A NUMERICAL ANALYSIS OF STRESS-STRAIN STATE OF CONTACTING ELASTIC BODIES WITH CRACKS

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ABSTRACT

This work describes a numerical analysis of two-dimensional stress-strain state of contacting elastic bodies with cracks. Unilateral contact conditions are given on a priori the unknown contact surfaces of the bodies and the face of the cracks. A variational-inequality formulation of such contact problems is obtained. Finite dimensional approximations of the problems are constructed by means of finite element methods. The complementary constraint method suggested by the author is used for modelling the displacement field in the vicinity of the crack tips. Some numerical results regarding the contact problems of the elastic bounded bodies with surface and buried cracks have been obtained. The influence of a contact effort distribution on stress intensity factors has been analyzed.

KEYWORDS

Stress intensity factors, contact problem, finite element methods.

INTRODUCTION

The indentation fracture is a subject of great interest both in theoretical and applied mechanics. The modern view of fracture mechanics of contacting bodies and the methods of its application was presented by Kolesnikov and Morozov (1989). The consideration of real contact stress distribution was stated to be necessary for predicting the behavior of cracks in contacting bodies. But as a rule, the question of contact interactions is reduced to the study of contact efforts, to the determination of contact hardness and contact size. The problem of stress distribution in contacting bodies in the presence of crack near the contact area has not been enough analyzed.

In this work the computational algorithm based on the finite element methods is developed for solving the two-dimensional contact problems of the system of elastic bounded bodies with cracks and for evaluating stress intensity factors. Using this algorithm some numerical results regarding the stress-strain state of cracked bodies are obtained.

STATEMENT OF THE PROBLEM

Let us consider a system of M contacting elastic bodies, which occupy the two-dimensional bounded domain $\Omega^1,\dots,\Omega^M$ with regular boundary $\Gamma^1,\dots,\Gamma^M$. A rectangular Cartesian coordinate system Ox_1x_2 is employed. With respect to it the components of the displacement vector, of the deformation and stress tensors at a point $x=x(x_1,x_2)$ are $u_i(x)$, $\varepsilon_{ij}(x)$, $\sigma_{ij}(x)$ respectively.

The boundary Γ^m of the body Ω^m can be presented as the union of the four disjoint portions $\Gamma=\Gamma^m_u\cup\Gamma^m_\sigma\cup\Gamma^m_c\cup\Gamma^m_p$. On the portion Γ^m_u the displacements $g^m_i(x)$ are given; on the portion Γ^m_σ the surface tractions $P^m_i(x)$ are given; Γ^m_c denotes the set of the limiting areas on which the body Ω^m may come in contact with the other bodies; Γ^m_p stands for the crack faces of the body Ω^m. The boundary conditions on Γ^m_c and Γ^m_p represent the conditions of the unilateral contact without friction

$$\begin{aligned}
&\sigma_n(x^m) = \sigma_n(x^k) \le 0;\\
&\sigma_t(x^m) = \sigma_t(x^k) = 0;\\
&u_n(x^m) + u_n(x^k) \le \Psi^{mk}(x^m)\\
&\sigma_n(x^m)\Big[u_n(x^m) + u_n(x^k) - \Psi^{mk}(x^m)\Big] = 0;\\
&x^m\in\Gamma^m_c,\ x^k(x^m)\in\Gamma^k_c;
\end{aligned} \tag{1}$$

$$\begin{aligned}
&\sigma_n(x^+) = \sigma_n(x^-) \le 0;\\
&\sigma_t(x^+) = \sigma_t(x^-) = 0;\\
&u_n(x^+) + u_n(x^-) \le 0;\\
&\sigma_n(x^+)\Big[u_n(x^+) + u_n(x^-)\Big] = 0;\\
&x^+\in\Gamma^m_p,\ x^+(x^-)\in\Gamma^m_p,
\end{aligned} \tag{2}$$

where $\Psi^{mk}(x^m)$ is the normalized initial gap between the bodies Ω^m and Ω^k at a point $x^m\in\Gamma^m_c$; $x^k(x^m)\in\Gamma^k_c$ is the point of the intersection of the boundary Γ^k_c with the normal to the boundary Γ^m_c at the point x^m; indices "n" and "t" denote the normal and tangential components of vector respectively.

So the problem is to determine function $u_i(x)$, $\varepsilon_{ij}(x)$, $\sigma_{ij}(x)$, satisfying in the domain $\Omega^1,\dots,\Omega^M$ the equilibrium equations with volume forces ρF_i, the generalized Hooke's law, and boundary conditions specified above. In addition it is necessary to determine actual contact areas and stress intensity factors.

VARIATIONAL FORMULATION OF THE PROBLEM

For solving the problems a variational method is used. Let us consider direct product of Sobolev's spaces

$$H = [W^1_2(\Omega^1)]^3\otimes\dots\otimes[W^1_2(\Omega^M)]^3$$

and extract the set of admissible displacements

$$V = \Big\{ v=(v^1,\dots,v^M)\in H:\ v^m_i(x) = g^m_i(x),\ x\in\Gamma^m_u;\\
v^m_n(x^m) + v^k_n(x^k) \le \Psi^{mk}(x^m),\ x^m\in\Gamma^m_c,\ x^k(x^m)\in\Gamma^k_c;\\
v^m_n(x^+) + v^m_n(x^-) \le 0,\ x^+\in\Gamma^m_p,\ x^-(x^+)\in\Gamma^m_p \Big\}. \tag{3}$$

To simplify the notation we introduce the following designations

$$a(u,v)=\sum_{m=1}^{M}\int_{\Omega^m} a^m_{ijkl}\varepsilon_{ij}(u^m)\varepsilon_{kl}(v^m)d\Omega \tag{4}$$

$$L(v)=\sum_{m=1}^{M}\Big\{\int_{\Omega^m}\rho F^m_i v^m_i d\Omega + \int_{\Gamma^m_\sigma}P^m_i v^m_i d\Gamma\Big\} \tag{5}$$

where a^m_{ijkl} are the elasticities of the material of which the body Ω^m is composed. Using the technique developed by Duvaut and Lions (1980) and assuming that

$$a^m_{ijkl}\in L_\infty(\Omega^m),\quad g_i\in H^{1/2}(\Gamma^m_k),\quad P_i\in L_2(\Gamma^m_\sigma),\\
\rho F^m_i\in L_2(\Omega^m),\quad \Psi^{mk}\in H^{1/2}(\Gamma^m_c),\ m=1,2,\dots,M, \tag{6}$$

we can prove the following statements.

The solution u of the problem considered in the differential formulation satisfies the variational inequality

$$a(u,v-u) \ge L(v-u),\quad \forall v\in V,\ u\in V. \tag{7}$$

If the solution of variational inequality (7) exists and has the second order derivatives (at least generalized), it satisfies the equations and the boundary conditions of the problem considered in the differential formulation.

The variational inequality (7) is equivalent to the nonlinear programming problem

$$\inf_{v\in V}\Big\{ J(v) = \frac{1}{2}a(v,v) - L(v)\Big\}. \tag{8}$$

The results obtained by Duvaut and Lions (1980) may also be used to ~~be~~ analyzed the issue of existence and uniqueness of solutions to (7).

A FINITE ELEMENT APPROXIMATION OF THE PROBLEM

The formulation of the considered problem as the nonlinear programming problem (8) is used for elaborating the computational algorithm. The problem (8) is approximated by the finite element methods, with the triangular elements of the first order being used. On the faces of cracks and on the limiting contact areas Γ_c^m and Γ_c^k nodal points are placed opposite each other. In such a case the convex closed set of nodal admissible displacement is obtained as

$$V_h = \Big\{ z=(u_1(P^1),u_2(P^1),u_1(P^2),\dots,u_2(P^N)) \in R^{2N}: \quad u_i(P^*)=g_i(P^*),\ P^*\in \Gamma_u^m; \quad u_n(P^*)+u_n(P^{**})\le \Psi^{mk}(P^*),\ P^*\in \Gamma_c^m;\ P^{**}(P^*)\in \Gamma_c^k; \quad u_n(P^*)+u_n(P^{**})\le 0,\ P^*\in \Gamma_p^m;\ P^{**}(P^*)\in \Gamma_p^m \Big\}, \tag{9}$$

where $P^1,\dots,P^N$ are nodal points; z is nodal displacement vector; R^{2N} is 2N-dimensional Euclidean spaces. It is essential to be noted that the normal and tangential components of displacement vector are used as nodal displacement for the nodal points placed on Γ_c and Γ_p. The energy functional $J(v)$ is approximated by the function of several variables

$$J_h(z) = \frac{1}{2} z^T A z - B^T z, \tag{10}$$

where A is square $[2N\times 2N]$ stiffness matrix; B is the 2N-dimensional vector of external forces.

In this manner we obtain the quadratic programming problem

$$\min_{z\in V_h} J_h(z) \tag{11}$$

as a result of the finite element approximation.

THE COMPLEMENTARY CONSTRAINT METHOD

A near crack tip displacement field is known to be described by the asymptotic expression

$$u_i(x)=u_{i0}+ K_1 F_i(x) + K_2\Phi_i(x), \tag{12}$$

where K_1 and K_2 are stress intensity factors; u_{i0} are the components of a crack tip displacement vector; $F_i(x)$, $\Phi_i(x)$ are the known functions. The complementary constraint method suggested by Bobylev (1988) is used for modelling that displacement field . The main purpose of this method is to construct the approximate solution of a crack problem, which satisfies the asymptotic expression (12) at the nodes of some crack tip vicinity.

Let ω be a small region in the vicinity of a crack tip and include this tip. It follows from asymptotical expression (12) that a region located at a crack tip has the 4 degrees of freedom. If the region ω contains the $L\ge 3$ nodes, its discrete model, on the other hand, has the 2L degrees of freedom. Hence, it is necessary to impose the 2L-4 complementary constraints on the nodal displacements to make an approximate solution satisfy the relations (12) at all the nodes of the region ω.

The procedure being used for the constructing complementary constraints is as follows. The 2L-2 equalities of the type (12) are written for the nodes of the region ω with the exception of the crack tip node. On excluding the parameters $K_1$and K_2 from these equalities we obtain the desired constraints in the terms of linear equalities.

The failure of crack faces to interpenetrate in the vicinity of crack tip can be expressed as

$$K_1 \ge 0. \tag{13}$$

This condition may be converted into the linear inequality for nodal displacements, with expression (12) being used.

Imposing the complementary constraints on the admissible nodal displacement separates the convex closed subset V_h^* from the set V_h. The elements of the subset V_h^* satisfy the asymptotical expression (12) in the meaning stated above. In accordance with the complementary constraints method the nonlinear programming problem (12) is replaced by the following problem

$$\inf_{z\in V_h^*} J_h(z). \tag{14}$$

Imposing the complementary constrains on the admissible nodal displacements was shown to be equivalent to changing an original shape function system. In the modified system the finite element approximations of the asymptotical displacement field (12) obtained with the original shape function system are used as shape functions different from zero in the vicinity of a crack tip.

THE COMPUTATIONAL ALGORITHM

The modification of projected conjugate gradient method suggested by Bobylev (1987) is used for solving the quadratic programming problem (14). That modification is different from the well-known algorithm described by Pshenichny and Danilin (1975) in the following aspects. Firstly testing the condition of increasing the dimension of a working subspace is done at every iteration. Secondly the linear combination of an antigradient and the projection of the descent direction of a latest iteration on a new working subspace is used as a descent direction when the dimension of the working subspace is changed. The computational experiments carried out confirm the expedience of such modification.

A number of numerical results regarding the contact problems of elastic bounded bodies with surface and buried cracks have been obtained, with proposed technique being used. This paper, we describes the results referring to the brittle bounded body compressed by the two elastic bodies as shown in Fig.1.

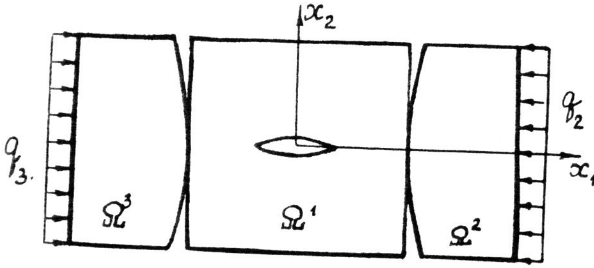

Fig.1.The system of contacting bodies

The bodies considered occupy the two-dimensional domains

$$\Omega^1=\left\{ x=(x_1,x_2)\in R^2:\ -h_1\le x_1\le h_1;\ -b_1\le x_2\le b_2 \right\};$$

$$\Omega^2=\left\{ x=(x_1,x_2)\in R^2:\ h_1+k_{11}x_2^4+k_{12}x_2^2\le x_1\le h_1+h_2;\ -b_1\le x_2\le b_2 \right\};$$

$$\Omega^3=\left\{ x=(x_1,x_2)\in R^2:\ -h_1-h_3\le x_1\le -h_1-k_{21}x_2^4-k_{22}x_2^2;\ -b_1\le x_2\le b_2 \right\}.$$

The thickness t of all the bodies is assumed to be equal to 1.

The uniform pressure $-q_2$ having the resultant force -Q is applied on the face $x_1=h_1+h_2$ of the body Ω^2, and the uniform pressure q_3 having the resultant force Q is applied on face $x_1=-h_1-h_3$ of the body Ω^3.

At first the case when the isotropic and homogeneous body Ω^1 is free of cracks was analyzed.It follows from the obtained results that non-uniform stress field is formed in the elastic bounded body being compressed axially by the elastic bodies making contact with it. The regions of tensile stresses may appear. The tensile stresses at the edge of the contact area amount to 0.13 of maximum contact pressure value p. This is in agreement with the result obtained with Hertzian solution. The tensile stresses reach their maximum value in the centre of the body and are normal to the axis of compression. They amount to 0.18p and exceed essentially the same value of half-space, which is 0.01-0.015p (Cherepanov, 1974). Compressive stresses come up to their maximum value on contact surfaces. Shearing stresses attain their maximum value 0.49p in the centre of the body. The ratio of maximum value of the tensile stresses and the tensile stresses reach 0.37.

Thus in this case the most probable opening mode crack initiation region is located in the central part of the body, with crack imperfections being distributed in the solid uniformly.

Taking into account the results presented above we further on consider the case when the body Ω^1 has the crack being located on the line segment $[-1;1]$ of the X_1-axis. For computational purpose the following data have been used

$$b_1=b_2=b_3=h_1=h_2=h_3=h=0.005\text{m}.$$

The materials of all the bodies are the same. Young's modulus and Poisson's ratio are assumed to be 10^4MPa and 0.23 respectively.

Fig.2 shows stress intensity factors K_1 related to the crack length ($l_1=l_2=l$). Curve 1 (where the compressive force is Q=0.01MN) and curve 2 (Q=0.05MN) conform to the initial gap being the quadratic function ($k_{11}=k_{21}=0$, $k_{12}=k_{22}=0.25\cdot10^{-2}h^{-2}$). Curve 3 (Q=0.1MN) and curve 4 (Q=0.05MN) correspond to the initial gap being the power function of the fourth order ($k_{11}=k_{21}=0.625\cdot10^{-2}h^{-4}$, $k_{12}=k_{22}=0$). Fig.3 shows the stress intensity factor K_1 as a function of compressive force. Curve 1 (where the crack sizes are $l_1=l_2=0,166h$) and curve 2 ($l_1=l_2=0,333h$) conform to the initial gap being the quadratic function. Curve 3 ($l_1=l_2=0,166h$) and curve 4

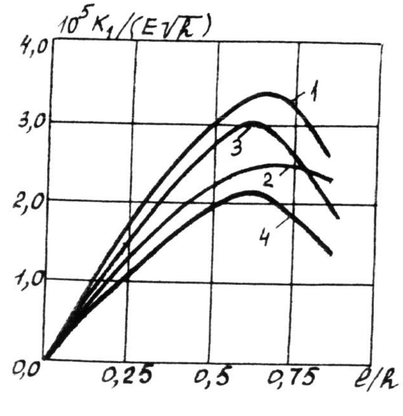

Fig.2.Stress intensity factors K_1 related to crack length

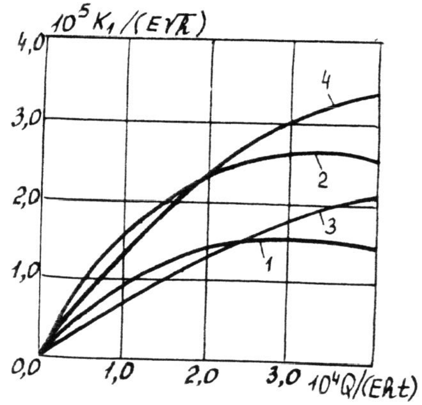

Fig.3.The stress intensity factor K_1 as a function of compressive force

($l_1=l_2=0,333h$) correspond to the initial gap being the power function of the fourth order.

The analysis of the obtained results reveals that the cracks being located in the central part of the compressed body and being parallel to the compression axis are opening mode cracks. If a crack tip is placed in the all-roundly compressed region being located under the contact surface, the crack intensity factor K_1 corresponding to this tip increases alongside with the growing of the compressive load. Hence, when the compressive load reaches its critical value crack begins growing instably, and the compressed body fails.

Thus, the brittle bounded body compressed axially by elastic bodies may fail along the planes which are parallel with a compression axis due to an opening mode crack growing in a tensile stress region.

CONCLUSION

The compression test carried out with a direct stress machine is one way to determine the strength of such brittle materials as concrete, pig iron, rock. Results described above show that due to the contact pressure being distributed uniformly the regions of tensile stresses may originate in a brittle body compressed axially, and the test bar may fail along the plane which is parallel to the compression axis. A real contact stress distribution obtained by solving corresponding contact problem should be taken into account in determining the strength of brittle materials with a compression test.

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