CALCULATION OF ENERGY INTEGRAL FOR BODIES WITH NOTCHES AND CRACKS

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ABSTRACT

In this paper the correlation between the energy J-integral and stress intensity factor is considered on the base of stress-strain concentration coefficient in elastic-plastic range for body with notch and crack encounted from notch.

KEYWORDS

Non-linar fracture mechanics, J-integral, consentration of stress and strain, stress intensity factor.

INTRODUCTION

The application of numerical methods in the fracture mechanics gives us a possibility of elastic or elastic-plastic fields in the vicinity of any stress concentrator. At the same time the number of analytical elastic-plastic solution for the estimation of the stress concentration effect for bodies with different geometry is limited. That is why the evaluation of approximate methods for the analysis of elastic-plastic bodies with the stress concentrators is an important engineering problem. In this publication we use the energy J-integral for the derivation of approximate formulas for the analysis of bodies with notches and cracks. After this the possible ways of the application of the obtained formulas to the analysis of bodies with different geometry (including a crack outgoing from the stress concentrator) and to the analysis of the critical state of a body with a short crack and the stress and strain fields near the crack tip are considered.

CALCULATION OF J-INTEGRAL FOR THE BODY WITH NOTCH

Let us consider a body with a notch under external loading. The notch tip is the semi-circle of the radius ρ (fig. 1). To estimate the stress and strain concentration we will use the Cherepanov-Rice energy integral. The integration path is the contour of the notch tip which is free of stresses. Then the J-integral can be represented by the following simple expression:

$$J=\int_{-\pi/2}^{\pi/2} W(\theta)\rho\cos\theta\, d\theta \qquad (1)$$

Here $W(\theta)$ is the specific work of deformation, θ is the angular coordinate of point at the notch contour in the polar coordinate system (the pole is the center of the notch tip circle). Using the following form for the specific work of strain near the contour of the notch tip in the expression (1)

$$W = W_{max}\cdot\cos\theta \qquad (2)$$

we obtain such expression for the J-integral (Hajinski,1983)

$$J=(\pi/2)\, W_{max}\,\rho \qquad (3)$$

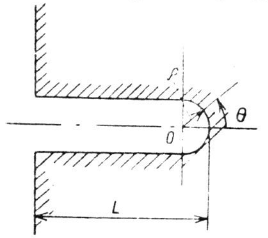

Fig.1. The notch geometry

From equation (3) it is possible to obtain the precise values of the stress intensity factor for the typical cases of tensile and bending loading.
Let us substitute the specific work of strain W_{max} for θ = 0 in the relation (3) by the specific work of distortion strain $\int \sigma_i\, d\varepsilon_i$ for strain-hardening materials. If $\varepsilon_{imax}\geq\varepsilon_0$ and the stress state is uniaxial then W_{max} is equal to

$$W_{max}=\int_0^{\varepsilon_0}\sigma_i\, d\varepsilon_i+\int_{\varepsilon_0}^{\varepsilon_{imax}}\sigma_i\, d\varepsilon_i=\frac{\sigma_0^2}{2E}+\frac{\sigma_*}{1+m}\left(\varepsilon_{imax}^{1+m}-\varepsilon_0^{1+m}\right). \qquad (4)$$

The deformation diagram is approximated by power function

$$\sigma_i=E\varepsilon_i,\ \sigma_i<\sigma_0;\qquad \sigma_i=\sigma_*\varepsilon_i^{m},\ \sigma_i\geq\sigma_0 \qquad (5)$$

where $\sigma_*=(\alpha\sigma_0)/\varepsilon_0^{m}$, m, α are the empirical values, $\varepsilon_0=\sigma_0/E$, E is the modulus of elasticity, σ_0 is the yield stress, the index "i" denotes the stress and strain intensity. Using the expression (4) we can obtain from the formula (3)

$$J=\frac{\pi}{2}\rho\left[\frac{\sigma_0^2}{2E}+\frac{\sigma_*}{1+m}\left(\varepsilon_{imax}^{1+m}-\varepsilon_0^{1+m}\right)\right]. \qquad (6)$$

This expression allows us to find the maximal stress and strain intensity near the notch tip using the radius of curvature, the mecanical properties and energy J-integral determined by means of experiment or calculation.
Let us express the right-hand side of the formula (6) in terms of stress and strain concentration factors. According to the definition these factors near the notch tip for θ = 0 are equal to:

$$K_\varepsilon=\varepsilon_{imax}/\varepsilon_i,\qquad K_\sigma=\sigma_{imax}/\sigma_i \qquad (7)$$

where and are the applied stress and strain intensity. Taking into account the relation (5) and (7), the expression (6) can be written in the following form:

$$J=\frac{\pi}{2}\rho\begin{cases}\dfrac{\sigma_0^2}{2E}+\dfrac{\sigma_*}{1+m}\left[\dfrac{\sigma_i^2}{\sigma_* E}K_\varepsilon K_\sigma-\left(\dfrac{\sigma_0}{E}\right)^{1+m}\right], & \dfrac{1}{\alpha_\sigma}\leq\dfrac{\sigma_i}{\sigma_0}<1,\\[2ex] \dfrac{\sigma_0^2}{2E}+\dfrac{\sigma_*}{1+m}\left[\left(\dfrac{\sigma_i}{\sigma_*}\right)^{\frac{1+m}{m}}K_\varepsilon K_\sigma-\left(\dfrac{\sigma_0}{E}\right)^{1+m}\right], & \dfrac{\sigma_i}{\sigma_0}\geq 1.\end{cases} \qquad (8)$$

Let us eliminate the term $K_\varepsilon K_\sigma$ from the relation (8) with the help of the Neuber-Maghutov (Maghutov, 1981) relation for the elastic-plastic deforming of material in the region of concentration:

$$\frac{K_\varepsilon K_\sigma}{\alpha_\sigma^2}=F, \qquad (9)$$

where $F=F(\alpha_\sigma,\sigma_i/\sigma_0,m)=(\alpha_\sigma\sigma_i/\sigma_0)^{-n(1-m)[1-(\bar\sigma_i-(\frac{1}{\alpha_\sigma}))]}$

Here the following notation is used: $\bar\sigma_i=\sigma_i/\sigma_0$, α_σ is the theoretical stress concentration factor, n is the function of α_σ and $\bar\sigma_i$ (it is usually considered as a constant equal to 0.5).
At the early stage of elastic-plastic deformation near the notch tip for $1/\alpha_\sigma\leq\bar\sigma_i<1$

$$J=\frac{\pi}{2}\rho\left\{\frac{\sigma_0^2}{2E}+\frac{\sigma_*}{1+m}\left[\frac{\sigma_i^2}{\sigma_* E}\alpha_\sigma^2 F-\left(\frac{\sigma_0}{E}\right)^{1+m}\right]\right\}. \qquad (10)$$

At the second stage of deforming $\bar\sigma_i\geq 1$ and the relation (8) can be written in the following form:

$$J=\frac{\pi}{2}\rho\left\{\frac{\sigma_0^2}{2E}+\frac{\sigma_*}{1+m}\left[\left(\frac{\sigma_i}{\sigma_*}\right)^{\frac{1+m}{m}}\alpha_\sigma^2 F-\left(\frac{\sigma_0}{E}\right)^{1+m}\right]\right\}. \qquad (11)$$

Let us consider the variation of the J-integral due to the variation of its parameters. It is convenient to use the relative value J/J_e, where $J_e=\pi\rho\alpha_\sigma^2\sigma_i^2/4E$ is the elastic value of the J-intagral ($\bar\sigma_i<1/\alpha_\sigma$). The empirical value α is taken equal to 1. Then for $1/\alpha_\sigma\leq\bar\sigma_i<1$

$$J/J_e=(\alpha_\sigma\bar\sigma_i)^{-2}\left(1-\frac{2}{1+m}\right)+\frac{2F}{1+m}, \qquad (12)$$

and for $\bar\sigma_i\geq 1$

$$J/J_e=(\alpha\bar\sigma_i)^{-2}\left(1-\frac{2}{1+m}\right)+\frac{2F}{(1+m)}\bar\sigma_i^{\frac{1-m}{m}} \qquad (13)$$

The behavior of J/J_e for increasing $\bar\sigma_i$, m and α_σ (n = 0.5) is shown in fig. 2. The value of J is increasing with increase of $\bar\sigma_i$ and the increase of material ability for plastic deforming (i. e. with the decrease of m). The value of J practically does not depend on α_σ. The value of J is equal to the elastic value J_e for $\bar\sigma_i\leq 1/\alpha_\sigma$. In the stress range 1/ < < 1 the J-integral is practically constant and does not deviate far from J_e. The value of J begins to increase when the applied stress becomes more than the yield stress.

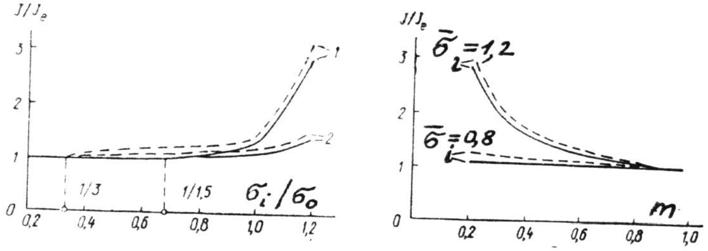

Fig.2. The energy integral for notch body.
1 - m = 0.2, 2 - m = 0.5;
------- $-\alpha_\sigma = 1.5$, - - - - $-\alpha_\sigma = 3$; $J_e = K^2/E$.

Now consider the calculation of the J-integral for the case of the body with the notch and the crack outgoing from it (fig.3). We assume that the crack is short, i. e. the crack length is less than the radius of curvature ρ of the notch. Therefore, such crack is placed in the "shadow" of the local stress peak due to the concentration near the notch. We consider this crack as an intermediate crack between the "infinitely long" crack (in this case the concentration factor is equal to $\alpha_\sigma = \sqrt{1+(L/\ell)}$, where L is the notch length) and the "infinitely short" crack (in this case the concentration factor is equal to α_σ).

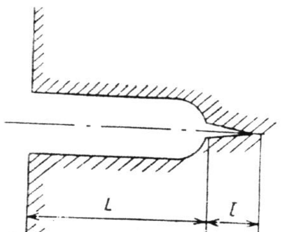

Fig.3. The notch geometry with short crack.

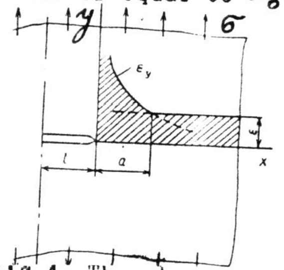

Fig.4. The strain distribution near crack tip in plane.

Let us use a rather effective interpolation method of Neuber. According to this we can use the same formulas (10) and (11), substituting by the following stress concentration factor:

$$\alpha_{\sigma *} = 1 + \left\{ \frac{(\alpha_\sigma - 1)^2 (\alpha_\sigma^\infty - 1)^2}{(\alpha_\sigma - 1)^2 + (\alpha_\sigma^\infty - 1)^2} \right\}^{1/2}$$

CALCULATION OF J-INTEGRAL FOR BODY WITH CRACK

Let us consider the body with a crack. The crack is interpreted here as a thin notch. In the limit case $\rho \to 0$ the notch transforms into the crack and the relations (10) and (11) give us the expression of the J-integral the body

with the crack-type notch

$$J = \begin{cases} \lim\limits_{\rho \to 0} \dfrac{\pi \rho \alpha_\sigma^2 F}{2(1+m)} \dfrac{\sigma_i^2}{E}, & \dfrac{1}{\alpha_\sigma} < \bar{\sigma}_i < 1 \\ \lim\limits_{\rho \to 0} \dfrac{\pi \rho \alpha_\sigma^2 F \sigma_*}{2(1+m)} \left(\dfrac{\sigma_i}{\sigma_*} \right)^{\frac{1+m}{m}}, & \bar{\sigma}_i \geq 1 . \end{cases} \tag{15}$$

The passage to the limit $\rho \to 0, \alpha_\sigma \to \infty$ of two expressions $\rho \alpha_\sigma^2$ (when F = 1, $\bar{\sigma}_i < 1/\alpha_\sigma$) and $\rho \alpha_\sigma^2 F$ is to give the finite value. That is why the limit of F for $\alpha_\sigma \to \infty$ (or $\sigma_{i\,max} \to \infty$) is bounded and an order $(\sigma_{i\,max}/\sigma_0)^{æ}$, $æ = n(1-m)(\bar{\sigma}_i - 1)$ Since the value of n is a undetermined function of α_σ and $\bar{\sigma}_i$, it is probable that when α_σ is significant the power is small (æ << 1) and therefore the limit value of F is equal to 1. It is also worth to note that $K = \lim\limits_{\rho \to 0} \frac{1}{2}\sqrt{\pi \rho}\, \alpha_\sigma \sigma$ In the case of uniaxial stress state this leads to the well-known expression for J in terms of the stress intensity factor K in the elastic region ($J = G = K^2/E$, $\bar{\sigma}_i < 1/\alpha_\sigma$). For large enough stress:

$$J = \begin{cases} \dfrac{2}{1+m} \dfrac{K^2}{E}, & 1/\alpha_\sigma < \bar{\sigma}_i < 1 \\ \dfrac{2}{1+m} \dfrac{K^2}{\sigma_*} \left(\dfrac{\sigma_i}{\sigma_*} \right)^{\frac{1+m}{m}}, & \bar{\sigma}_i \geq 1 \end{cases} \tag{16}$$

Let us pay attention to the fact that the range of the applied stress starts from 0 ($1/\alpha_\sigma \to 0$ when $\alpha_\sigma \to \infty$). Therefore, there is no elastic state near the crack tip at any $\bar{\sigma}_i$ (it is natural for real material). That is why the usual formula introduced for the elastic body $J = J_e = K^2/E$ is some idealization.

Now we will use another method for calculating the J-integral. The approximate formula for the J-integral in the case of plane stretched by the stress with the single crack of length 2l (fig.4) can be obtained using the method of sections (Matvienko, Morozov, 1984). Let us write the balance of forces. Here we assume that the force that is not transmitted by the crack is counterbalanced by additional force of the stress concentration near the crack tip (the origin of coordinates is at the crack tip):

$$\int_0^a \sigma_y \, dx = \sigma \ell \tag{17}$$

The stress σ_y and the strain ε_y on the line of crack extension before its tip for strain-hardening material are taken in accordance with the HRR-model:

$$\sigma_y = \sigma_* \left(\frac{J}{\sigma_* I_m x} \right)^{\frac{m}{1+m}} \tilde{\sigma}_y(0, m),$$
$$\varepsilon_y = \left(\frac{J}{\sigma_* I_m x} \right)^{\frac{1}{1+m}} \tilde{\varepsilon}_y(0, m), \tag{18}$$

where I_m is the undimensional value that depends on the stress state and the hardening power: m. $\tilde{\sigma}_y(0,m)$ and $\tilde{\varepsilon}_y(0,m)$ are the constant values close to 1. The length a is determined by the condition of equality of the strain ε_y (on the line of crack prolongation before the crack tip) and the applied strain ε when $x = a$, i.e.

$$
\begin{aligned}
\sigma_y &= \sigma_*(\sigma/E)^m, & \sigma &< \sigma_0 \\
\sigma_y &= \sigma, & \sigma &\geq \sigma_0 .
\end{aligned}
\tag{19}
$$

These conditions allow us to write:

$$
a = \begin{cases} \dfrac{J}{\sigma_* I_m}\left(\dfrac{E}{\sigma}\right)^{1+m}\left(\tilde{\sigma}_y(0,m)\right)^{\frac{1+m}{m}}, & \sigma < \sigma_0 \\[2ex] \dfrac{J}{\sigma_* I_m}\left(\dfrac{\sigma_* \tilde{\sigma}_y(0,m)}{\sigma}\right)^{\frac{1+m}{m}}, & \sigma \geq \sigma_0 \end{cases}
\tag{20}
$$

Using in the equation of balance (17) the stress σ_y in the form (18) and the value a according to (20), we have the approximate formulas for the J-integral for the case of extension of the plane with the crack:

$$
J = \begin{cases} \dfrac{I_m}{(1+m)(\tilde{\sigma}_y(0,m))^{\frac{1+m}{m}}}\,\dfrac{\sigma^2 \ell}{E}, & \sigma < \sigma_0, \\[2ex] \dfrac{\sigma_* I_m}{1+m}\left(\dfrac{\sigma}{\sigma_* \tilde{\sigma}_y(0,m)}\right)^{\frac{1+m}{m}} \ell, & \sigma \geq \sigma_0 \end{cases}
\tag{21}
$$

Let us put the J-integral from the formulas (21) in the expression (20) we obtain the size of the region of strain concentration near the crack tip:

$$
a = \begin{cases} \dfrac{\sigma^{1-m} E^m}{\sigma_* (1+m)}\,\ell, & \sigma < \sigma_0, \\[2ex] \dfrac{\ell}{1+m}, & \sigma \geq \sigma_0 \end{cases}
\tag{22}
$$

In the case of ideally elastic-plastic material the value is determined from the following relations: $a = \ell$ for $m = 0$ and $a = \ell/2$ for $m = 1$. One can see (taking into account that $K = \sigma\sqrt{\pi \ell}$) that the expressions (16) and (21) correlate. To carry out a comparison let us consider a case $\sigma_i = \sigma$ and let us rewrite the formulas (16) for the plane with crack:

$$
J = \begin{cases} \dfrac{2K^2}{(1+m)E}, & \sigma/\sigma_0 < 1, \\[2ex] \dfrac{2K^2}{(1+m)\sigma_*}\left(\dfrac{\sigma}{\sigma_*}\right)^{\frac{1-m}{m}}, & \sigma/\sigma_0 \geq 1 \end{cases}
\tag{23}
$$

The behaviour of J/J_e for the body with the crack is similar to this one for the body with the notch (fig.5): the value of J/J_e is increasing with the decrease of hardening and increase of applied stress. But in this case unlike the body with the notch the value $J/J_e = 2/(1+m)$ is constant for the material with fixed m when $\sigma/\sigma_0 < 1$. When we pass from the plane to other body shapes and types of loading we shoud write the stress intensity factor in the formulas (23) for the J-integral taking into account the correction function Y (of K-calibration) according to the formula $K = \sigma\sqrt{\pi\ell}Y$. Thus, the obtained formulas (23) for calculation of the J-intagral allow us to use widely the energy integral conception both in the experimental estimation of the fracture toughness of materials and in the analysis of structural integrity of constructions since it is enough to know the stress intensity factor, the applied load, the crack length and the mechanical properties of material (E, σ_*, and m) for determination of J.

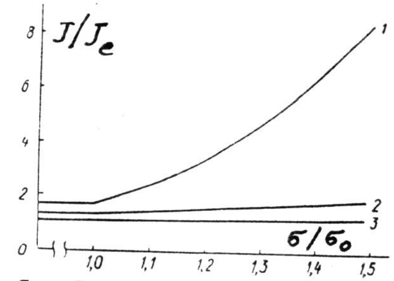

Fig.5. The energy integral for cracked body. 1 - m = 0.2, 2 - m = 0.5, 3 - m = 0.8.

REFERENCES

Hajinski G.M., (1983) Calaulation of concentrators when elastic-plasttic deforming, In: Rastchety na prochnost, v.24, p. 102-112, Mashinostroenie, Moscow. (In Russian).

Mahutov N.A., (1981) Deformation criteria of fracture and calculation of strength of constraction elements, Mashinostroenie, Moscow. (In Russian).

Matvienko Yu.G., Morozov E.M., (1984) Criteria of non-linear fracture mechanics and stress state near the crack tip, Problemy prochnosty, No. 11, pp. 10-13, (In Russian). (English trnsl. - Strength of materials, Plenum press).