CONTINUOUS THEORY OF DEFORMING AND FRACTURE OF DAMAGED MATERIALS

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ABSTRACT

The paper contains a systematic formulation of the theory purposed to the description of deformation and fracture of an anisotropic damaged media regarded as two-phase one. The stress-strain relations are constructed and damage evolution equations which takes into account the possibility of interphase mass transfer are derived. Two criteria of fracture are formulated on the basis of energetic approach. The first of them coincides with Lemaitre's failure criterion in the case of isotropic damage and corresponds to the hypothesis of the total specific deformation energy. The second one corresponds to the hypothesis of the specific energy of the shape changing for virgin (undamaged) materials.

KEYWORDS

Damage, anisotropy, two-phase media, damage evolution, stress-strain relations, fracture criterion.

TWO-PHASE DAMAGED MEDIA AND DAMAGE MEASURES

Let us consider the actual configuration $\mathcal{K}_t$ of the body $\mathcal{V}$, containing spatially distributed microdefects. Let us also adopt that there exists a representative volume of $\mathcal{V}$ - an elementary sphere dV that is quite big to contain great number of internal imperfections, but too small to provide the homogeneity of overall stress-strain state. We will consider as imperfections not only the microdefects embedded in matrix, but also, following Lemaitre(1985), those parts of matrix material, that have lost their load carrying capacity due to concentration of microstress, interaction of neighboring defects and so on. Therefore, the total volume $d\hat{V}^{\square}$ of internal imperfections in sphere dV consists not only of hollows, but also a certain mass $dm^{\square}$ possessing some defect phase density $\rho^{\square}$, averaged through $d\hat{V}^{\square}$ The remainder of the matrix material (carrying phase) contains some mass $dm^{*}=dm-dm^{\square}$ with density ρ^{*}. The total mass dm of the representative volume dV posses-ses density ρ averaged through dV. The carrying phase $d\hat{V}^{*}$ will be considered as homogeneous, isotropic and satisfying to Hooke's law in elastic range. It is easily to derive the relation between phase densities and that of the whole media:

$$\rho = \vartheta\rho^{*}+(1-\vartheta)\rho^{\square} \tag{1}$$

where $\vartheta=d\hat{V}^{*}/dV$ may be regarded here as volume damage measure. In initial configuration $\mathcal{K}_{o}$ of body $\mathcal{V}$ his representative volume dV_{o} contains both the carrying phase dV^{*}_{o} with mass dm^{*}_{o}, and the defect phase $dV^{\square}_{o}$ with mass $dm^{\square}_{o}$. It should be noticed that although mass conservation law is valid for whole damaged media, it obviously not take place for each phase separately. During deformation process some additional parts of the matrix material may lose their carrying capacity due to development of stress concentration zones and so on. So in the actual configuration the phase masses are $dm^{*}=dm^{*}_{o}-dm_{t}$ and $dm^{\square}=dm^{\square}_{o}+dm_{t}$, where dm_{t} is the part of mass being transformed from carrying state to defect one. For simplicity we shall call this transformation as "mass transfer". Similarly, let us adopt for current phase volumes $d\hat{V}^{*}=dV^{*}-dV^{*}_{t}$ and $d\hat{V}^{\square}=dV^{\square}+dV^{\square}_{t}$. The first items at right-hand parts of those equalities, that may be called, analogically to Nigmatoulin(1978) as "external" current phase volumes, are satisfying mass conservation law for each separate phase, i.e. $dm^{*}_{o}=\rho^{*}dV^{*}$, $dm^{\square}_{o}=\rho^{\square}dV^{\square}$. Accordingly, the mass transfer value can be found with help of the second items: $dm_{t}=\rho^{*}dV^{*}_{t}=\rho^{\square}dV^{\square}_{t}$. The total volume $dV=d\hat{V}^{*}+d\hat{V}^{\square}$ can be presented now as

$$dV = \frac{dm^{*}}{dm^{*}_{o}}\,dV^{*}+\frac{dm^{\square}}{dm^{\square}_{o}}\,dV^{\square} \tag{2}$$

Planar Damage Measures and Damage Tensor. Let $dS^{\square}_{n}$ signify cross-sectional area of defects in some central section dS_{n} of the representative volume dV, that defines by the normal $\mathbf{n}$. According to Lemaitre(1985), this area includes not only cross-section of microholes and microcracks, but also depends upon stress concentration and interaction between nearest defects. The measure of damage Ω_{n} in this section may be defined as corrected by abovementioned factors effective load carrying area, $dS^{*}_{n}=dS_{n}-dS^{\square}_{n}$, related to central cross-sectional area dS_{n}, i.e. $\Omega_{n}=dS^{*}_{n}/dS_{n}$. If spatial distribution of internal imperfections is orthotropic one, there exist the sections determined by normals $\mathbf{n}_{p}$, p=1,2,3, where damage measure Ω_{n} reaches it's main values Ω_{p}. Those three quantities we shall call as planar damage measures. The symmetric second-rank damage tensor ω may be defined by usual way

$$\omega = \sum_{q}\Omega_{q}\mathbf{n}_{q}\mathbf{n}_{q} \tag{3}$$

We will postulate the relation between volume ϑ and planar Ω_{p} measures in the form

$$\vartheta = \sqrt{\Omega_{1}\Omega_{2}\Omega_{3}} \tag{4}$$

From a geometrical point of view, ϑ is the relative volume of some ellipsoid possesses main sectional areas $\Omega_{q}dS$, q=1,2,3, in the representative volume dV

Two Kinds of Phase Deformation Rates. As Nigmatoulin(1978) pointed out, deformation in each phase is caused not only by the displacements at outer bounds, that are determined by both external field of rates and phase material properties, but also by displacement of interphase surfaces inside the representative volume being explained as a result of mass transfer. Following the mentioned work, we will introduce for each m-th phase both external deformation rate tensor $\dot{\varepsilon}^{(m)}$ and real deformation rate tensor $\hat{\dot{\varepsilon}}^{(m)}=\dot{\varepsilon}^{(m)}+\dot{\varepsilon}^{(m)}_{t}$, where $\dot{\varepsilon}^{(m)}_{t}$ takes into account displacement of the substance of m-th phase at the surface of phases separation. Therefore, together with representative volume change rate, $d\dot{V}=dV\,tr(\dot{\varepsilon})$, where ε is overall deformation tensor, we can similarly present for separate phase the rate of change of the "external" phase volume as $d\dot{V}^{*}=dV^{*}tr(\dot{\varepsilon}^{*})$ or $d\dot{V}^{\square}=dV^{\square}tr(\dot{\varepsilon}^{\square})$ by using external deformation rate tensor. Now by time differentiating of (2) and by using the relations between volumes, masses and densities, we found

$$tr(\dot{\varepsilon}) = \vartheta\,tr(\hat{\dot{\varepsilon}}^{*})+(1-\vartheta)\,tr(\hat{\dot{\varepsilon}}^{\square}), \tag{5}$$

where

$$tr(\hat{\dot{\varepsilon}}^{*})=tr(\dot{\varepsilon}^{*})-d\dot{m}_{t}/dm^{*},\quad tr(\hat{\dot{\varepsilon}}^{\square})=tr(\dot{\varepsilon}^{\square})+d\dot{m}_{t}/dm^{\square} \tag{6}$$

It is clear that we can write $\hat{\dot{\varepsilon}}^{*}=\dot{\varepsilon}^{*}+\dot{\varepsilon}^{*}_{t}$, for carrying phase and similarly for defect one. Here $\dot{\varepsilon}^{*}_{t}$ arises due to mass transfer and should be represented by an appropriate way. Let us assume the rate of mass transfer $d\dot{m}_\iota$ to be proportional both to the mass of carrying phase dm^* and the difference between effective and nominal mean stresses, i.e. $d\dot{m}_\iota=3(\sigma^*_m-\sigma_m)dm^*/\eta$, where η is a constant value with dimension of viscosity, and stress tensors σ^* and σ, so as their mean values, will be introduced later. Then following representation

$$\dot{\varepsilon}^*_\iota = \dot{\hat{\varepsilon}}^* - \dot{\varepsilon}^* = -(\sigma^*-\sigma)/\eta \tag{7}$$

will satisfy both the first from eqs. (6) and prepositions made above. It should be noticed that in absence of the damage ($\sigma^*=\sigma$) mass transfer is vanish and $\dot{\varepsilon}^*_\iota=0$.

DAMAGE EVOLUTION

The relations between phase densities and "external" phase volumes were formulated above; as it follow from them, equalities $\dot{\rho}^*=-\rho^* tr(\dot{\varepsilon}^*)$, $\dot{\rho}^\square=-\rho^\square tr(\dot{\varepsilon}^\square)$ are valid for each phase separately. Differentiating eq. (1) by time and adding eqs. (5) and (6) to result, we can easy derive evolution law for volume damage measure:

$$\dot{\vartheta} = -\vartheta tr(\dot{\varepsilon}-\dot{\hat{\varepsilon}}^*) = (1-\vartheta) tr(\dot{\varepsilon}-\dot{\hat{\varepsilon}}^\square). \tag{8}$$

which rises from mass conservation law for whole damaged media and bears in mind interphase mass transfer. From other hand, differentiating eq. (4) and equating the result to (8) gives

$$\sum_s \frac{\dot{\Omega}_s}{\Omega_s} = -2 tr(\dot{\varepsilon}-\dot{\hat{\varepsilon}}^*) - 2\frac{(1-\vartheta)}{\vartheta} tr(\dot{\varepsilon}-\dot{\hat{\varepsilon}}^\square). \tag{9}$$

Developing the approach proposed earlier by the authors (Fedenko and Yanko, 1987) for porous media, we shall use the first from eqs. (8) and (9) henceforth. Owing to the form of their right-hand sides it may be presumed that damage tensor evolution law takes the form

$$\dot{\omega} = \mathcal{P}:(\dot{\varepsilon}-\dot{\hat{\varepsilon}}^*) \tag{10}$$

where $\mathcal{P}$ is fourth-rank tensor depending on ω. The most common representation of such tensor is described in many works and, particularly, by Cowin(1985):

$$\mathcal{P} = a_1 II + a_2 \mathcal{J} + b_1 I\omega + b_2 \omega I + 4b_3 \mathcal{W} + c_1 \omega\omega + c_2 I(\omega\cdot\omega) +$$
$$+ c_3(\omega\cdot\omega)I + 4c_4 \hat{\mathcal{W}} + d_1 \omega(\omega\cdot\omega) + d_2 (\omega\cdot\omega)\omega + e(\omega\cdot\omega)(\omega\cdot\omega). \tag{11}$$

Here $a_1, a_2, \ldots, d_2, e$ are some coefficients that may depend of main invariants of ω; $I, \mathcal{J}$ are second-rank and fourth-rank unit tensors, accordingly;

$$\mathcal{W}_{ijkl}=(\omega_{ik}\delta_{jl}+\omega_{jk}\delta_{il}+\omega_{il}\delta_{jk}+\omega_{jl}\delta_{ik})/4,$$

and $\hat{\mathcal{W}}$ has similar form, if ω_{mn} are replaced by $\omega_{ms}\omega_{sn}$.

Because $\dot{\Omega}_p=\dot{\omega}:(n_p n_p)$, we found

$$\dot{\Omega}_p = [(a_1+b_2\Omega_p+c_3\Omega_p^2)I+(a_2+4b_3\Omega_p+4c_4\Omega_p^2)(n_p n_p)+ \tag{12}$$
$$+(b_1+c_1\Omega_p+d_2\Omega_p^2)\omega+(c_2+d_1\Omega_p+e\Omega_p^2)(\omega\cdot\omega)]:(\dot{\varepsilon}-\dot{\hat{\varepsilon}}^*).$$

Substitution (12) into (9) gives following four restrictions for coefficients

$$3a_1 T+3b_2+4b_3+3c_3 S=-2; \qquad a_2=0; \tag{13}$$
$$3b_1 T+3c_1+4c_4+3d_2 S=0, \quad c_2 T+d_1+eS=0.$$

Here $S=1/3\sum\Omega_q=1/3\, tr(\omega)$, $T=1/3\sum\Omega_q^{-1}=1/3\, tr(\omega^{-1})$. Additional considerations are necessary to determine all coefficients introduced.

STRESS-STRAIN RELATIONS

Let us assume there exists Helmholtz free energy $\rho\Psi$, what is a joint scalar invariant of damage (ω) and strain (ε) tensors. It is postulated to be a homogeneous function of degree two in deformation ε:

$$\rho\Psi = 1/2\,\varepsilon:\mathcal{K}^\bullet:\varepsilon,$$

where for fourth-rank elasticity tensor $\mathcal{K}^\bullet$ we can use the most common representation like (11) once again. In the frames of the thermodynamics of irreversible processes by usual procedure we have

$$\sigma = \rho\frac{\partial\Psi}{\partial\varepsilon} = \mathcal{K}^\bullet:\varepsilon, \tag{14}$$

where σ is overall (nominal) stress tensor in damaged media. Such authors as Lemaitre(1985), Chaboche(1988), Wang and Chow (1989) and many others are adopted the effective moduli for damaged media in form $E_p=[1-(1-\Omega_p)^m]^n E^*$, where m,n are constant values, and E^* is Young modulus for isotropic undamaged material. Particularly, some authors assume $m=n=1$ so that $E_p=\Omega_p E^*$ In this case it is appropriate to represent the effective shear modulus of damaged media as $\mu_{pq}=(\Omega_p+\Omega_q)\mu^*/2$, where μ^* is the carrying phase shear modulus. It can be proved in frames of suppositions used above that stiffness tensor $\mathcal{K}^e$ in (14) should be in form

$$\mathcal{K}^e = a\mathbf{I}\mathbf{I}+2\mu^*\mathcal{W}+b(\mathbf{I}\omega+\omega\mathbf{I})+c\omega\omega, \qquad (15)$$

and functions a, b and c are of follow kind:

$$a = 3\varkappa\nu^* S, \quad b = \varkappa(2-\nu^*), \quad c = 3\varkappa\nu^* T;$$

$$\varkappa = 2\mu^*\nu^*[(2-\nu^*)^2-9\nu^{*2}ST]^{-1},$$

where ν^* is Poisson's ratio of virgin material. If stress tensor and damage tensor are coaxial (this assumption is not essential for further formulation), the total specific energy of deformation takes the form

$$\rho u = \left[(1+\nu^*)\Sigma\, \Omega_p^{-1}\sigma_{pp}^2-3\nu^*\sigma_m\Sigma\, \Omega_p^{-1}\sigma_{pp}\right]/(2E^*), \qquad (16)$$

and $\sigma_m=1/3\,tr(\sigma)$ is mean overall stress. Differentiating the first from eq.(14) by the time we find

$$\dot{\sigma} = \mathcal{K}^e:\dot{\varepsilon}+\mathcal{M}\dot{\omega}-\sigma tr(\dot{\varepsilon}), \quad \mathcal{M} = \rho\partial^2\Psi/(\partial\varepsilon\partial\omega). \qquad (17)$$

The system of equations (7),(8),(10),(17) will have two additional expressions to be full. The first of them is Hooke's law for virgin material,

$$\dot{\sigma}^* = \mathcal{L}^e:\dot{\varepsilon}^*-\sigma^* tr(\dot{\varepsilon}^*), \qquad (18)$$

where σ^* is effective stress tensor applied to carrying phase material, and $\mathcal{L}^e=\lambda^*\mathbf{I}\mathbf{I}+2\mu^*\mathcal{J}$ is virgin elastic tensor. Moreover, the relation between effective (σ^*) and nominal (σ) stresses and their rates may be written here similarly to Murakami and Sanomura(1985) result

$$\sigma = \mathcal{W}:\sigma^*, \qquad \dot{\sigma} = \mathcal{W}:\dot{\sigma}^*+\mathcal{T}^*:\dot{\sigma}. \qquad (19)$$

The expression for fourth-rank tensor $\mathcal{W}$ was written above, and the form of tensor $\mathcal{T}^*$ coincides with $\mathcal{W}$, after ω_{mn} are replaced

by $\dot{\sigma}^*_{mn}$. It is worth to notice that just external deformation rate tensor $\dot{\varepsilon}^*$ have been used in equation (18) because mass conservation law is valid for traditional elasticity.

FRACTURE CRITERION FOR ANISOTROPIC MEDIA

Now it possible to formulate some criteria of brittle fracture for an anisotropic damaged solid. To this purpose we shall use an energetic approach which was proposed earlier by Lemaitre(1985) for isotropic damaged media. Let us assume that kinetic equation (12) for planar damage measures has only non-zero coefficient $c_3=-2/(3S)$, which value may be obtained from (13). By means of (8) we found $\dot{\Omega}_p=2\Omega_p^2\dot{\vartheta}/(3S\vartheta)=-\Omega_p^2\dot{\Phi}/[S(1-\Phi)]$, where new scalar variable $\Phi=1-\vartheta^{2/3}$ is introduced. In the case of isotropy $(\Omega=\Omega_p=1-\Phi)$ this variable is similar to the isotropic planar damage measure used by Lemaitre. Determining from (16) the rate of total deformation energy $\rho\dot{u}$ and substituting the last expression for $\dot{\Omega}_p$ to result we have

$$\rho\dot{u} = \Sigma\frac{\partial(\rho u)}{\partial\sigma_{qq}}\dot{\sigma}_{qq}-y\dot{\Phi}, \qquad -y = \frac{\sigma_D^2}{2E^*S(1-\Phi)},$$

where

$$\sigma_D = \sigma_e\left[\frac{2}{3}(1+\nu^*)+3(1-2\nu^*)\left(\frac{\sigma_m}{\sigma_e}\right)^2\right]^{1/2}. \qquad (20)$$

and σ_e is the Von Mises equivalent nominal stress. The variable σ_D was introduced by Lemaitre(1985), who called it as "equivalent stress of damaged media". If spatial damage distribution is isotropic $(S=1-\Phi=\Omega)$, the variable $(-y)$ will coincide with Lemaitre's result.

As Lemaitre suggested, the damage accumulation process causes arising of macrocracks under the critical value y_c of variable $(-y)$, which is the characteristic parameter of material given; it may be received from the test on simple loading. The criterion being formulated is similar to hypothesis of the total specific deformation energy for undamaged materials and coincides with it when damage vanishes

However, for undamaged and low damaged solids the hypothesis of the specific shape changing energy is widely used to describe fracture. At the same time the hydrostatic part of the stress tensor has essential influence on failure of a strong deteriorated materials. With purpose to join this two approaches let us subtract from total specific energy ρu those part ρu^{*v}, that expends on pure hydrostatic deformation of carrying phase only (but not of whole damaged media). Let us assume $\sigma^*_{pp}=\sigma^*_m$ ($p=1,2,3$), where σ^*_m is mean effective stress, so that it follow from (19) $\sigma_{pp}=\Omega_p\sigma^*_m$, and from (16) we have

$$\rho u^{*v} = 3(1-2\nu^*)S\sigma^{*2}_m/(2E^*).$$

Now using the value $\rho\hat{u}=\rho(u-u^{*v})$ instead of ρu, we can obtain by an similar way

$$\rho\dot{\hat{u}} = \sum \frac{\partial(\rho\hat{u})}{\partial\sigma_{pp}}\dot{\sigma}_{pp} - z\dot{\Phi}, \qquad -z = \frac{\sigma^2_F}{2E^*S(1-\Phi)},$$

where

$$\sigma_F=\sigma_e\Big\{\frac{2}{3}(1+\nu^*)+ +3(1-2\nu^*)\Big[\Big(\frac{\sigma_m}{\sigma_e}\Big)^2+\Big(Q-2S^2\Big)\Big(\frac{\sigma^*_m}{\sigma_e}\Big)^2\Big]\Big\}^{1/2}, \qquad (21)$$

and $Q=1/3\,tr(\omega\cdot\omega)$. The fracture criterion may be formulated again by using of the concept of critical value z_c for variable $(-z)$. As it evident, the last item in (21) vanishes when damage is absent, so that this criterion coincides in this case with appropriate one for virgin solids.

Fracture criteria proposed here that's include by means of variables S, Q and Φ all three main invariants of the damage tensor ω, are simplest from possible those in the frames of the theory presented here. More sophisticated approaches may be realized by keeping more non-zero coefficients in kinetic equation (12) for planar damage measures.

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