiber composites, whose ratios of elastic and shear moduli are such that the respective values of their eigenangle approach the limiting value for isotropic materials $\omega_i = 125.36°$ [20], the respective elliptic $(\sigma_1\sigma_2)$-intersections tend to become circular by reducing the lengths of their major axes [21].

By taking advantage of this property we examine the failure loci in the transverse $(\sigma_1\sigma_2)$-plane of symmetry for the three versions of the criterion. The transverse intersection of the general quadric surface, by the plane of symmetry of the material is given by, in the $(\sigma_1\sigma_2)$-principal stress coordinate frame:

$$H_{11}(\sigma_1^2+\sigma_2^2) + 2H_{12}\sigma_1\sigma_2 + h_1\sigma_1 + h_2\sigma_2 - 1 = 0 \qquad (38)$$

The same equation for any off-axis of symmetry Oxy- frame is transformed to :

$$H_{xx}\sigma_x^2 + H_{yy}\sigma_y^2 + 2H_{xy}\sigma_x\sigma_y + h_x\sigma_x + h_y\sigma_y - 1 = 0 \qquad (39)$$

where the coefficients H_{xx} H_{yy} H_{xy} h_x h_y are expressed by [2] :

$$H_{xx} = U_1 + U_2\cos2\theta + U_3\cos4\theta$$

$$H_{yy} = U_1 - U_2\cos2\theta + U_3\cos4\theta$$

$$H_{xy} = (U_4 - U_3\cos4\theta) \qquad (40)$$

$$h_x = h_y = \frac{1}{2}(h_1+h_2) = h_1 = h_2$$

where θ is the angle subtended by the principal direction of the material and the direction of loading. The coefficients U_i are given by [2] :

$$U_1 = \frac{1}{8}[3H_{11} + 3H_{22} + 2H_{12} + H_{66}]$$

$$U_2 = \frac{1}{2}(H_{11} - H_{22}) = 0 \qquad (41)$$

$$U_3 = \frac{1}{8}[H_{11} + H_{22} - 2H_{12} - H_{66}]$$

$$U_4 = \frac{1}{8}[H_{11} + H_{22} + 6H_{12} - H_{66}]$$

In these relations the following expressions for the off-diagonal coefficient H_{12} are valid for the three versions of criteria studied here :

(i) *For the (EPFS) :*

$$H_{12} = \frac{1}{2}(H_{33} - 2H_{11})$$

(ii) *For (TH) :*

$$H_{12} = -\frac{H_{11}}{2} \quad \text{and} \qquad (42)$$

(iii) *For (TW) with $H_{ij}=0$, $(i\neq j)$:*

$$H_{12} = 0$$

Introducing relations (41) and (42) into relations (40) we derive for the three criteria the following expressions :

(i) *For the (EPFS)-criterion :*

$$H_{xx} = \frac{1}{8}\{(4H_{11}+H_{33}+H_{66}) + (4H_{11}-H_{33}-H_{66})\cos4\theta\}$$

$$H_{xy} = \frac{1}{8}\{(3H_{33}-4H_{11}-H_{66}) - (4H_{11}-H_{33}-H_{66})\cos4\theta\} \qquad (43)$$

(ii)*For the (TH)-criterion :*

$$H_{xx} = \frac{1}{8}\{(5H_{11}+H_{66}) + (3H_{11}-H_{66})\cos 4\theta\}$$

$$H_{xy} = -\frac{1}{8}\{(H_{11}+H_{66}) + (3H_{11}-H_{66})\cos 4\theta\} \qquad (44)$$

and

(iii) *For the (TW)-criterion with $H_{ij}=0$ (for $i\neq j$) :*

$$H_{xx} = \frac{1}{8}\{(6H_{11}+H_{66}) + (2H_{11}-H_{66})\cos 4\theta\}$$

$$H_{xy} = \frac{1}{8}(2H_{11}-H_{66})(1-\cos 4\theta) \qquad (45)$$

For an arbitrary biaxial in-plane loading mode of the plate with $\sigma_x\neq\sigma_y$ relation (39) becomes:

$$H_{xx}(\sigma_x^2+\sigma_y^2) + 2H_{xy}\sigma_x\sigma_y + h_1(\sigma_x+\sigma_y) - 1 = 0 \qquad (46)$$

whereas, for an equal biaxial in-plane loading of the plate (hydrostatic loading), with $\sigma_x=\sigma_y=\sigma$ relation (46) becomes :

$$2(H_{xx}+H_{xy})\sigma^2 + 2h_1\sigma - 1 = 0 \qquad (47)$$

Relation (47) for the three different criteria studied becomes :

(i) *For the (EPFS) :*	$H_{33}\sigma^2 + 2h_1\sigma - 1 = 0$	
(ii)*For the (TH) :*	$H_{11}\sigma^2 + 2h_1\sigma - 1 = 0$	(48)
(iii)*For the (TW) :*	$2H_{11}\sigma^2 + 2h_1\sigma - 1 = 0$	

Relations (48) indicate at once that any **in-plane hydrostatic loading of a plate corresponding to a transverse intersection of the fiber-reinforced composite, so that the fiber axes are parallel to the thickness of the plate, is invariant and independent of the angle of orientation of loading with respect to the principal directions of the composite.**

Relation (46) implies that the elliptic intersections of the failure loci by the $(\sigma_1\sigma_2)$-transverse isotropic plane have their major axes, lying along the bisector of the first quadrant $(\sigma_1 O\sigma_2)$ - right angle, equal to the respective major axes of the

ellipses corresponding to an hydrostatic loading of the plate where $\sigma_x=\sigma_y=\sigma$, since for them also the same equality of applied stresses σ_x, σ_y is also valid. This is valid for any loading mode of the plate, where either the external loading is either on-axis ($\theta=0°$) or off-axis ($\theta\neq0°$) oriented, relatively to the principal material directions.

On the contrary, the other principal axes of the ellipses (minor axes) depend on the θ-angle with the maximum deviations appearing at $\theta=45°$.

Figures 4 and 5 present the elliptic intersections of the three failure loci studied for the oriented polycarbonate, whose failure stresses in simple tension or compression are given by the values (36). Figure 4 corresponds to angle $\theta=0°$, whereas Fig.5 to angle $\theta=45°$. Similarly, Figs 6 and 7 give the same intersections of the failure loci for the oriented polypropylene, whose properties are given in refs. [13] and [14], for angles $\theta=0°$ and $\theta=45°$.

By comparing the ellipses of either pair of figures one may realize at once the coincidence of the major axes of the respective ellipses, for different θ-angles, whereas their minor axes differ significantly. It is wothwhile remarking that, while the oriented polycarbonate is a compression strong material ($tr\mathbf{h}=(h_3+2h_1) > 0$), the oriented polypropylene is a tension-strong material ($tr\mathbf{h}=h_3+2h_1<0$). The larger differences in the failure stresses between the various criteria appear for the (OPC) in the third quadrant, whereas for the (OPP) in the first quadrant, where either material presents its maximum strength.

By comparing the failure stresses for hydrostatic loading of the plates of the materials corresponding to their transverse isotropic planes it is obvious that the ratios of these stresses for the various versions of the failure tensor polynomial criterion are considerable. These ratios are stronger for the mode of loading for which the material is more resistant, that is this difference in failure stresses under hydrostatic in-plane loading becomes higher in the compression-compression zone of loading for compression-strong materials and in the tension-tension quadrant for the tension-strong materials.

Thus, for the oriented polycarbonate the ratios of failure stresses under hydrostatic in-plane compression is given by :

$$\sigma_{(EPFS)} : \sigma_{(TH)} : \sigma_{(TW)} = (-73.10) : (-51.12) : (-33.64) \text{ MPa}$$

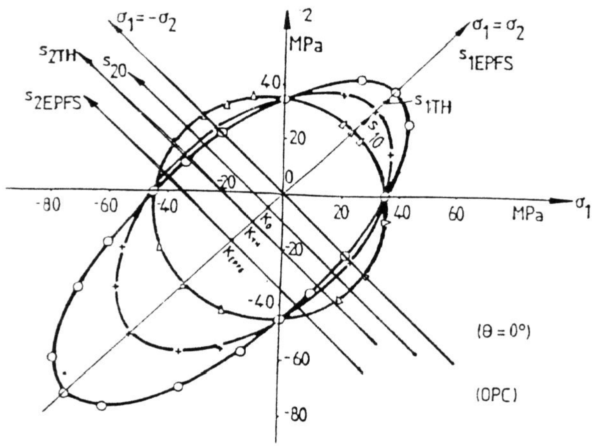

Fig.4 The $(\sigma_1\sigma_2)$-principal stress intersections with θ=0° for oriented polycarbonate according to the three versions of failure tensor polynomial criterion (EPFS): The elliptic paraboloid failure surface, (TH) : The Tsai-Hahn criterion and (TW) : The Tsai-Wu with H_{ij}=0 (i≠j) criterion.

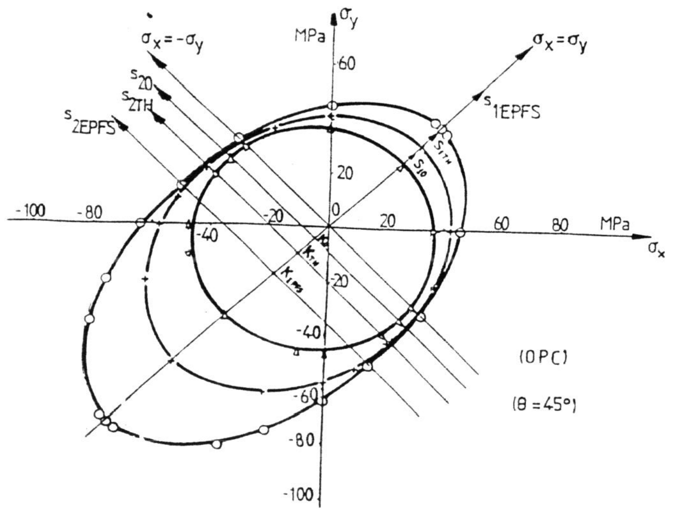

Fig.5 The $(\sigma_x\sigma_y)$-transverse isotropic plane intersections with θ=45° for oriented polycarbonate according to the three versions of failure tensor polynomial criterion (EPFS) : The elliptic paraboloid failure surface, (TH) : The Tsai-Hahn criterion and (TW) : The Tsai-Wu with H_{ij}=0 (i≠j) criterion.

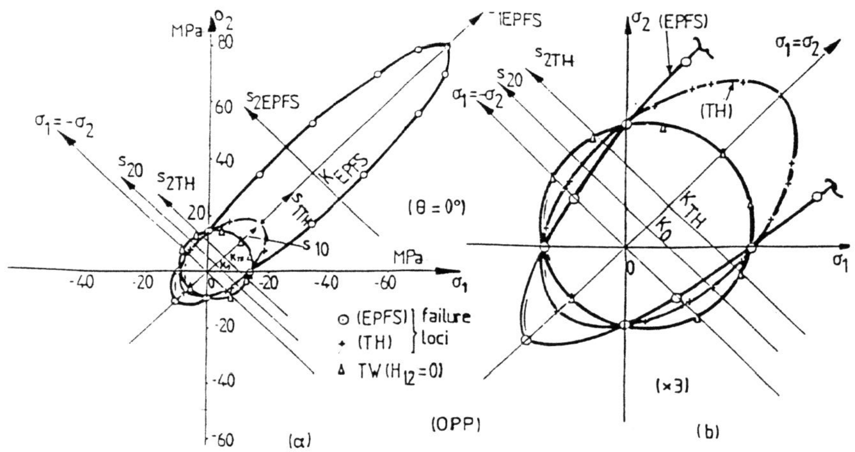

Fig.6 The $(\sigma_1\sigma_2)$-principal stress intersections with θ=0° for oriented polypropylene according to the three versions of failure tensor polynomial criterion (EPFS): The elliptic paraboloid failure surface, (TH) : The Tsai-Hahn criterion and (TW) : The Tsai-Wu with H_{ij}=0 (i≠j) criterion.

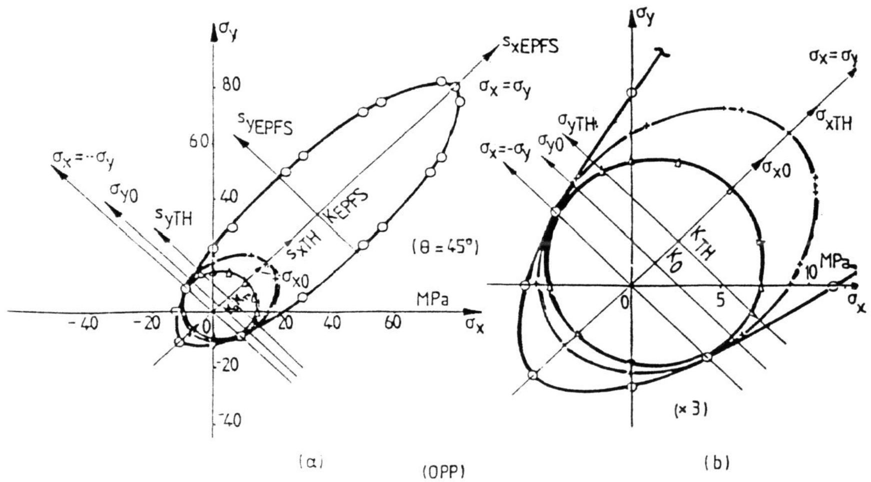

Fig.7 The $(\sigma_x\sigma_y)$-transverse isotropic plane intersections with θ=45° for oriented polypropylene according to the three versions of failure tensor polynomial criterion (EPFS) : The elliptic paraboloid failure surface, (TH) : The Tsai-Hahn criterion and (TW) : The Tsai-Wu with H_{ij}=0 (i≠j) criterion.

whereas for the oriented polypropylene the ratios of failure stresses under hydrostatic in-plane tension are :

$$\sigma_{(EPFS)} : \sigma_{(TH)} : \sigma_{(TW)} = (80.124) : (17.83) : (11.144) \text{ MPa}$$

It is clear from these differences in failure stresses in mildly anisotropic materials that an efficient and decisive type of test, in order to decide which of the various versions of the failure tensor polynomial criteria is closer to reality, is the hydrostatic in-plane tension or compression testing of specimens of the material cut-off along the transversely isotropic plane. Since the test is rather easy to be executed in a triaxial type of testing machine, it constitutes the critical test for selecting the appropriate form of the criterion.

For highly anisotropic materials, as they are the carbon glas- and other fiber-composites the difference of failure stresses of plates corresponding to their transverse isotropic plane and subjected to lateral in-plane hydrostatic tension or compression is much higher. Thus, for the carbon-fiber, epoxy matrix composite under the commercial indication as T-300/5208 graphite epoxy composite and a fiber volume fraction $\upsilon_f = 0.40$ we have the following results.

(i) For the typical fiber reinforced composite the failure stresses in simple tension and compression are given by [3, 22] :

$$\sigma_{T3} = 1340 \text{ MPa} \qquad \sigma_{T1} = \sigma_{T2} = 51.90 \text{ MPa}$$

$$\sigma_{C3} = 932 \text{ MPa} \qquad \sigma_{C1} = \sigma_{C2} = 233.0 \text{ MPa}$$

(ii) For woven-fabric laminae consisting of plain-weave carbon fiber reinforcements the experimental data obtained from regular coupon type specimens gave [23] :

$$\sigma_{T1} = \sigma_{T2} = 465 \text{ MPa} \qquad \sigma_{T3} = 80 \text{ MPa}$$

$$\sigma_{C1} = \sigma_{C2} = 88 \text{ MPa} \qquad \sigma_{C3} = 264 \text{ MPa}$$

It is clear from these data that the σ_i (i=1,2) directions lie at the isotropic plane which is weak for the typical fiber-reinforced composite and strong for the plane-weave of the fabric.

Evaluating now for the two types of composites, the failure ratios under lateral in-plane hydrostatic compression we can readily find that :

(i) For the T300/5208 graphite-epoxy fiber-reinforced composite :

$$\sigma_{(EPFS)} : \sigma_{(TH)} : \sigma_{(TW)} = (-37,335.93) : (-392.15) : (-209.53) \text{ MPa}$$

(ii) For the T300/5208 graphite-epoxy woven-fabric composite :

$$\sigma_{(EPFS)} : \sigma_{(TH)} : \sigma_{(TW)} = (-437.85) : (-806.02) : (-425.72) \text{ MPa}$$

It is worthwhile mentioning that, while the ratios of failure stresses under lateral in-plane hydrostatic loading for the typical fiber-reinforced composite are very large, for the woven-fabric composite the ratios for $\sigma_{(EPFS)}$ and $\sigma_{(TW)}$ are very close together, fact which can be explained by the infinitesimal influence of the weak σ_3-axis failure stresses on the strong failures on the plane of weave.

These large differences in strengths given by the different versions of the tensor failure polynomial criterion in plates corresponding to transverse cross-sections on the isotropic plane of a transversely isotropic material constitute a certain technique for deciding which version corresponds to reality, corroborating existing experimental results and they justify the acceptance of this method for the selection of the best criterion.

Furthermore, since biaxial tests of lateral hydrostatic tension or compression in thin plates are rather easy to be executed and yield reliable results, the method is additionally convenient to be applied in a laboratory facility disposing a triaxial type of testing machine commonly used in soil and rock mechanics.

A series of such tests was executed with plates 10 mm thick of dimensions 50 x 50 mm, cut-off from prismatic specimens of polymeric materials tested after they were subjected to a mechanical and thermal treatment to create the appropriate orientation. As for the composites, tests in the specimens were cut at a transverse direction to their fiber direction. Only for oriented polypropylene, which was subjected to a biaxial hydrostatic tension the specimens were more complicated having the shape of a greek-cross with protrusions, used for applying a uniform tension stress along each side.

The preliminary experimental results derived from the lateral biaxial hydrostatic compression gave for polycarbonate $\sigma_{(OPC)}$=-72.10 MPa, for polypropylene $\sigma_{(OPP)}$ = 78 MPa, whereas for the T300/5208 graphite epoxy composite we could not attain failure although hydrostatic pressures of the 1000 MPa were applied to the specimens. On the contrary, for the T300/5208 graphite epoxy woven-fabric composite failure stresses of $\sigma_{(WF)}$ = -450 MPa were achieved.

Although these experiments were only preliminary, indicated clearly that the experimental data are in favor of the results derived from the (EPFS)-criterion than from any other version of the failure tensor polynomial criterion. However, further sophisticated experimental evidence with convenient equipment is needed for rising the accuracy of the experimental results.

5. CONCLUSIONS

It has been established that the most powerful failure criteria for anisotropic bodies finding a broad application to the study of failure of fiber composites are the criteria based on tensor polynomials. Three versions of this criterion were briefly examined and their differences were pointed out. These criteria are the Tsai-Wu criterion with the simplification discussed by Narayanaswani and Adelman [3], the Tsai-Hahn criterion [2] and the elliptic paraboloid failure criterion introduced by the author [5-8].

The constitutive equations of these criteria were described and the important intersections of the surfaces expressing these criteria were established. A comparison of their principal diagonal intersections defined by their axes of symmetry and the strong (fiber) axes of the materials gave immediately that the Tsai-Wu criterion with $H_{ij}=0$ for $i\neq j$ is unacceptable since its principal diagonal intersection is a closed (elliptic) curve which is in contradiction with the general experimental evidence that the strength of these materials in hydrostatic compression is many times larger than any other failure limit.

On the other hand, the Tsai-Hahn criterion having its axis of symmetry inclined to the hydrostatic axis and depending on the particular mechanical properties of each material is not in conformity with results consisting of an hydrostatic pressure superimposed with either a tensil or compressive simple loading where the effect of this additional loading should by independent of direction.

Finally, the elliptic paraboloid failure criterion presents intersections whose shapes and positions are in satisfactory agreement with existing experimental evidence.

The strong (σ_3)- weak (σ_1 or σ_2) - intersections of all three criteria give doubtful results not allowing a striking superiority of one of them oven the other two criteria, but they are in fairly satisfactory agreement with experimental evidence in this important plane which represents the plane of anisotropic sheets containing the fibers of the materials and therefore are important in applications. This coincidence is mainly due to the fact that all three ellipses representing the loci of these criteria in the (σ_3,σ_1) principal stress plane should pass from the same points along the principal stress axes, thus not allowing large ground for significant discrepancies.

It was shown in this paper that the other principal stress plane, the isotropic (σ_1, σ_2) - plane, presents on the contrary significant differences in the failure loci for the three versions of the criterion thus allowing a safecomparison between the results of the three criteria, as compared also to experimental evidence, especially along the diagonal σ_1=σ_2 (tension or compression) loading.

Again a comparison of the three versions of the criterion with isotropic plates cut-off from three dimensional specimens perpendicularly to the fiber direction indicated clearly the superiority of the (EPFS)-criterion over the two other versions.

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