BEHAVIOUR OF THE STRESSES NEAR PLANE WEDGESHAPED DEFECTS

G.A. MORARI
Polytechnical Institute, Kishinev, Moldova

G. Ya. POPOV
State University, Odessa, Ukraine

ABSTRACT

Stresses or displacements behaviour in an infinite body near the apex of a plane wedgeshaped defect is being studied in the present work on the basis of a discontinuity solutions method.

KEYWORDS

Elasticity, defect, crack, inclusion, stress singularity, discontinuity solutions method, integral equation method

THE DISCONTINUITY SOLUTIONS FOR SPACE

Suppose that there is in elastic medium in a coordinate system r , θ , z in a plane $z = 0$ is present a defect, i.e. region ω , at intersection with which the field of displacements and stresses undergoes discontinuity. Let us introduce the following notations for jumps

$$
\begin{aligned}
& u_r(r,\theta,-0) - u_r(r,\theta,+0) = \langle u_r \rangle \\
& u_\theta(r,\theta,-0) - u_\theta(r,\theta,+0) = \langle u_\theta \rangle \\
& u_z(r,\theta,-0) - u_z(r,\theta,+0) = \langle u_z \rangle \\
& \sigma_z(r,\theta,-0) - \sigma_z(r,\theta,+0) = \langle \sigma_z \rangle \\
& \tau_{zr}(r,\theta,-0) - \tau_{zr}(r,\theta,+0) = \langle \tau_{zr} \rangle \\
& \tau_{z\theta}(r,\theta,-0) - \tau_{z\theta}(r,\theta,+0) = \langle \tau_{z\theta} \rangle
\end{aligned}
\tag{1}
$$

The solution of equations in theory of elasticity including the given jumps, i.e. discontinuity solution, was obtained in the paper by G.A.Morari and G.Ya.Popov (1990). This solution may be represented in the following form

$$\{U^{\circ}\} = \| K_{11}^{(z)} \| \{S_u\} + \| K_{12}^{(z)} \| \{S_\sigma\}$$
$$\{\sigma^{\circ}\} = \| K_{21}^{(z)} \| \{S_u\} + \| K_{22}^{(z)} \| \{S_\sigma\}. \qquad (2)$$

Here $\{U^{\circ}\} = \| u_r \; u_\theta \; u_z \|^T$, $\{\sigma^{\circ}\} = \| \sigma_z^{\circ} \; \tau_{zr}^{\circ} \; \tau_{z\theta}^{\circ} \|^T$ are accordingly the displacement and stress vectors in elastic medium from the given jumps (1), only those components are being left which are necessary to the boundary problems; $\{S_u\} = \| \langle u_\rho \rangle \; \langle u_\eta \rangle \; \langle u_z \rangle \|^T$, $\{S_\sigma\} = \| \langle \tau_{z\rho} \rangle \; \langle \tau_{z\eta} \rangle \; \langle \sigma_z \rangle \|^T$ are accordingly jumps vectors of the corresponding functions in the point $r = \rho$, $\theta = \eta$, $z = z$; $\| K_{ij}^{(z)} \|$ are matrixes with dimensions 3x3, for example

$$\| K_{21}^{(z)} \| = \left\| \begin{matrix} T_{31}^{(z)} & T_{32}^{(z)} & T_{33}^{(z)} \\ T_{51}^{(z)} & T_{52}^{(z)} & T_{53}^{(z)} \\ T_{61}^{(z)} & T_{62}^{(z)} & T_{63}^{(z)} \end{matrix} \right\| \qquad (3)$$

Integral operators act according to the rule

$$T_{ij}^{(z)} f = \iint_{\omega} t_{ij}(r, \rho, \mu, z) f(\rho, \eta) \rho \, d\rho \, d\eta, \quad \mu = \theta - \eta \qquad (4)$$

Expressions for another matrixes and corresponding elements have been given in the above - mentioned paper by G.A.Morari and G.Ya.Popov (1990), for example

$$t_{51} = -(4\pi)^{-1} \mu R^{-3} [(4\varkappa - 3) + 3(1 - 2\varkappa) R^{-2} r (r - \rho\cos\mu) +$$
$$+ 30(1 - \varkappa) R^{-4} \rho z^2 (\rho - r\cos\mu)] \sin\mu$$

$$t_{52} = (4\pi)^{-1} \mu R^{-3} [(4\varkappa - 3)\cos\mu - 3(1 - 2\varkappa) R^{-2} (r - \rho\cos\mu)(\rho - r\cos\mu) +$$
$$+ 30(1 - \varkappa) R^{-4} r \rho z^2 \sin^2\mu]$$

$$t_{61} = (4\pi)^{-1} \mu R^{-3} [(4\varkappa - 3)\cos\mu + 3(1 - 2\varkappa) R^{-2} r \rho \sin^2\mu -$$
$$- 30(1 - \varkappa) R^{-4} z^2 (r - \rho\cos\mu)(\rho - r\cos\mu)]$$

$$t_{62} = (4\pi)^{-1} \mu R^{-3} [(4\varkappa - 3) + 3(1 - 2\varkappa) R^{-2} \rho (\rho - r\cos\mu) +$$
$$+ 30(1 - \varkappa) R^{-4} r z^2 (r - \rho\cos\mu)] \sin\mu$$

$$(R = (r^2 + \rho^2 - 2r\rho\cos\mu)^{1/2})$$

Here μ - shear modulus; $\varkappa = 1/2\,(1-2\nu)/(1-\nu)$; ν - Poisson's ratio.

Discontinuity solutions permit to obtain a system of integral equations for unknown jumps on condition of a defect. In general case, this system has the sixth order. Not to consider on general case, we shall examine the case when in the plane z=0 there is a defect in the form of wedge-shaped crack subjected to displacement shear.

THE PROBLEM CONCERNING WEDGE-SHAPED CRACK BY DISPLACEMENT SHEAR

Let us assume that the crack is located in the plane z = 0 and occupies the region $|\theta| \leq \alpha$, $0 \leq r < \infty$. The strained state of the body is such that only displacement jumps u_r, u_θ appear on the crack surfaces, i.e. it is in the state of shear. The strained state of space is presented in the form of a sum of the basic one caused by an external load and perturbed by presence of the crack

$$\tau_{z\theta} = \tau_{z\theta}^{*} + \tau_{z\theta}^{\circ}\,; \quad \tau_{zr} = \tau_{zr}^{*} + \tau_{zr}^{\circ}$$

where the function of the bases state is marked with asterisk.

Let us assume, that the crack surfaces are free from stresses, i.e.

$$\tau_{z\theta}(r, \theta, \pm 0) = \tau_{zr}(r, \theta, \pm 0) = 0 \qquad (5)$$

The system of integral equations for unknown jumps $\langle u_r \rangle, \langle u_\theta \rangle$ will be obtained by means of (2) realizing the conditions on the defect (5)

$$T_{51}^{(o)} \langle u_\rho \rangle + T_{52}^{(o)} \langle u_\eta \rangle = f_1(r, \theta)$$
$$T_{61}^{(o)} \langle u_\rho \rangle + T_{62}^{(o)} \langle u_\eta \rangle = f_2(r, \theta) \qquad (6)$$

The operators $T_{ij}^{(o)}$ are calculated according to (4) when z=0; functions $f_i(r,\theta)$ are dependent on the basic state of the elastic body.

The system (6) allows to find the unknown jumps $\langle u_r \rangle$, $\langle u_\theta \rangle$. It may be as well used for studying the displacements behaviour near apex of the wedge-shaped crack.

We introduce the notations

$$\varphi_1(\eta) = \int_0^\infty \langle u_\rho \rangle (\rho, \eta) \rho^{s-1} d\rho, \quad \varphi_2(\eta) = \int_0^\infty \langle u_\eta \rangle (\rho, \eta) \rho^{s-1} d\rho \qquad (7)$$

Multiplying (6) by r^s and executing the integration on r in the limits $(0, \infty)$ we shall obtain a system of one-dimensional integral equations

$$\int_{-\alpha}^{\alpha} \| K \| \{\phi\} d\eta = \{F\}, \quad \{\phi\} = \| \varphi_1 \; \varphi_2 \|^T \qquad (8)$$

Matrix elements $\|K\|$ have the form

$$k_{11}(\gamma) = -S^*(2\varkappa s - s - 1)\, P_{s-1}^{-1}(-\cos\gamma)$$

$$k_{12}(\gamma) = S^*\left[(2\varkappa s - s - 1)\,\mathrm{ctg}\,\gamma\, P_{s-1}^{-1}(-\cos\gamma) - (1-2\varkappa)(1+s)\csc\gamma\, P_s^{-1}(-\cos\gamma)\right]$$

$$k_{21}(\gamma) = S^*\left[(2\varkappa - 2\varkappa s + s - 2)\,\mathrm{ctg}\,\gamma\, P_{s-1}^{-1}(-\cos\gamma) + (1-2\varkappa)(1+s)\csc\gamma\, P_s^{-1}(-\cos\gamma)\right]$$

$$k_{22}(\gamma) = S^*\left[2\varkappa(1-s) + s - 2\right] P_{s-1}^{-1}(-\cos\gamma); \quad S^* = \pi s/\sin\pi s$$

Here $P_\nu^\mu(x)$ are associate Legendre's functions and divergent integrals can be understood in the meaning of finite part of divergent integrals.

Using the results of H.Bateman and A.Erdelyi's (1977) reference-book we can show, that

$$k_{11}(\gamma) = 2(2\varkappa s - 1 - s)/(s-1)\gamma + k_{11}^*(\gamma)$$

$$k_{12}(\gamma) = 4(\varkappa - 1)/(s-1)\gamma^2 + 2\varkappa s\, \ln(|\gamma|/\alpha) + k_{12}^*(\gamma)$$

$$k_{21}(\gamma) = 2/(1-s)\gamma^2 - (4\varkappa - 3)\, s\, \ln(|\gamma|/\alpha) + k_{12}^*(\gamma)$$

$$k_{22}(\gamma) = 2\left[2\varkappa(1-s) + s - 2\right]/(1-s)\gamma + k_{22}^*(\gamma),$$

where $k_{ij}^*(\gamma)$ are continual functions.

After reducing the integration interval to the length (-1,1) the system solution is being found in the form

$$\varphi_1(\alpha\theta) = \sqrt{1-\theta^2}\sum_{m=1}^{N} X_m U_{m-1}(\theta)$$

$$\varphi_2(\alpha\theta) = \sqrt{1-\theta^2}\sum_{m=1}^{N} Y_m U_{m-1}(\theta) \qquad (9)$$

where $U_m(x)$ are Tchebysheff's polynomials of the second kind.

Continual parts of kernels $K_{ij}^*(\alpha\gamma)$ we shall approximate with truncated Fourier series by Tchebysheff's polinomials.

$$k_{\lambda\beta}^*(\alpha\gamma) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{(\lambda,\beta)}\, T_{i-1}(\theta)\, U_{j-1}(\eta)$$

Coefficients $a_{ij}^{(\lambda,\beta)}$ may be calculated by means of the well-known formulas which can be found in the reference-book by V.I.Krylov (1967)

$$a_{ij}^{(\lambda,\beta)} = \frac{1}{N(N+1)(1+\delta_{i1})}\sum_{p=1}^{N}\sum_{q=1}^{N} k_{\lambda\beta}^*(\alpha\lambda_p - \alpha\lambda_q)\,\cdot$$

$$\cdot \sin^2\lambda_q\; T_{i-1}(\lambda_p)\, U_{j-1}(\lambda_q)$$

$$\left(\lambda_p = \cos\left[(2p-1)\pi/2N\right],\ \lambda_q = \cos\left[q\pi/(N+1)\right]\right)$$

Substituting functions (9) in the system (8) and multiplying after that by $T_{k-1}(\theta)/\sqrt{1-\theta^2}$ and by integrating in the limits of (-1,1) we shall get a system of algebraic equations for determination of unknown coefficients X_m, Y_m

$$\tfrac{1}{2}a_1 X_{k-1} + \tfrac{1}{4}(1+\delta_{1k})\sum_{m=1}^{N} a_{km}^{(1,1)} X_m + \tfrac{1}{2}a_3\mu_{k-1}(Y_k - Y_{k-2}) +$$

$$+\sum_{m=1}^{N}\left[-a_2 m\rho_{k-1,m-1} + \tfrac{1}{4}(1+\delta_{1k})\, a_{km}^{(1,2)}\right] Y_m = b_{1,k} \qquad (10)$$

$$\tfrac{1}{2}\mu_{k-1} a_5 (1+\delta_{1k})(X_k - X_{k-2}) + \sum_{m=1}^{N}\left[-m a_4 \rho_{k-1,m-1} + \tfrac{1}{4}(1+\delta_{1k})\, a_{km}^{(2,1)}\right] X_m +$$

$$+\tfrac{1}{2}a_6 Y_{k-1} + \tfrac{1}{4}(1+\delta_{1k})\sum_{m=1}^{N} a_{km}^{(2,2)} Y_m = b_{2,k}; \quad k = 1,2,\ldots,N$$

The following notations are used in the system (10)

$a_1 = 2(2\varkappa s - s - 1)/(s-1)\alpha$; $a_2 = (4\varkappa - 1)/(1-s)\alpha^2$; $a_3 = -2\varkappa s$

$a_4 = 2/(s-1)\alpha^2$; $a_5 = (4\varkappa - 3)s$; $a_6 = 2\left[2\varkappa(1-s) + s - 2\right]/(1-s)\alpha$

$\mu_0 = -\tfrac{1}{2}\ln 2$; $\mu_m = -1/2m$, $m = 1,2,\ldots$

$\rho_{k,m} = 0$, if $k+m$ is even

$\rho_{k,m} = 1$, if $m > k-1$ and $k+m$ is odd

$\rho_{0,m} = [1 + (-1)^m]/2$; $X_k = Y_k = 0$, $k \leq 0$

If the determinant of system (10) is put equal to 0 we shall get the equation for determining the parameter s. The computations show that in the interval $1 < s < 1{,}5$ there are two near roots. The values s are given in table 1, when $\varkappa = 0{,}3$ for various α.

Table 1

$\alpha\pi^{-1}\cdot 10^3$	125	250	375	500	625	750	875
$s\cdot 10^2$	182	173	164	150	141	127	106
	176	166	158	150	133	115	103

So, the jumps $\langle u_r\rangle$, $\langle u_\theta\rangle$ behave as $O(r^{s-1})$ when $r \to 0$. The displacements U_r, U_θ have the same singularity while the stresses behave as $O(r^{s-2})$.

Considerable difficulties can arise when calculating the root with direct methods. Therefore, it is expedient to apply the programmes of minimization of the determinant, preliminarily representing the system matrix in the form $\|U\|\cdot\|G\|\cdot\|V\|$ where $\|U\|$, $\|V\|$ are the rotation matrixes and $\|G\|$ is a diagonal matrix. Then the determinant is equal to det $\|G\|$. These applied programmes have been taken from the book by E.Forsyte et al. (1977).

The singularities near the apex of the wedge-shaped defect of thin rigid or flexible inclusion were being studied.

In conclusion we shall mention, that the behaviour of stresses on the apex near the plane wedgeshaped crack at normal discontinuity (the jump undergoes only the displacement U_z) was being studied in the paper by K.Takakuda (1985).

REFERENCES

E.Forsyte et al. (1977). Computer Methods for Mathematical Computations, 280, Prentice-Hall.

V.I.Krylov (1967). Approximate Calculation of Integrals (in Russian), 500, Nauka.

G.A.Morari and G.Ya.Popov (1990). To the Construction of Discontinual Solutions in the Spatial Theory of Elasticity. Journal of Applied Mathematics and Mechanics (in Russian), Vol.54, Issue 5, 798-806.

K.Takakuda (1985). Stress Singularities near Crack Front Edges. Bulletin of JSME, Vol.28, No.236, 225-231.