BOUNDARY INTEGRAL EQUATION METHOD IN FRACTURE MECHANICS

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ABSTRACT

A general approach to constructing integral representations of general solutions and boundary integral equations of boundary value problem of elasticity theory for regions with cuts is proposed. It involves the use of generalized functions theory, and in particular of the surface delta function. Three-dimensional and plane problems of elastostatics for an anisotropic medium with a set of arbitrary cracks are analized simultaneously. An axisymmetric deformation of a transversely isotropic elastic body with cracks is also studied.

KEYWORDS

Fracture mechanics, elasticity theory, anisotropic solid with cracks, three-dimensional problems, plane problems, axisymmetric problems, boundary integral equation method.

INTRODUCTION

The method of boundary integral equations, also known as the boundary element method, is widely used for solving boundary value problems of mathematical physics. It has gained especially wide acceptance in the numerical analysis of elasticity problems with multiply connected regions that contain cavities (holes) of quite arbitrary shape. In fracture mechanics one has to solve boundary value problems of elasticity for regions with cracks or cuts. Although a crack can be treated as a limiting case of a cavity, the direct application of boundary integral equations in crack problems leads to their degeneration. Therefore, boundary value problems for regions with cracks require a special consideration. To reduce a problem to boundary integral equations it is necessary to know an integral representation of its general solution in terms of boundary values of some quantities. Such representation of the displacement field in an elastic body in terms of boundary values of displacements and tractions is given by the well-known Somigliana integral identity based on the Betti reciprocal work theorem. From this one can obtain an integral representation of the general solution for crack problems by means of a limiting process in terms of displacements discontinuities on the crack surface (Cruse, 1988; Balaš et al., 1989). Sometimes this version of the boundary integral equation method is called the method of potential theory (Parton and Perlin, 1977; Kit and Khai, 1989). In two-dimensional crack problems admitting application of the complex variable theory the singular integral equation method is also successfully employed (Panasyuk et al., 1976; Savruk, 1981; Kit and Kryvtsun, 1983; Savruk et al.,1989).

In this paper we propose an approach to the direct construction of integral representations of general solutions and of boundary integral equations of elasticity problems for regions with cuts. The three-dimensional and the plane problem of elastostatics for an anisotropic medium with a set of cracks are analysed. The axisymmetric deformation of a transversely isotropic elastic body with cracks is also studied. The approuch proposed here is of a general nature and can be extended to other multidimensional problems of mathematical physics for regions with cuts.

THREE-DIMENSIONAL AND PLANE PROBLEMS

Let us consider simultaneously three-dimensional and plane problems of elasticity theory for a homogeneous anisotropic body with cuts on the surface S. For this purpose we shall assume that the equations presented below hold in the Euclidean space R^n, where $n=2$ corresponds to the plane problem, and $n=3$ to the three-dimensional problem.

We introduce the generalized surface delta function δ_S which is concentrated on the surface S and defined by the expression (Kecs and Teodorescu, 1978; Vladimirov, 1981)

$$(\mu\delta_S, f)=\int_{R^n} \mu(x)\delta_S f(x)dx=\int_S \mu(\xi)f(\xi)dS, \qquad (1)$$

where $\mu(\xi)$ is a continuous function on S; $dx=dx_1 \dots dx_n$ is a volume element, and dS is a surface element. The following relationship holds as well

$$(-\partial_i(\mu\delta_S), f)=(\mu\delta_S, \partial_i f)=\int_S \mu(\xi)\partial_i f(\xi)dS, \quad \partial_i=\partial/\partial x_i. \qquad (2)$$

In the case of a piecewice-continuous function $f(x)$ of a point x of the space R^n with coordinates $(x_1,\dots,x_n)$ that has a discontinuity on the surface S, the partial derivatives in the generalized sense will be expressed by

$$\partial_i f=\{\partial_i f\}+[f]_S n_i \delta_S, \quad i=1,\dots,n. \qquad (3)$$

Here $\{\partial_i f\}$ is the classical derivative of the function $f(x)$;

$n_i=\cos(n,x_i)$ is the direction cosine of the normal to the surface S at the point $\xi\in S$; $[f]_S=f^+(\xi)-f^-(\xi)$, $\xi\in S$; the superscripts + and - indicate the limiting values of the function on the surface S in accordance with the direction of the normal n ($f^+(\xi)$ is the limiting value of $f(x)$ for $x\to\xi$ from the side towards which the normal n is directed).

Assuming that the stress components $\sigma_{ij}=\sigma_{ij}(x)$ $(i,j=1,\dots,n)$ are discontinuos functions on the surface S and taking under consideration the relation (3), we write down the equilibrium equations in the generalized sense

$$\partial_j\sigma_{ij}+X_i=[t_i]_S\delta_S.$$

Here $X_i=X_i(x)$ are components of volume forces; $t_i=t_i(x)$ are components of the stress vector on a plane with a normal n, defined by the relationship

$$t_i(x)=\sigma_{ij}(x)n_j=\hat{T}_{ik}(n_x,\partial_x)u_k(x), \qquad (4)$$

$$\hat{T}_{ik}(n_x,\partial_x)=c_{ijkl}n_j(x)\partial_l,$$

where $u_k(x)$ are components of the displacement vector, and c_{ijkl} are elastic constants. In the same way, assuming the displacements being discontinuous on the S, the Hooke's law is written down in the generalized sense as well

$$\sigma_{ij}=c_{ijkl}(\partial_l u_k-[u_k]_S n_l\delta_S).$$

As a result, the equilibrium equations in displacements take the form

$$\hat{L}_{ik}u_k+X_i=[t_i]_S\delta_S+c_{ijkl}\partial_j([u_k]_S n_l\delta_S), \quad \hat{L}_{ik}=c_{ijkl}\partial_j\partial_l. \qquad (5)$$

The components $U_{ij}=U_{ij}(x,y)$ of the tensor of fundamental solutions satisfy the system of equations

$$\hat{L}_{ik}U_{mk}=-\delta_{im}\delta(x-y),$$

where δ_{im} is Kronecker's delta, and $\delta(x)$ is the delta function. With the aid of the fundamental solutions $U_{ij}(x,y)$ and of the formulas (1) and (2), we obtain a solution of the nonhomogeneous equations (5) in the form

$$u_k(x)=\int_{R^n} X_i(y)U_{ik}(x,y)dy-\int_S([t_i(\eta)]_S U_{ik}(x,\eta)-$$

$$-[u_i(\eta)]_S T_{ik}(x,\eta))dS_\eta \qquad (6)$$

with

$$T_{ik}(x,y)=\hat{T}_{im}(n_y,\partial_y)U_{mk}(x,y).$$

Let us consider boundary value problems when on the faces of the cut S the displacements (the first basic problem)

$$u_k^{\pm}(\xi)=v_k(\xi)\pm\gamma_k(\xi), \quad \xi\in S \qquad (7)$$

or the stresses (the second basic problem)

$$t^{\pm}_k(\xi)=p_k(\xi)\pm\mu_k(\xi),\ \ \xi\in S \qquad (8)$$

are specified. Stresses are assumed to vanish at infinity. By satisfying the boundary conditions (7) and (8) with the aid of the integral representation of the displacement components $u_k(\mathbf{x})$ (6), we obtain boundary integral equations. In the case of the first basic problem we have the equations

$$2\int_S(\gamma_\ell(\eta)T_{\ell k}(\xi,\eta)-\mu_\ell(\eta)U_{\ell k}(\xi,\eta))dS_\eta=v_k(\xi)-\int_{R^n}X_\ell(y)U_{\ell k}(\xi,y)dy,\qquad \xi\in S \qquad (9)$$

for finding the unknown components of the jump of stresses $2\mu_\ell(\xi)$. For the second basic problem we obtain the hypersingular integral equations

$$2\int_S(\gamma_\ell(\eta)S_{l\ell}(\xi,\eta)-\mu_\ell(\eta)D_{l\ell}(\xi,\eta))dS_\eta=p_l(\xi)-\int_{R^n}X_\ell(y)D_{l\ell}(\xi,y)dy,\qquad \xi\in S \qquad (10)$$

for finding the unknown components of the jump of displacements $2\gamma_\ell(\xi)$. Here

$$S_{l\ell}(\mathbf{x},\mathbf{y})=\hat{T}_{lk}(n_x,\partial_x)T_{\ell k}(\mathbf{x},\mathbf{y}),\quad D_{l\ell}(\mathbf{x},\mathbf{y})=\hat{T}_{lk}(n_x,\partial_x)U_{\ell k}(\mathbf{x},\mathbf{y}).$$

In the case of the plane problem both equations (9) and (10) can be reduced to singular integral equations (Savruk, 1981). In the abcence of the jump of stresses ($\mu_\ell(\xi)=0$) the equations (10) coincide with those published earlier (Balaš et al., 1989).

AXISYMMETRIC PROBLEMS

Axisymmetric problem of elasticity for a transversely isotropic body, when the plane of isotropy is normal to the axis of revolution, is decomposed into two independent problems of axisymmetric torsion and axisymmetric deformation. In the case of an isotropic elastic body with arbitrary cracks both problems were studied earlier (Savruk, 1989, 1990). It is known that the axisymmetric torsion problem of a homogeneous anisotropic body of revolution is reduced to the similar one of an isotropic body by means of a simple change of spatial variables (Lekhnitskii, 1977). The case of axisymmetric deformation of a transversely isotropic body requires a special consideration.

We assume that the distribution of stresses in a transversely isotropoc body of revolution is symmetrical about the axis of revolution. Let us introduce a cylindrical coordinate system (r, φ, z) with z-axis coinsiding with the axis of the body. Then the problem of the axisymmetric deformation of such bodies is reduced to finding two components of the displacement vector u_r, u_z and four

components of the stress tensor σ_{rr}, $\sigma_{\varphi\varphi}$, σ_{zz} and σ_{rz}. Let a system of axisymmetric cuts on smooth surfaces of revolution be located in an elastic medium. Denote by S a set of cut contours in the half-plane $\Pi=\{r\geq 0;\ -\infty<z<\infty\}$. Assuming that the stresses are discontinuous on the contour S, we write down the equilibrium equations in the generalized sense

$$\partial_\beta\sigma_{\alpha\beta}+\tau_\alpha/r+X_\alpha=[t_\alpha]_S\delta_S,$$

where $\tau_r=\sigma_{rr}-\sigma_{\varphi\varphi}$, $\tau_z=\sigma_{rz}$; $\partial_\beta=\partial/x_\beta$, $x_r=r$, $x_z=z$. Here and henceforth, the Greek substrips take the values r and z. In this case the Hooke's law takes the following form (Lekhnitskii, 1977)

$$\sigma_{rr}=A_{11}\varepsilon_{rr}+A_{12}\varepsilon_{\varphi\varphi}+A_{13}\varepsilon_{zz},$$

$$\sigma_{\varphi\varphi}=A_{12}\varepsilon_{rr}+A_{11}\varepsilon_{\varphi\varphi}+A_{13}\varepsilon_{zz},$$

$$\sigma_{zz}=A_{13}\varepsilon_{rr}+A_{13}\varepsilon_{\varphi\varphi}+A_{33}\varepsilon_{zz},\quad \sigma_{rz}=A_{44}\varepsilon_{rz},$$

where A_{11}, A_{12},... are elastic constants;

$$\varepsilon_{rr}=\partial_r u_r-[u_r]_S n_r\delta_S,\quad \varepsilon_{zz}=\partial_z u_z-[u_z]_S n_z\delta_S,$$

$$\varepsilon_{rz}=\partial_z u_r+\partial_r u_z-[u_r]_S n_z\delta_S-[u_z]_S n_r\delta_S,\quad \varepsilon_{\varphi\varphi}=u_r/r.$$

The components of the stress vector $t_\alpha(\mathbf{x})$ on a plane with a normal n are defined by a formula similar to (4), where operators $T_{\alpha\beta}(n_x,\partial_x)$ have the form

$$\hat{T}_{rr}(n_x,\partial_x)=A_{11}n_r\partial_r+A_{44}n_z\partial_z+A_{12}n_r/r,$$

$$\hat{T}_{rz}(n_x,\partial_x)=A_{13}n_r\partial_z+A_{44}n_z\partial_r,$$

$$\hat{T}_{zr}(n_x,\partial_x)=A_{44}n_r\partial_z+A_{13}n_z\partial_r+A_{13}n_z/r,$$

$$\hat{T}_{zz}(n_x,\partial_x)=A_{44}n_r\partial_r+A_{33}n_z\partial_z.$$

The equilibrium equations in displacements in the generalized sense are written down as follows

$$L_{\alpha\beta}u_\beta+X_\alpha=[t_\alpha]_S\delta_S+\Lambda^{+}_{\alpha\beta\gamma}(\partial_x)([u_\beta]_S n_\gamma\delta_S), \qquad (11)$$

where

$$L_{rr}=A_{11}(\partial_r^2+(1/r)\partial_r-1/r^2)+A_{44}\partial_z^2,$$

$$L_{rz}=(A_{13}+A_{44})\partial_r\partial_z,\quad L_{zr}=(A_{13}+A_{44})(\partial_r\partial_z+(1/r)\partial_z),$$

$$L_{zz}=A_{44}(\partial_r^2+(1/r)\partial_r)+A_{33}\partial_z^2;$$

$$\Lambda^{\pm}_{rrr}(\partial_x)=A_{11}\partial_r\pm(A_{11}-A_{12})/r,\quad \Lambda^{\pm}_{rrz}(\partial_x)=A_{44}\partial_z,$$

$\Lambda^{\pm}_{rzr}(\partial_x)=A_{44}\partial_z$, $\Lambda^{\pm}_{rzz}(\partial_x)=A_{13}\partial_r$, $\Lambda^{\pm}_{zrr}(\partial_x)=A_{13}\partial_z$,

$\Lambda^{\pm}_{zrz}(\partial_x)=\Lambda^{\pm}_{zzr}(\partial_x)=A_{44}(\partial_r\pm 1/r)$, $\Lambda^{\pm}_{zzz}(\partial_x)=A_{33}\partial_z$.

Fundamental solutions $U_{\gamma\beta}(x,y)$ $(y=(r',z'))$ satisfying

$$L_{\alpha\beta}U_{\gamma\beta}=-\delta_{\alpha\gamma}\delta(x-y)$$

are expressed in terms of elliptic integrals (Savruk, 1993). Making use of them and of relations (1), (2), and (3), we obtain a solution of equations (11) in the form

$$u_\beta(x)=\int_\Pi X_\alpha(y)U_{\alpha\beta}(x,y)dy+\int_S([u_\alpha(\eta)]_S T_{\alpha\beta}(x,\eta)-$$
$$-[t_\alpha(\eta)]_S U_{\alpha\beta}(x,\eta))dS_\eta \qquad (12)$$

with

$T_{\alpha\beta}(x,y)=T^-_{\gamma\alpha}(n_y,\partial_y)U_{\gamma\beta}(x,y)$, $T^-_{\gamma\alpha}(n_y,\partial_y)=n_\beta(y)\Lambda^-_{\gamma\alpha\beta}(\partial_y)$.

Satisfying the boundary conditions (7) and (8) with the aid of the integral representation of displacements $u_\beta(x)$ (12), we arrive at two systems of boundary integral equations

$$2\int_S(\gamma_\alpha(\eta)T_{\alpha\beta}(\xi,\eta)-\mu_\alpha(\eta)U_{\alpha\beta}(\xi,\eta))dS_\eta=$$
$$=v_\beta(\xi)-\int_\Pi X_\alpha(y)U_{\alpha\beta}(\xi,y)dy,\ \ \xi\in S; \qquad (13)$$
$$2\int_S(\gamma_\alpha(\eta)S_{\alpha\beta}(\xi,\eta)-\mu_\alpha(\eta)D_{\alpha\beta}(\xi,\eta))dS_\eta=$$
$$=p_\beta(\xi)-\int_\Pi X_\alpha(y)D_{\alpha\beta}(\xi,y)dy,\ \ \xi\in S \qquad (14)$$

in the unknown jumps of stresses $2\mu_\alpha(\xi)$ (13) (the first basic problem) and of displacements $2\gamma_\alpha(\xi)$ (14) (the second basic problem). Here

$S_{\alpha\beta}(x,y)=\hat{T}_{\beta\gamma}(n_x,\partial_x)T_{\alpha\gamma}(x,y)$, $D_{\alpha\beta}(x,y)=\hat{T}_{\beta\gamma}(n_x,\partial_x)U_{\alpha\gamma}(x,y)$.

In the case of a system of flat coplanar cracks in the plane z=0 the systems (13) and (14) are decomposed into four independent equations which can be written in the form

$$\int_S F_n(r')r'dr'\int_0^\infty J_n(tr')J_n(tr)dt=f_n(r),\ \ r\in S; \qquad (15)$$

$$\int_S G_n(r')r'dr'\int_0^\infty t^2J_n(tr')J_n(tr)dt=g_n(r),\ \ r\in S; \qquad (16)$$

where n=0,1. Here $F_n(r)$ and $G_n(r)$ are unknown functions; $f_n(r)$ and $g_n(r)$ are specified functions; $J_n(t)$ is the Bessel function of order n. When the S is an internal $(0\leqslant r\leqslant R)$ or external $(R\leqslant r<\infty)$ circular region the equations (15) and (16) are solved in closed form for arbitrary $n\geqslant 0$ (Savruk, 1993). In the case of a penny-shaped cut we have

$$F_n(r) = -\frac{2r^{n-1}}{\pi}\frac{d}{dr}\int_r^R\frac{xdx}{x^{2n}\sqrt{x^2-r^2}}\frac{d}{dx}\int_0^x\frac{t^{n+1}f_n(t)dt}{\sqrt{x^2-t^2}};$$

$$G_n(r) = \frac{2r^n}{\pi}\int_r^R\frac{dt}{t^{2n}\sqrt{t^2-r^2}}\int_0^t\frac{x^{n+1}g_n(x)dx}{\sqrt{t^2-x^2}} \qquad (17)$$

with $G_n(r)$ satisfying the condition $G_n(R)=0$.

The last equation of (17) provides the possibility to write the closed-form expressions of the stress intensity factors K_I and K_{II} under the action of volume forces in a solid and arbitrary nonself-equilibrium tractions on faces of the penny-shaped crack. When the volume forces are absent $(X_r(x)=X_z(x)=0)$ these expressions take the form

$$K_I = -\frac{2}{\sqrt{\pi R}}\int_0^R\frac{rp_z(r)dr}{\sqrt{R^2-r^2}},$$

$$K_{II} = -\frac{2}{R\sqrt{\pi R}}\left[\int_0^R\frac{r^2p_r(r)dr}{\sqrt{R^2-r^2}}+b_1\int_0^R r\mu_z(r)dr\right]$$

with

$$b_1=(\pi/2)(A_{11}/A_{33}-c^2)/\sqrt{2(a^2+c^2)},$$
$$a^2=(A_{11}A_{13}-2A_{13}A_{44}-A_{13}^2)/(2A_{33}A_{55}),\ \ c^4=A_{11}/A_{33}.$$

The case of an external circular crack $(S=\{r\geqslant R;\ z=0\})$ can be considered in a similar manner (Savruk, 1993). The corresponding expressions of stress intensity factors are

$$K_I = -\frac{2}{\sqrt{\pi R}}\left[\int_R^\infty\frac{rp_z(r)dr}{\sqrt{r^2-R^2}}+b_2\int_R^\infty\mu_r(r)dr\right],$$

$$K_{II} = -2\sqrt{(R/\pi)}\int_R^\infty\frac{p_r(r)dr}{\sqrt{r^2-R^2}} \qquad (18)$$

with

$$b_2=2A_{13}(1/A_{44}-1/A_{33}-A_{13}/(A_{33}A_{44})+1/(A_{33}c^2))/\sqrt{2(a^2+c^2)}.$$

It should be noted that the solution (18) is obtained under the condition that the displacements are equal to zero at infinity. To avoid such a restriction one should carry out an additional analysis, similar to the isotropic case (Stallybrass, 1981).

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