DETERMINATION OF THERMOSTRESSED STATE AT THE BODY WITH INCLUSIONS IN TWO DIMENSIONAL CASE

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ABSTRACT

In the reaserchwork the thermostressed state of the body with inclusions which can be both thin and limitedsizes is investigated. Solution of the problem is reduced to the system of Prandl singular integral differential equations (PSIDE). In the case of circular disk and straight thin inclusion the system is solved numerically by the method of mechanical quadratures. Numerical analysis of stress intensity factors (SIF) is presented.

KEYWORDS

Thin inclusion, heat transfer, thermal flow, stress, strain, disk, complex potential, stress intensity factor (SIF), Prandl singular integral differential equations (PSIDE), displacement

INTRODUCTION

Suppose a disk and thin inclusion of arbitrary form are soldered in elastic isotropic infinite plate. The plate is influenced by uniformally distributed at infinity stretching forces N and N which act in reciprocally perpendicular directions and force N forms together with axis OX angle β. Thermal flow q and temperature T_o are given at infinity. On the material boundary line the conditions of ideal mechanical and thermal contact are observed. It is supposed that plate bases are thermoisolated. Let L be the line which limits the disk, L be the middle line of the thin inclusion (L is the contour of Lyapunov type). We choose the Cartesian coordinates XOY originating in any point of the disk. Let quantities assotiated with the disk be denoted by index 'O' and quantities assotiated with the thin inclusion be denoted by index '1'. Fig. 1

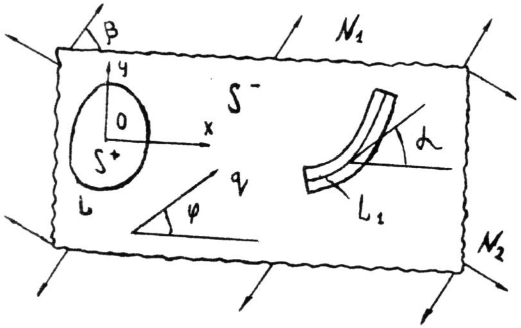


Fig. 1. Plate general view.

HEAT TRANSFER PROBLEM

To determine temperature field T(x,y) the following equations are used (Prusov, 1975)

$$T(x;y)=2\mathrm{Re}F_o(z);$$

$$\frac{\partial T}{\partial s} = F(z)e^{i\alpha}+\overline{F(z)}e^{-i\alpha} \; ;$$

$$\frac{\partial T}{\partial n} =[F(z)e^{i\alpha}+\overline{F(z)}e^{-i\alpha}]i \; ; \qquad (1)$$

where $F(z)=\tilde{F}_o'(z)$, $z=x+iy$; $\tilde{F}_o(z)$ is holomorphic function; $z\in L$. α is angle between OX and tangent to L in the point z; s is the arch coordinate, n is the normal to L in the point z.
Taking into consideration the thinness of the inclusion and (1) the following equations are obtained.

$$\frac{\partial T^+}{\partial s} - \frac{\partial T^-}{\partial s} = 2h\ \rho_k'(u), \quad \frac{\partial T^+}{\partial n} - \frac{\partial T^-}{\partial n} = -2h\ g_k'(u), \quad u\in L_1 \qquad (2)$$

$$\frac{\partial T^+}{\partial s} + \frac{\partial T^-}{\partial s} = 2[F_k^*(u)e^{i\alpha}+ \overline{F_k^*(u)}e^{-i\alpha}];$$

$$\frac{\partial T^+}{\partial n} + \frac{\partial T^-}{\partial n} = 2i[F_k^*(u)e^{i\alpha}- \overline{F_k^*(u)}e^{-i\alpha}];$$

where

$$g_k'(u)= F_k^{*\prime}(u)e^{i\alpha}+ \overline{F_k^{*\prime}(u)}e^{-i\alpha}; \rho_k'(u)= [F_k^{*\prime}(u)e^{i\alpha}- \overline{F_k^{*\prime}(u)}e^{-i\alpha}]i$$

$F_k(u)$ is unknown function. Sings "+" and "-" to values of corresponding quantities on inclusions margins.
Let us present the complex potential F(z) for the plate in the following way:

$$F(z)=F_1(z)+F_2(z)+C_o$$

where $F_1(z)$ is piecewise holomorphic function with breakline L_1; $F_2(z)$ is piecewise holomorphic function with breakline L_2;

$$F_2(z)=\frac{1}{2\pi i}\int_L \frac{\varphi_1(t)dt}{t-z} \; ; C_o=-\frac{1}{2k}qe^{-i\varphi} \; ;$$

k is coeffitient of thermal conductivity.
On the material boundary line the following conditions of ideal temperature contact are observed :

$$\frac{\partial T}{\partial s} = \left(\frac{\partial T}{\partial s}\right)_1 \; ; \qquad k\frac{\partial T}{\partial n} =k_1\left(\frac{\partial T}{\partial n}\right)_1 \qquad (4)$$

Taking into consideration (4) and (2) boundary problem for determinating F(z) is obtained.

$$[F_1(u)e^{i\alpha}+ \overline{F_1(u)}e^{-i\alpha}]=2hg'(u)n_1 i \; ;$$

$$[F_1(u)e^{i\alpha}- \overline{F_1(u)}e^{-i\alpha}]=2h\rho'(u) \; ; \qquad (5)$$

where

$$g'(u)=g_k'(u) - g_o'(u) \; ; \rho'(u)=\rho_k'(u) - \rho_o'(u) \; ;$$

$$g_o'(u)= [F_{2k}(u)e^{i\alpha}+ \overline{F_{2k}(u)}e^{-i\alpha}]\varepsilon \; ;$$

$$\rho_o'(u)= [F_{2k}(u)e^{i\alpha}- \overline{F_{2k}(u)}e^{-i\alpha}]\varepsilon i \; ;$$

$$F_{2k}(z)=F_2(z)+C_o \; ; \quad F_k(z)=F_k^*(z)-\varepsilon F_{2k}(z) \; ;$$

$$\varepsilon = \min(1;n_1^{-1}) \; ; \quad n_1=k_1/k$$

On solving (5) we obtain

$$F_1(z)=\frac{1}{2\pi}\int_L e^{-i\alpha(t)} \frac{n_1 g'(t)-i\rho'(t)}{t-z}dt \; ; \qquad (6)$$

Let us present the complex potential $F_o(z)$ for the disk in the following way:

$$F_o(z) = F_2(z) + F_1(z) + C_o$$

Let us present $\varphi_1(t)$ in the following way:

$$\varphi_1(t)=-ir(t)e^{-i\alpha(t)}$$

Then the first equation (4) is satisfied automatically, and from the second equation (4) the following for finding r(t) is obtained:

$$i(k-k_o)\,\mathrm{Im}\left[\left(\frac{1}{\pi}\int_L \frac{r(t)e^{-i\alpha(t)}}{t-u}\,dt + F_{1o}(u)\right)e^{i\alpha(u)}\right] - \tag{7}$$

$$-\,(k_o+k)r(t) = 0\,, \qquad (u\in L);$$

Taking into consideration (4) and satisfying (3) following system of equations for finding unknown function F (u) is obtained:

$$\frac{h}{\pi}\,\mathrm{Re}\left[\int_{L_1}\frac{n_1 g'(t)-i\rho'(t)}{t-u}\,dt - (1-\varepsilon)/\pi\cdot\right.$$

$$\cdot\mathrm{Re}\left[\left(\int_L \frac{r(t)e^{-i\alpha(t)}}{t-u}\,dt + C_o\right)e^{i\alpha(u)}\right]=F_k(u)e^{i\alpha}+\overline{F_k(u)}e^{-i\alpha}, \quad u\in L_1$$

$$\frac{h}{\pi}\,\mathrm{Im}\left[\int_{L_1}\frac{n_1 g'(t)-i\rho'(t)}{t-u}\,dt - (1-\varepsilon n_1)/\pi\cdot\right. \tag{8}$$

$$\cdot\mathrm{Im}\left[\left(\int_L \frac{r(t)e^{-i\alpha(t)}}{t-u}\,dt + C_o\right)e^{i\alpha(u)}\right]=-n_1 i[F_k(u)e^{i\alpha}-\overline{F_k(u)}e^{-i\alpha}]$$

So the final system of equations consists of (7), (8) and the conditions of the heat flow being equal to zero and the temperature being unchangeable on its going round the inclusion contour:

$$\int_{L_1} g'(t)dt=0, \quad \int_{L_1} \rho'(t)dt=0;$$

The temperature complex potential can be obtained from $F_k(z)$.

THERMOELASTICITY PROBLEM

Let us introduce complex potentials $\Phi(z)$ and $\Psi(z)$ On any curve L the following equations are observed (Prusov, 1975)

$$N+iT=\Phi(t) + \overline{\Phi(t)} + \frac{\partial\bar t}{\partial t}\left[t\overline{\Phi'(t)} + \overline{\Psi(t)}\right];$$

$$\frac{\partial}{\partial t}(u+iv)=\kappa\Phi(t) - \overline{\Phi(t)} - \frac{\partial\bar t}{\partial t}\left[t\overline{\Phi'(t)} + \overline{\Psi(t)}\right] + H\tilde\Psi(t); \tag{1}$$

where $\tilde\Psi(t)$ is the complex thermal potential.
On the material boundary line the following conditions of ideal mechanical contact are observed.

$$\left(N+iT\right)^{\pm} = \left(N+iT\right)_1^{\pm}\ ; \quad \frac{\partial}{\partial t}\left(u+iv\right)^{\pm}=\frac{\partial}{\partial t}\left(u+iv\right)_1^{\pm}\ ; \tag{2}$$

Let us present the complex potentials $\Phi(t)$ and $\Psi(t)$ in the following way:

$$\Phi(t) = \Phi_1(t)+\Phi_2(t)+\Gamma\ ; \quad \Psi(t)=\Psi_1(t)+\Psi_2(t)+\Gamma'\ ;$$

where $\Gamma = \frac{1}{4}(N_1+N_2)$, $\Gamma'=-\frac{1}{2}(N_1-N_2)e^{-2i\beta}$

$\Phi_1(t)$, $\Psi_1(t)$ and $\Phi_2(t)$, $\Psi_2(t)$ are piecewise holomorphic functions with the breaklines L_1 and L.
Therefore we can present $\Phi_2(t)$ and $\Psi_2(t)$ in the following way:

$$\Phi_2(z)=\frac{1}{2\pi i}\int_L \frac{Q(t)dt}{t-z}\ ;$$

$$\Psi_2(z)=-\frac{1}{2\pi i}\left(\int_L \frac{\overline{Q(t)}d\bar t}{t-z} + \int_L \frac{\bar t Q(t)dt}{(t-z)^2}\right)\ ; \tag{3}$$

Where Q(t) is unknown function (Theocaris and Ioakimidis,1977).Then the first equation (2) is satisfied automatically on the line L The following equation for finding Q(t) is obtained on satisfying the second condition (2) on the L.

$$aQ(u) + \frac{b}{2\pi i}\int_L \frac{\overline{Q(t)}d\bar t}{\bar t-\bar u} - \frac{c}{2\pi i}\int_L \frac{Q(t)dt}{t-u} + \tag{4}$$

$$+\frac{c}{2\pi i}\,\frac{\partial t}{\partial \bar t}\left(\int_L \frac{\overline{Q(t)}dt}{t-u} + \int_L \frac{(\bar t-\bar u)Q(t)dt}{(t-u)^2}\right)=R(u)\ , \quad u\in L$$

where $a=\frac{1}{2}\left[\kappa_o+1 + \frac{\mu_o}{\mu}(\kappa+1)\right]$, $b=\frac{\mu_o\kappa-\mu\kappa_o}{\mu_o}$, $c=1-\frac{\mu_o}{\mu}$,

$$R(u)=\left(-\kappa_o+\frac{\mu_o}{\mu}\kappa\right)\Phi_{1o}(u) +\left(\frac{\mu_o}{\mu}-1\right)\left\{\overline{\Phi_{1o}(u)}+\right.$$

$$\left.+\frac{\partial\bar u}{\partial u}\left(u\overline{\Phi_{1o}(u)} + \overline{\Psi_{1o}(u)}\right)\right\} + \frac{\mu_o}{\mu}H\tilde\Psi(u)-H_o\tilde\Psi_o(u)$$

Suppose complex potentials $\Phi^*_k(u)$; $\Psi^*_k(u)$ and $\tilde\Psi_{ok}(u)$ determine thermostressed state in the thin inclusion, then

$$\left(N+iT\right)^{+} - \left(N+iT\right)^{-}=2ihK_1'(t)$$

$$\frac{\partial}{\partial t}(u+iv)^{+} - \frac{\partial}{\partial t}(u+iv)^{-} = \frac{ih}{\mu_1}\left(M_1'(t) + \Psi'_{ok}(t)\right) \tag{5}$$

$$(N+iT)^{+} + (N+iT)^{-} = 2\left(\Phi(t) + \overline{\Phi_k^*(t)} + e^{-2i\alpha}R_k^*(t)\right) \tag{6}$$

$$\frac{\partial}{\partial t}(u+iv)^{+} + \frac{\partial}{\partial t}(u+iv)^{-} = \frac{1}{\mu_1}\left(\kappa_1\Phi_k^*(t) - \overline{\Phi_k^*(t)} - e^{-2i\alpha}R_k^*(t) + H\tilde{\Psi}_{ok}(t)\right)$$

where

$$R(t)=t\overline{\Phi'(t)}+\overline{\Psi(t)};\quad K_1'(t)=\frac{\partial}{\partial s}\Phi_k^*(t)+\frac{\partial}{\partial s}\Phi_k^*(t)+e^{-2i\alpha}\frac{\partial}{\partial s}R_k^*(t);$$

$$M_1'(t)=\kappa_1\frac{\partial}{\partial s}\Phi_k^*(t)-\frac{\partial}{\partial s}\Phi_k^*(t)-e^{-2i\alpha}\frac{\partial}{\partial s}R_k^*(t);\quad \tilde{\Psi}'_{ok}(t)=\frac{\partial}{\partial s}\tilde{\Psi}_{ok}(t);$$

s is the arch coordinate,

$$\Phi_k(u)=\Phi_k^*(u)-w\Phi_{2k}^*(u);\quad R_k(u)=R_k^*(u)-wR_{2k}^*(u);$$

Taking into consideration (2) from (5) boundary problem is obtained. On solving which we obtain:

$$\Phi_1(z)=h\gamma\int_{L_1}\frac{f_1(t)dt}{t-z};\quad R_1(z)=-h\gamma\left[\int_{L_1}\frac{\overline{f_1(t)}}{(\bar{t}-\bar{z})^2}(t-z)d\bar{t}+\int_{L_1}\frac{f_2(t)d\bar{t}}{\bar{t}-\bar{z}}\right];$$

where

$$f_1(t)=K'(t)+n_1M'(t);\quad f_2(t)=-\kappa K'(t)+n_1M'(t);$$

$$K'(t)=K_1'(t)-K_*'(t);\quad M'(t)=M_1'(t)-M_*'(t);\quad \gamma=1/((1+\kappa)\cdot\pi)$$

$$K_*'(t)=\left[\frac{\partial}{\partial s}\Phi_{2k}^*(t)+\frac{\partial}{\partial s}\Phi_{2k}^*(t)+e^{-2i\alpha}\frac{\partial}{\partial s}R_{2k}^*(t)\right]w;$$

$$M_*'(t)=\left[\kappa_1\frac{\partial}{\partial s}\Phi_{2k}^*(t)-\frac{\partial}{\partial s}\Phi_{2k}^*(t)-e^{-2i\alpha}\frac{\partial}{\partial s}R_{2k}^*(t)\right]w;$$

$$\Phi_{2k}(z)=\Phi_2(z)+\Gamma;\quad \Psi_{2k}(z)=\Psi_2(z)+\Gamma;\quad w=\min(1;n_1^{-1});\quad n_1=\mu/\mu_1$$

Unknown functions $f_1(t)$ and $f_2(t)$ are being found from the following equations which are obtained from (6)

$$\Phi_k(u)+\overline{\Phi_k(u)}+e^{-2i\alpha}R_k(u)-2h\gamma\,\mathrm{Re}\left[\int_{L_1}\frac{f_1(t)dt}{t-u}\right]+e^{-2i\alpha}h\gamma\cdot$$

$$\cdot\left[\int_{L_1}\frac{\overline{f_1(t)}}{(\bar{t}-\bar{u})^2}u\,d\bar{t}-\int_{L_1}\frac{f_2(t)d\bar{t}}{\bar{t}-\bar{u}}\right]=(1-w)\left[\Phi_{2k}(u)+\overline{\Phi_{2k}(u)}+e^{-2i\alpha}R_{2k}(u)\right]$$

$$n_1\left(\kappa_1\Phi_k^*(t)-\overline{\Phi_k^*(t)}-e^{-2i\alpha}R_k^*(t)\right)-\kappa h\gamma\int_{L_1}\frac{f_1(t)dt}{t-u}+h\gamma\int_{L_1}\frac{\overline{f_1(t)}d\bar{t}}{\bar{t}-\bar{u}}- \tag{7}$$

$$-e^{-2i\alpha}h\gamma\left[\int_{L_1}\frac{\overline{f_1(t)}}{(\bar{t}-\bar{u})^2}u\,d\bar{t}-\int_{L_1}\frac{f_2(t)d\bar{t}}{\bar{t}-\bar{u}}\right]=(\kappa_o-\kappa_1n_1w)\Phi_{2k}(u)-$$

$$-\left[\overline{\Phi_{2k}(u)}+e^{-2i\alpha}R_{2k}(u)\right](1-wn_1)-i\gamma_1n_1\quad;$$

where γ_1 is the angle inclusion turn as arigid whole.

Conditions of contour displacement unchangeability, of equality to zero of main vector and main momentumon their going round the inclusi on contour have the form:

$$\int_{L_1}K'(u)du=0\;;\quad \int_{L_1}M'(u)du=0\;;\quad \mathrm{Im}\int_{L_1}\bar{u}\,K'(u)du=0\;; \tag{8}$$

So (4),(7) and (8) form the final system for the determination unknown functions K(u), M(u) and Q(u).

In particular cases, mentioned in literature cases can be obtained.(Grilitskij, Opanasovich and Tysovskij,1982)

THE CASE OF CIRCULAR DISK AND STRAIGHT THIN INCLUSION

In this case the problems of linear conjugation on circular line of material boundary can be solved analitically. And the final system of equations can be rewriten:

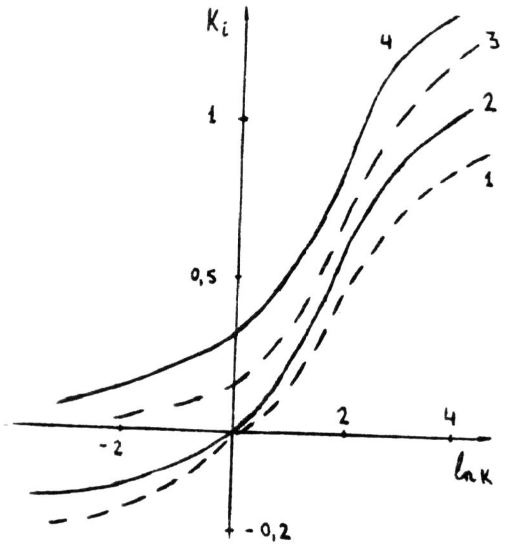
$$\sum_{i=1}^{4}\left[a_{ij}u_i(x)+b_{ij}\int_{-1}^{1}\frac{u_i'(t)dt}{t-x}+\int_{-1}^{1}L_{ij}(t,x)u_i'(t)dt\right]=p_j(x)$$

where a_{ij}, b_{ij}, $L_{ij}(t,x)$, $p_j(x)$ are known constants and functions, $L_{ij}(t,x)$ are regular functions.

This system is solved numerically by the mechanic quadrature method. Values SIF obtained in boundary cases are in good agreement with those known in literature (Grilitskij, Opanasovich and Tysovskij, 1982).

Calculation was made with the following values of input parameteres: $h/l=0.1$, $R/l=2$, $z_o/l=4$, $\kappa_o=\kappa_1=\kappa$ Fig. 2.

Where 1 and 2 marks known in literature case, 3 and 4 marks unknown in literature case when the plate is influenced by force N_1 and temperature $T_o= 5$. 1 and 3 determine the undimensional SIF (which are marked by K_i) at the right tip of the inclusion, 2 and 4 determine the SIF at the left tip of the inclusion. $K=\mu/\mu_1$.



Undimensional SIF K_1 equals SIF / ($N_1\sqrt{l}$)

Fig. 2. Defendence K_i of $K=\mu/\mu_1$

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