CRACK PROPAGATION CRITERIA IN PLATES AND SHELLS UNDER FINITE DEFORMATIONS

L.M. ZUBOV

Mechanical-Mathematical Faculty of Rostov State University, Zorge 5, Rostov-on-Don, 344104, Russia

ABSTRACT

The path-independent energy curvilinear integrals for non-linear elastic plates and cylindrical shells subjected to large bending and extensional deformation are constructed in this paper. The contour integrals obtained are similar to the Eshelby-Cherepanov-Rice integrals and define the resistance force caused by advance of a through crack or other defect. New specialized form of the equations of general nonlinear theory of shells, found by author, is used for constructing this integrals. With the invariant integrals, the energy criteria of through crack propagation in thin-walled structures were formulated.

KEYWORDS

Through crack, strong bend, invariant energy integrals

GENERAL RELATIONS IN NONLINEAR THEORY OF SHELLS

Let the radius-vector $\mathbf{r}(q^1, q^2)$ denote the positions of points of the median surface, σ_0, of a thin-walled shell in the undeformed state, where q^α are the curvilinear coordinates on σ_0. An unit normal to σ_0 is denoted by $\mathbf{n}$, and vectors of principal and reciprocal bases are denoted by $\mathbf{r}_\alpha$ and $\mathbf{r}^\beta$, respectively, where $\mathbf{r}_\alpha = \partial\mathbf{r}/\partial q^\alpha$, $\mathbf{r}_\alpha \cdot \mathbf{r}^\beta = \delta^\beta_\alpha$, $\mathbf{r}^\beta \cdot \mathbf{n} = 0$ $(\alpha, \beta = 1, 2)$. Let $g_{\alpha\beta} = \mathbf{r}_\alpha \cdot \mathbf{r}_\beta$ and $b_{\alpha\beta} = - \mathbf{r}_\alpha \cdot \partial\mathbf{n}/\partial q^\beta$ be the coefficients of first and second quadratic forms of the surface σ_0. The median surface, Σ_0, of a deformed shell has the radius-vector $\mathbf{R}(q^1, q^2)$, the unit normal $\mathbf{N}(q^1, q^2)$, and the base vectors $\mathbf{R}_\alpha(q^1, q^2) = \partial\mathbf{R}/\partial q^\alpha$ and $\mathbf{R}^\beta$, where $\mathbf{R}^\beta \cdot \mathbf{N} = 0$, $\mathbf{R}^\beta \cdot \mathbf{R}_\alpha = \delta^\beta_\alpha$ (δ^β_α is the Kronecker symbol). The coefficients of the first and second quadratic forms of the surface Σ_o will be denoted by $G_{\alpha\beta} = \mathbf{R}_\alpha \cdot \mathbf{R}_\beta$, $B_{\alpha\beta} = -\mathbf{R}_\alpha \cdot \partial\mathbf{N}/\partial q^\beta$.

The equilibrium equations of shell have the following force-moment form (Koiter, 1966; Pietraszkievicz, 1977; Zubov, 1982)

$$\nabla_\alpha (T^{\alpha\beta} - M^{\alpha\delta} B^\beta_\delta) - B^\beta_\delta \nabla_\alpha M^{\alpha\delta} + f^\beta = 0$$

$$\nabla_\alpha \nabla_\beta M^{\alpha\beta} + B_{\alpha\beta}(T^{\alpha\beta} - B^\alpha_\delta M^{\delta\beta}) + f = 0 \qquad (1)$$

$$\mathbf{f}' = f^\alpha \mathbf{R}_\alpha + f\mathbf{N}, \quad B^\alpha_\delta = G^{\alpha\beta} B_{\beta\delta}, \quad G_{\alpha\beta} G^{\beta\gamma} = \delta^\gamma_\alpha$$

Here, $T^{\alpha\beta}$ and $M^{\alpha\beta}$ are the components of symmetric tensors of membrane forces and bending moments; ∇_α is the symbol of covariant derivative in $G_{\alpha\beta}$-metric; $\mathbf{f}'$ is the vector of a load distributed over Σ_o.

For elastic shell, the following relations are valid (Galimov, 1951; Zubov, 1982)

$$\eta (G/g)^{1/2}\, T^{\alpha\beta} = \partial W/\partial \varepsilon_{\alpha\beta}, \quad \eta (G/g)^{1/2}\, M^{\alpha\beta} = -\,\partial W/\partial \text{æ}_{\alpha\beta}$$

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(G_{\alpha\beta} - g_{\alpha\beta}), \quad \text{æ}_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta}, \qquad (2)$$

$$G = |G_{\alpha\beta}|, \quad g = |g_{\alpha\beta}|, \quad \eta = \begin{cases} 1 & \alpha = \beta \\ 2 & \alpha \neq \beta \end{cases},$$

where W is the specific elastic energy of shell (per unit area of σ_o), $\varepsilon_{\alpha\beta}$ and $\text{æ}_{\alpha\beta}$ are the components of tangential and bending strains, respectively.

The force boundary conditions at the shell edge may be written as follows (Zubov, 1982)

$$\sqrt{G/g}\; m_\alpha (T^{\alpha\beta} - 2B^\beta_\delta M^{\alpha\delta}) = e(l^\beta - B^\beta_\alpha L^\alpha) \quad (\beta = 1, 2)$$

$$\sqrt{G/g}\; m_\alpha m_\beta M^{\alpha\beta} = e m_\alpha L^\alpha$$

$$\sqrt{G/g}\; m_\beta \nabla_\alpha M^{\alpha\beta} + \frac{d}{ds}\left[\sqrt{G/g}\; e^{-2} t^\delta m_\beta G_{\alpha\delta} M^{\alpha\beta}\right] = \qquad (3)$$

$$e l + \frac{d}{ds}(e^{-1} t^\delta L_\delta), \quad e = dS/ds$$

$$\mathbf{l}' = l^\beta \mathbf{R}_\beta + l\mathbf{N}, \quad \mathbf{L}' = L^\alpha \mathbf{R}_\alpha = L_\beta \mathbf{R}^\beta$$

In the equations (3) $\mathbf{m} = m_\alpha \mathbf{r}^\alpha$ and $\mathbf{t} = t^\beta \mathbf{r}_\beta$ are the unit vectors of normal and tangent to contour γ_o bounding the shell surface, σ_o, in undeformed state. After deforming, the curve γ_o is transformed into the contour Γ_o bounding the shell surface, Σ_o, in deformed state. The vectors $\mathbf{l}'$ and $\mathbf{L}' \times \mathbf{N}$ are the densities of an external force and an external moment, respectively, distributed along the contour Γ_o; ds and dS are the arc elements of γ_o and Γ_o, respectively.

Let us consider in three-dimensional space a tensor $\Phi(q^1, q^2)$ of an arbitrary rank and introduce for it the following differential operations

$$\text{grad}\, \Phi = \mathbf{r}^\alpha \otimes \partial\Phi/\partial q^\alpha, \quad \text{div}\, \Phi = \mathbf{r}^\alpha \cdot \partial\Phi/\partial q^\alpha \qquad (4)$$

$$\text{Grad}\, \Phi = \mathbf{R}^\alpha \otimes \partial\Phi/\partial q^\alpha$$

Define the unsymmetric tensors of forces and moments for an elastic shell as follows (Zubov, 1982)

$$\mathbf{D} = \sqrt{G/g}\; (T^{\alpha\beta} - M^{\alpha\delta} B^\beta_\delta)\mathbf{r}_\alpha \otimes \mathbf{R}_\beta \qquad (5)$$

$$\mathbf{H} = \sqrt{G/g}\; M^{\alpha\beta} \mathbf{r}_\alpha \otimes \mathbf{R}_\beta$$

Using the equations (2), (5), we can establish the following relations

$$\mathbf{D} = \partial W/\partial \mathbf{F}, \quad \mathbf{H} = \partial W/\partial \mathbf{K} \qquad (6)$$

$$\mathbf{F} = \text{grad}\, \mathbf{R} = \mathbf{r}^\alpha \otimes \mathbf{R}_\alpha, \quad \mathbf{K} = \text{grad}\, \mathbf{N}$$

The equilibrium equations (1) are reduced to the following simple form (Zubov, 1989)

$$\text{div}[\mathbf{D} + (\text{div}\, \mathbf{H}) \cdot (\text{Grad}\, \mathbf{r})\mathbf{N}] + \mathbf{f} = 0, \quad \mathbf{f} = \sqrt{G/g}\; \mathbf{f}' \qquad (7)$$

The tensors **D** and **H** being introduced by Zubov (1982) are analogous to Piola stress tensor in nonlinear theory of elasticity (Lurie, 1980).

The boundary conditions (3) in terms of tensors **D** and **H** are written as follows

$$\mathbf{m}\cdot[\mathbf{D} + (\mathrm{div}\,\mathbf{H})\cdot(\mathrm{Grad}\,\mathbf{r})\,\mathbf{N}] + \partial(H_{ms}e^{-1}\mathbf{N})/\partial s =$$

$$\mathbf{l} + \partial(L_s e^{-1}\mathbf{N})/\partial s, \qquad H_{m\mu} = L_{\mu} \tag{8}$$

$$H_{ms} \equiv \mathbf{m}\cdot\mathbf{H}\cdot\boldsymbol{\tau}, \quad H_{m\mu} \equiv \mathbf{m}\cdot\mathbf{H}\cdot\boldsymbol{\mu}, \quad \mathbf{L} = e\mathbf{L}', \quad \mathbf{l} = e\mathbf{l}'$$

$$\mathbf{L} = L_{\mu}\boldsymbol{\mu} + L_{\mu}\boldsymbol{\tau}$$

Here, $\boldsymbol{\mu}$ is a normal to the boundary Γ_o of the surface Σ_o ($\boldsymbol{\mu}\cdot\mathbf{N} = 0$), $\boldsymbol{\tau}$ is an unit tangential vector to the contour Γ_o, $\mathbf{l}$ and $\mathbf{L}\times\mathbf{N}$ are the intensities (per unit length of the contour γ_o) of force and moment loads distributed over shell edge.

INVARIANT CONTOUR INTEGRALS

Let us suppose that external surface load, $\mathbf{f}$, acting upon unit surface area of σ_o consists of two parts: $\mathbf{f} = \mathbf{f}_o + \rho(q^1,q^2)\mathbf{N}$, where $\mathbf{f}_o$ = const is an uniform dead load; $\rho\mathbf{N}$ is a follower normal load. For nonlinear elastic plate, i.e., when σ_o is a plane, consider the following contour integral

$$\mathbf{I} = \oint_{\gamma} \left[\,\mathbf{m}(W - \mathbf{f}_o\cdot\mathbf{R}) - \mathbf{m}\cdot(\mathbf{D}\cdot\mathbf{F}^T + \mathbf{H}\cdot\mathbf{K}^T)\right] ds \tag{9}$$

The following theorem is valid: the integral **I** is equal to zero for any closed piecewise-smooth contour γ_o bounding the domain σ, in which plate material is homogeneous and tensors **F** and **K** are continuous differential functions of the coordinates q^1, q^2.

Using the divergence theorem, the integral (9) is transformed as follows

$$\mathbf{I} = \int_{\sigma} \left[\mathrm{grad}\,W - \mathbf{f}_o\cdot\mathbf{F}^T - \mathrm{div}(\mathbf{D}\cdot\mathbf{F}^T + \mathbf{H}\cdot\mathbf{K}^T)\right] d\sigma \tag{10}$$

Taking into account the equalities

$$\mathbf{N}\cdot\mathbf{F}^T = 0, \quad \mathbf{K}^T = -\,(\mathrm{Grad}\,\mathbf{r})\cdot(\mathrm{grad}\,\mathbf{F})\cdot\mathbf{N} \tag{11}$$

$$\mathbf{K}\cdot\mathrm{div}\,\mathbf{H} = \mathbf{F}\cdot\mathrm{div}[(\mathrm{div}\,\mathbf{H})\cdot(\mathrm{Grad}\,\mathbf{r})\mathbf{N}]$$

and relation: $\mathrm{grad}\,W = \mathbf{D}^T\cdot\cdot\,\mathrm{grad}\,\mathbf{F}^T + \mathbf{H}^T\cdot\cdot\,\mathrm{grad}\,\mathbf{K}^T$ resulting from (6) and plane homogeneity, we obtain

$$\mathbf{I} = -\int_{\sigma} \left\{\mathrm{div}\left[\mathbf{D} + (\mathrm{div}\,\mathbf{H})\cdot(\mathrm{Grad}\,\mathbf{r})\mathbf{N}\right] + \mathbf{f}_o\right\}\cdot\mathbf{F}^T d\sigma \tag{12}$$

instead of (10). From the equilibrium equations (7), the relation (12) and first equality in (11), it follows that

$$\mathbf{I} = \int_{\sigma} \rho\mathbf{N}\cdot\mathbf{F}^T d\sigma = 0$$

which proves the theorem.

If theorem conditions are failed in some domain $\sigma' \subset \sigma$, the integral **I** along the contour which encloses σ does not vanish in general. In this case, value of the integral is independent of the choice of closed contour γ enclosing σ. The subdomain σ may include nonhomogeneities, inclusions, holes, dislocations, disclinations, singular points of tensor fields **F**, **K**, and other defects. The vector integral **I** is similar to Eshelby integral (Eshelby, 1956) in three-dimensional theory of elasticity.

Another form of I-integral, resulting from (2) and (5), is of some interest

$$\mathbf{I} = \oint_{\gamma} m_{\alpha}\left[(W - \mathbf{f}_o\cdot\mathbf{R})\delta^{\alpha}_{\gamma} - (\partial W/\partial\varepsilon_{\alpha\beta})G_{\beta\gamma} + 2(\partial W/\partial\text{æ}_{\alpha\beta})B_{\beta\gamma}\right]\mathbf{r}^{\gamma} ds \tag{13}$$

Applying the equality

$$\mathbf{m}\cdot\mathbf{H}\cdot\mathbf{K}^T = [(\partial(H_{ms}\, e^{-1}\mathbf{N})/\partial s]\cdot\mathbf{F}^T - H_{m\mu}\mathbf{N}\cdot\partial\mathbf{F}^T/\partial\mu,$$

$$(\partial\Phi/\partial\mu \equiv \mu\cdot\mathrm{Grad}\,\Phi),$$

we obtain another representation of the invariant integral in nonlinear plate theory (Zubov, 1989)

$$I = \oint_\gamma \Big[\mathrm{m}(W - \mathbf{f}_o\cdot\mathbf{R}) - [\mathbf{m}\cdot\mathbf{D} + \partial(H_{ms}e^{-1}\mathbf{N})/\partial s]\cdot\mathbf{F}^T + H_{m\mu}\mathbf{N}\cdot\partial\mathbf{F}^T/\partial\mu\Big]\,ds \qquad (14)$$

Let us suppose that the plate contains an infinitely thin crack with the edges to be parallel to an unit vector $\mathbf{h}$. The crack borders are load-free, i.e., the conditions (8) are fulfilled on those when $\mathbf{l} = \mathbf{L} = 0$. In this case, basing on the theorem, the expression (14) and the equality $\mathbf{N}\cdot\mathbf{F}^T = 0$, it is proved that the integral $J = \mathbf{h}\cdot\mathbf{I}$ have the same value for all contours enclosing one of the crack ends and having the begin and the end at the different borders of a cut. For integral J to be independent on the choice of contour it's sufficient for material to be homogeneous only in crack direction. Moreover, J-integral invariance occurs place not only for plane plate but also for cylindrical shell of arbitrary cross-section if the crack is parallel to the cylinder generator. This is valid also for cutout of finite width with parallel borders. The integral J is similar to Cherepanov-Rice integral (Cherepanov, 1967; Rice, 1968).

Suggesting that $\mathbf{R} = \mathbf{r} + \mathbf{u}$ in (14), where $\mathbf{u}$ is a vector of displacements of the median surface σ_o, and taking into account the condition of equilibrium of an arbitrary part of a shell:

$$\int_{\sigma_o} \mathbf{f}\,d\sigma + \oint_\gamma \Big[\mathbf{m}\cdot[\mathbf{D} + (\mathrm{div}\,\mathbf{H})\cdot(\mathrm{Grad}\,\mathbf{r})\mathbf{N}] + \partial(H_{ms}e^{-1}\,\mathbf{N})/\partial s\Big]ds = 0\ ,$$

resulting from (8) in the absence of a follower load ($\rho = 0$), we obtain the following representation of the energy integral

$$I = \oint_\gamma \Big[\mathbf{m}(W - \mathbf{f}_o\cdot\mathbf{u}) - [\mathbf{m}\cdot\mathbf{D} + \mathbf{m}\cdot(\mathrm{Grad}\,\mathbf{u})^T\cdot(\mathrm{div}\,\mathbf{H})\mathbf{N} + \partial(H_{ms}e^{-1}\mathbf{N})/\partial s]\cdot(\mathrm{grad}\,\mathbf{u})^T + H_{m\mu}\mathbf{N}\cdot\partial(\mathrm{grad}\,\mathbf{u})^T/\partial\mu\Big]\,ds \qquad (15)$$

The expression (15) for the invariant contour integral would be appropriate for use in the case of small deformations, specifically in geometric-linear theory of plates and shells.

CRITERION OF CRACK PROPAGATION

Let the boundary γ_o of a plate or a shell consist of two parts: $\gamma_o = \gamma_1 \cup \gamma_2$. The part γ_1 is fixed or hinged; γ_2 is under dead force load. Then the potential energy of an elastic shell, with $\rho = 0$, is given by

$$\Pi = \int_{\sigma_o}(W - \mathbf{f}_o\cdot\mathbf{u})d\sigma - \int_{\gamma_2}\mathbf{l}\cdot\mathbf{u}ds$$

It may be proved that the energy variation conditioned by crack propagation by a value of $d\lambda$ can be written as

$$d\Pi = -J d\lambda \qquad (16)$$

Since the invariant energy integral, J, in nonlinear theory of shells characterizes, in accordance with (16), the energy release when the crack grows, this integral may be used as a criterion of crack propagation in plates and shells under bending, as in the case of plane stress state (Bróberg, 1971).

REFERENCES

Broberg K.B. (1971) Crack-growth criteria and non-linear fracture mechanics. - J. Mech. and Phys. Solids, v. 19, pp. 407-418.

Cherepanov G.P. (1967) On the propagation of cracks in continuous medium (in Russian).- Prikl. Mat. Mekh., v. 31, No 3, pp. 476-488.

Eshelby J.D. (1956) The continuum theory of lattice defects. - In: Solid State Physics, v.3, pp. 79-144, New-York: Academic Press.

Galimov K.Z. (1951) On the general theory of plates and shells with finite displacement and deformations (in Russian). - Prikl. Mat. Mekh., v.15. pp. 723-742.

Koiter W.T. (1966) On the non-linear theory of thin elastic shells. - Proc. Koninkl. Ned. Ak. Wet. Ser. B, v. 69, pp. 1-54.

Lurie A.I. (1980) Non-linear theory of elasticity (in Russian). Moscow: Nauka.

Pietraszkiewicz W. (1977) Introduction to the non-linear theory of shells. Ruhr-Univ., Inst. f. Mech. Mitt., 10, Boshum.

Rice J.R. (1968) A path independent integral and approximate analysis of strain concentration by notches and cracks. - Trans. ASME., J. Appl. Mech., v.35, pp. 376- 386.

Zubov L.M. (1982) Methods of nonlinear elasticity in shell theory (in Russian). Rostov-on-Don: Rostov Univ. Press.

Zubov L.M. (1989) Energetic contour integrals in nonlinear theory of elastic plates and shells (in Russian). - Dokl. AN SSSR, v.308, 1, pp. 56-58.