BEARING CAPACITY OF A RADIALLY CRACKED FLOATING PLATE INCLUDING CRACK-CLOSURE EFFECTS

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ABSTRACT

Progressive radial cracking of a floating plate and its consequences are studied. The material behavior is assumed to be elastic-brittle. The cracks are assumed to be 'relatively' long in the sense that the three dimensional contact problem can be described in a statically equivalent two-dimensional idealization. The number of cracks is supposed 'large enough' to permit a quasi-continuum approach rather than one involving the discussion of discrete sectors. The formulation incorporates the action of both bending and stretching as well as closure effects of the radial crack face contact. Fracture mechanics is used to explore the load-carrying capacity and the importance of the role of the crack-surface-interaction. In this paper the complete formulation is given.

KEYWORDS

floating plate, radial cracks, crack closure, plane-bending problem

INTRODUCTION

The bearing capacity of floating plates has received a great deal of attention in ice mechanics, wherein the majority of papers have considered the quasi-elastic response of fully intact ice covers (Hertz, 1884; Wyman, 1950; Meyerhof, 1960; Nevel, 1970 & 1972; Mohaghegh and Coon, 1975; Frederking and Gold 1976; Beltaos 1978)—the great number of papers in this area have been thoroughly reviewed by Kerr (1976). A thorough understanding of this topic is the key to the provision of winter transportation routes (Gold, 1971) and easy access to construction sites on rivers and lakes. Notwithstanding the large number of papers on the bearing capacity of ice, there does not exist an analysis of radial cracking and its consequences. The formation of these radial cracks is as follows: under increasing loads, a surface crack initiates at the bottom of the ice sheet. This crack propagates up through-the-thickness as well as radially. At some juncture, further cracking ensues such that eventually a multiply-radially-cracked zone has developed. Ultimately, circumferential cracking is caused by tension on the top surface of the ice. No study to date has included the effects of crack face interaction on the bearing capacity and breakup mechanisms of an ice sheet. There exists no rigorous explanation of the observation by many experimentalists that, under increasing load, typically five or six finite radial cracks occur before ultimate failure.

The crack face interaction that occurs after a number of radial cracks have 'popped in' produces a wedging action that allows the ice sheet to maintain a finite bearing capacity. This wedging action, or crack face interaction, should be more evident for thicker sheets. Analytically, this is a complicated fracture mechanics problem in that the loading at the crack-tip is not purely Mode I but, in all likelihood, involves a proportion of the other two modes; the crack face interaction is actually a complicated three-dimensional contact problem. The contact pressure distribution is unknown and acts over an unknown area. The constraint that the contact pressure be positive (compressive) or zero, thus excluding tensile tractions on the crack faces, in itself makes the problem nonlinear (even for small deformations and linearly elastic material behavior).

In this paper, progressive radial cracking of a floating plate and its consequences is studied. The material behavior is taken to be elastic-brittle. The cracks are assumed to be 'relatively' long in the sense that the three dimensional contact problem can be described in a statically equivalent two-dimensional idealization. The viewpoint adopted in this paper forms one extreme in which one supposes that the number of cracks formed permits a quasi-continuum approach rather than one involving the discussion of discrete sectors (a seminal paper in this regard is that by Hellan 1984). This supposition requires a formulation in which the action of both bending and stretching is treated as well as closure effects of the radial crack face contact. At the other extreme, it is necessary to model the deformation of a plate weakened by the presence of a few intersecting cracks only (the latter study is to be reported in a separate paper).

PROBLEM DESCRIPTION

The floating plate is subjected to a vertical concentrated load at the center. The radially cracked plate configuration is separated into three regimes: the crack surface interaction area (area I, $r \leq R_1$), the open crack area (area II, $R_1 < r \leq R_2$) and the unbroken or intact plate (area III, $r > R_2$) as indicated in Fig. 1a. The cracks are assumed to be uniformly distributed within regimes I and II ($r < R_2$). The in-plane interaction force S_θ is compressive in area I and is zero in area II; the formulation here prescribes that S_θ acting at $z = -e_f$ is statically equivalent to the crack closure forces acting within the crack surface interaction regime. Given that the wedging action (in-plane) of the closure forces may produce crushing, the local action of S_θ is described in terms of both e_f and e_c, where $z = -e_c$ identifies the transition from the crushed zone to intact ice. The local crushing caused by the wedging action can occur either below or above the neutral axis, depending on whether the plate is subjected to uplift ($w' \geq 0$) or is being pushed down ($w' \leq 0$), respectively. Note that $w(r)$ denotes the vertical deflection of the plate and $w' \equiv dw/dr$. In other words,

$$e_f \geq 0 \quad \text{if} \quad w' \leq 0 \quad \text{and} \quad e_f \leq 0 \quad \text{if} \quad w' \geq 0. \tag{1}$$

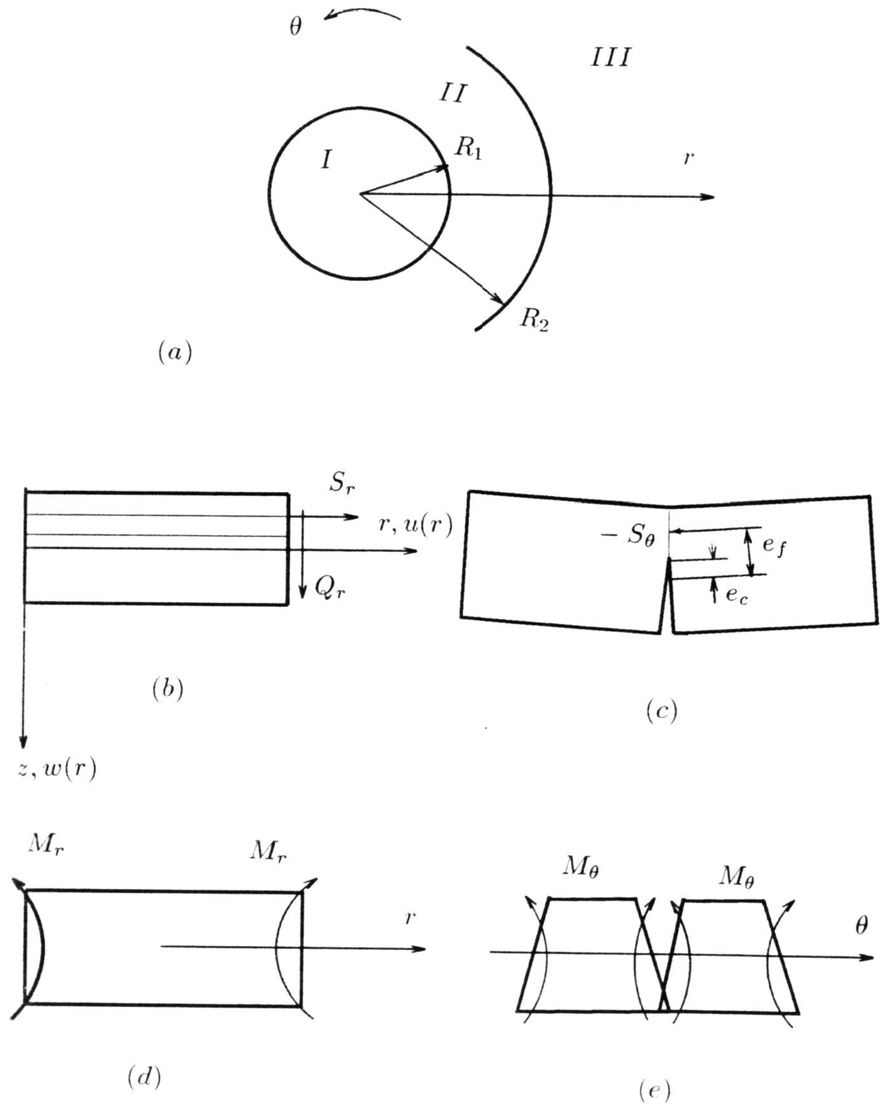

Fig. 1 Problem description: (a) three different regimes, (b,c) position of e_c and e_f, (d,e) moment convention

As an example, if crushing occurs and the crushing strength σ_c is assumed to be a positive constant and invariant with the area of crushing, the above details provide that

$$S_\theta = -\sigma_c[h + e_c \mathrm{sgn}(w')], \quad e_f = -\mathrm{sgn}(w')[h - e_c \mathrm{sgn}(w')]/2 \tag{2}$$

The upper bound of the load carrying capacity, given the radially cracked configuration, is given by assuming that in the crack surface interaction area I the membrane force S_θ is accommodated without any local crushing, *i.e.*, $e_c = e_f = \pm h$. On the other hand, the lower bound is obtained by assuming $e_c = e_f = 0$ (no crack face interaction). In the general case $e_f \neq 0$, and the force S_θ causes a bending moment M_θ; the occurrence of this eccentric in-plane force S_θ thus gives rise to a coupled plane-bending problem.

FORMULATION

I. Crack Surface Interaction Area ($0 \leq r \leq R_1$)

For the title axisymmetric problem, the tangential displacement u_θ is zero. Letting $w(r)$ and $u(r)$ be the vertical and radial displacement of the plate in the central plane respectively, and $u_r(r,z)$ be the radial displacement for arbitrary z. Using thin plate theory, the circumferential strain is given by $\epsilon_\theta = u_r(r, -e_c)/r$, where the radial displacement $u_r(r,z)$ at $z = -e_c$ is given by

$$u_r(r, -e_c) = u(r) + e_c w'(r). \tag{3}$$

The key to the formulation is the coupling between the in-plane (u, S_r, S_θ) and out-of-plane (w, M_r, M_θ, Q_r) quantities. This plane-bending coupling occurs solely through the expressions for M_θ and ϵ_θ:

$$M_\theta = S_\theta e_f, \qquad S_\theta \leq 0, \tag{4a, b}$$

$$\epsilon_\theta = (u + e_c w')/r = (S_\theta - \nu S_r)/2Eh + e_c(M_\theta - \nu M_r)/EI, \tag{4c}$$

where $I = 2h^3/3$. In addition,

$$EIw'' = M_r - \nu M_\theta, \tag{5a}$$

$$2Ehu' = S_r - \nu S_\theta. \tag{5b}$$

At any point in the plate (in any regime) the equations of equilibrium are

$$S_\theta = (rS_r)', \tag{6a}$$

$$M_\theta = (rM_r)' + rQ_r, \tag{6b}$$

$$(rQ_r)' - \rho g r w = -rq, \tag{6c}$$

where $'$ represents the derivative with respect to r. In the above equations, E, ν, ρ, h, q are the plate's Young's modulus, Poisson's ratio, density, half-thickness and lateral load intensity, respectively, while g is the acceleration due to gravity.

II. Open Crack Regime ($R_1 < r < R_2$)

In the open crack area, since $S_\theta = 0$, and $M_\theta = 0$, equations (4) and (5) quickly reduce to

$$EI(rw'')''/r + \rho g w = q, \quad \text{and} \quad (ru')' = 0. \tag{7a, b}$$

The solution of (7a,b) and the subsequent use of (5b) give

$$\frac{EIw(r)}{P\ell^2} = B_1 \mathrm{nev}_0(\bar r) + B_2 \mathrm{nev}_1(\bar r) + B_3 \mathrm{nev}_2(\bar r) + B_4 \mathrm{nel}_1(\bar r), \tag{8a}$$

$$u(r) = B_5 h \log(r/h) + B_6 h, \quad S_r = B_5\, 2Eh^2/r. \tag{8b}$$

where the constants $B_1, \ldots B_4$ may determined by boundary conditions and

$$\bar r = r/\lambda, \qquad \lambda^4 = EI/\rho g. \tag{9a, b}$$

The solution in (8a) and the functions $\mathrm{nev}_i(\bar r)$ (i=0,1,2) and $\mathrm{nel}_1(\bar r)$ were introduced and defined by Nevel (1958).

In this problem, it is important to note that in the vicinity of the load (as $r \to 0$) and at the transition radii (R_1 and R_2) the displacements $u = u_r$, $w = u_z$ and slope dw/dr are continuous, as are the forces S_r, Q_r and moment M_r. The stress field at infinity must be isotropic, *i.e.*, $S_r = S_\theta \to 2h\sigma_\infty$. In the case of a non-isotropic stress field at infinity ($\sigma_x \neq \sigma_y$) the present axisymmetric formulation cannot be used. In addition, the deflection of the plate must tend smoothly to zero at infinity.

III. Intact or Uncracked Regime ($r \geq R_2$)

In the area $r \geq R_2$, the plate is intact, and therefore

$$D\Delta^2 w + \rho g w = q, \quad 2Eh\, u/r = S_\theta - \nu S_r. \tag{10a, b}$$

The solution to (10a) with the requisite behavior at infinity is

$$w(r) = C_1 \mathrm{ker}(\bar r) + C_2 \mathrm{kei}(\bar r). \tag{11a}$$

where

$$\bar r = r/\ell, \qquad \ell^4 = EI/(1-\nu^2)\rho g. \tag{11b, c}$$

With the use of (5c) and (6a), (10b) is reduced to

$$[r(rS_r)']' - S_r = 0. \tag{12}$$

The solution of (12) and subsequent use of (6a) and (10b) give

$$S_r = \sigma_{ref} 2h(C_3 + C_4 h^2/r^2), \quad S_\theta = \sigma_{ref} 2h(C_3 - C_4 h^2/r^2), \tag{13a, b}$$

$$E\,u = \sigma_{ref} r[(1-\nu)C_3 - (1+\nu)C_4 h^2/r^2]. \tag{13c}$$

in which σ_{ref} represents a reference strength of the plate material.

The Governing Integral Equation

The proposed solution at this juncture has led to a system of differential equations. In principle, this system could be solved using the appropriate boundary and interface conditions. The system of equations is, however, still coupled; the solution procedure would therefore be still unnecessarily convovulated. Rather, in a direct attempt to uncouple the above system of equations, an auxiliary unknown function $f(r)$ is introduced as follows to split (4c):

$$u/r = (S_\theta - \nu S_r)/2Eh + f(r), \tag{14a}$$

$$w' = r(M_\theta - \nu M_r)/EI - rf(r)/e_c. \tag{14b}$$

This approach ultimately requires the solution of an integral equation, the specifics of which are now derived.

Through (14a), (5b) and (6a) the plane problem is now described by

$$[r(rS_r)']' - S_r = -2Eh(rf(r)) \tag{15}$$

It is now very useful to note the following equation and its solution:

$$[r(rS_r^o)']' - S_r^o = \delta(\xi - r), \quad S_r^o(r,\xi) = -\frac{\xi}{2}\Big(\frac{1}{\xi^2} - \frac{1}{r^2}\Big)H(\xi - r), \tag{16}$$

where $\delta(\,)$ is Dirac's delta function and $H(\,)$ is the Heaviside function. The complete solution of (15) follows quickly as

$$S_r(r,\xi) = \sigma_{ref}2h(A_5 + \frac{h^2}{r^2}A_6) - 2Eh\int_r^{R_1}(\xi f(\xi))_{,\xi}S_r^o(r,\xi)d\xi \quad r > 0, \tag{17}$$

Using (5a), (6b,c) and (14b), the governing equation for the deflection is

$$\Delta^2 w + (\rho g/D)w = q/D - \phi, \tag{18a}$$

in which

$$\phi(r) = [\nu(rf(r))' - f(r)]'/re_c. \tag{18b}$$

As in the case of in-plane deformations, it is very useful to note the following equation and its solution:

$$\Delta^2 w_o + (\rho g/D)w_o = \delta(\xi - r), \tag{19a}$$

where

$$w_o(r,\xi) = \begin{cases} -\xi\ell^2 W_o(\bar\xi, \bar r), & r < \xi \\ 0, & r > \xi \end{cases} \tag{19b}$$

in which $\bar\xi = \xi/\ell$ and

$$W_o(\bar\xi,\bar r) = \mathrm{kei}(\bar\xi)\mathrm{ber}(\bar r) + \mathrm{ker}(\bar\xi)\mathrm{bei}(\bar r) - \mathrm{bei}(\bar\xi)\mathrm{ker}(\bar r) - \mathrm{ber}(\bar\xi)\mathrm{kei}(\bar r), \tag{19c}$$

The complete solution of (18a) follows quickly as

$$w(r) = \int_r^{R_1} w_o(\xi, r)\{q(\xi)/D) - \phi(\xi)\}d\xi + W_h(r), \quad r > 0 \tag{20a}$$

where

$$W_h(r) = A_1\mathrm{ker}(\bar r) + A_2\mathrm{kei}(\bar r) + A_3\mathrm{ber}(\bar r) + A_4\mathrm{bei}(\bar r). \tag{20b}$$

From (5a) and (14b)

$$M_\theta = D\Big(\frac{w'}{r} + \nu w''\Big) + \frac{D}{e_c}f(r). \tag{21}$$

Remembering that $q(r) = 0$ except in the vicinity of $\xi = 0$, and substituting (4a), (6a), (17) and (20a) into (21) yields

$$\int_r^{R_1}\Big[-\Big(\frac{1}{r}\frac{\partial w_o}{\partial r} + \nu\frac{\partial^2 w_o}{\partial r^2}\Big)\phi(\xi) + (1-\nu^2)\frac{3e_f}{h^2}\frac{\partial}{\partial r}(rS_r^o(r,\xi))(\xi f(\xi))_{,\xi}\Big]d\xi$$
$$= -\frac{f(r)}{e_c} + \frac{e_f}{D}\Big[\sigma_{ref}2h(A_5 - \frac{h^2}{r^2}A_6)\Big] - \Big(\frac{1}{r}W_h' + \nu W_h''\Big). \tag{22}$$

CONTINUITY CONDITIONS

The general solutions given in the last section contain many unknown constants which are determined by the continuity conditions operative at the boundary of each regime. Letting subscript $_-$ and $_+$ be the crossponding quantities at $r \to R_{1-}$ and $r \to R_{1+}$ respectively, the continuity conditions can be written as

$$u_- = u_+, \qquad S_{r-} = S_{r+}, \tag{23a, b}$$

$$w_- = w_+, \qquad M_{r-} = M_{r+}, \tag{23c, d}$$

$$w'_- = w'_+, \qquad Q_{r-} = Q_{r+}. \tag{23e, f}$$

Besides the above conditions, as $r \to 0$, the following concentrated load and kinematic conditions hold:

$$2\pi r\, Q_r(r) = P, \tag{24a}$$

$$u(r) = 0, \qquad w'(r) = 0. \tag{24b, c}$$

By examining (14a,b), (24b,c) are satisfied if $(S_\theta - \nu S_r)$, $(M_\theta - \nu M_r)$ and $f(r)$ are finite as $r \to 0$.

FRACTURE CRITERION

The remaining task is to formulate the crack propagation criterion. The main difficulty in this instance resides in the fact that the description of the crack tip is a three-dimensional problem. Assuming the crack length is large enough, the Griffith-Irwin-Orowan energy criterion provides a means to overcome this difficulty. For the title problem, the crack tips are located at the radius $r = R_2$, and the problem can be formulated as a one dimensional equation in areas I, II and III. The released energy under the variation of the crack tip position R_2 can be evaluated. Dividing the released energy by the number of the cracks (which really is finite), and comparing the energy with the solution for a few cracks only (a separate study ongoing), the critical number of cracks can be obtained—this will be reported in a separate study.

When R_2 grows, the relationship between R_2, load P and the deflection at the center can be explored by employing the Griffith-Irwin-Orowom energy criterion

$$\frac{1}{2}\left(P\frac{dw}{dR_2} - w\frac{dP}{dR_2}\right) = 4h\gamma n, \tag{25}$$

in which n is the number of the crack and γ is the energy release per unit when R_2 grows. In the numerical evaluation, n and γ should be given as initial parameters.

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