Plane-strain Crack-tip Fields for Power-law Hardening Pressure-sensitive Dilatant Materials

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ABSTRACT

Mode I crack-tip stress and strain fields are presented for pressure-sensitive dilatant materials under plane strain conditions. We adopt a yield criterion which is a linear combination of the effective stress and the mean stress. We assume that plastic deformation obeys the normality flow rule and material hardening follows a power-law relation. The low-hardening solutions of the crack-tip stress fields are found to agree with the corresponding perfectly plastic solutions.

KEYWORDS

Fracture; singularity fields; pressure-sensitivity; plastic dilatancy; polymers; ceramics.

INTRODUCTION

In classical plasticity theories, it is generally assumed that hydrostatic pressure has no effects on material plastic deformation, and plastic dilatancy is neglected. This type of plasticity theories, with a popularly used yield criterion due to Von Mises, is applicable mainly to dense metals. By constrast, rocks, concretes, soils and other porous materials exhibit plastic volumetric deformation and pressure-sensitive yielding at large strain. Recently, toughened polymers and ceramics due to their outstanding mechanical properties have attracted tremendous research attention. Experiments on these two classes of materials support a constitutive description which accounts for pressure-sensitive yielding and plastic dilatancy.

Richmond and Spitzig (1980) observed that for polyethylene and polycarbonate the flow stress has a linear dependence on hydrostatic stress. Carapellucci and Yee (1986) performed biaxial tension tests on glassy bisphenol A–polycarbonate and found that a modified Mises yield criterion with a dependence on the hydrostatic stress can fit their experimental data well. Recently, Sue and Yee (1988) investigated the toughening mechanisms in a multi-phase alloy of Nylon 6,6/Polyphenylene oxide, and found that there is a considerable amount of plastic volumetric change in the composite material due to the formation of crazes at large strain. They concluded that toughening of the material can be achieved by inducing a large amount of volumetric deformation due to crazing and subsequent shear localization of plastic deformation around a crack tip. The phenomenon of pressure-sensitive plastic yielding is also observed in transformation-toughened ZrO_2-containing ceramics, for example, see Chen and Reyes Morel

(1986).

Dominant asymptotic crack-tip field solutions based on the Mises yield criterion with no dependence on the hydrostatic stress have been given by Hutchinson (1968), Rice and Rosengren (1968) for pure-mode fields (HRR solutions) and by Shih (1974) for mixed-mode fields. In this study we investigate crack-tip stress and strain fields for pressure-sensitive dilatant materials. A yield criterion with a linear dependence on the hydrostatic stress and the normality plastic flow rule are adopted to take account for plastic dilatancy. Plane-strain mode I asymptotic crack-tip fields are obtained. The corresponding perfectly plastic solutions of the crack-tip fields are discussed in the light of the power-law hardening solutions.

CONSTITUTIVE RELATIONS

We adopt a simple yield criterion which is a linear combination of two stress invariants, the effective shear stress τ_e and the mean stress σ_m. The yield criterion is written as

$$\psi(\sigma_{ij}) = \tau_e + \mu\, \sigma_m = Q \tag{1}$$

where $\tau_e = (s_{ij}s_{ij}/2)^{1/2}$, $\sigma_m = (1/3)\sigma_{kk}$, $s_{ij} = \sigma_{ij} - \sigma_m\delta_{ij}$, and $\psi(\sigma_{ij})$ represents the current yield surface in the stress space. The material constant μ measures the pressure sensitivity of yielding. The yield criterion, equation (1), can be represented by a straight line in the τ_e-σ_m plane as sketched in Fig. 1. The characteristic yield strength Q can be taken to depend upon the plastic work W^p. We introduce the generalized effective shear stress τ_{ge} and the generalized effective tensile stress σ_{ge}, defined by $\sigma_{ge}/\sqrt{3} = \tau_{ge} = \tau_e + \mu\, \sigma_m$. The outward normal of the yield surface in the stress space is $s_{ij}/(2\tau_e) + \mu\delta_{ij}/3$.

A direct measurement of the pressure sensitivity factor μ relies on shear experiments under pressure. It can be also obtained from the difference between the compressive yield strength σ_c and the tensile yield strength σ_t through the relation $\mu = \sqrt{3}(\sigma_c - \sigma_t)/(\sigma_c + \sigma_t)$. Another method to determine μ is to perform compressive tests under pressure p. Let σ_c^0 denote the compressive strength without pressure, and σ_c^p the compressive strength under pressure p. When experimental data can be fitted by the linear relation $\sigma_c^p = \sigma_c^0 + \alpha p$, the value of μ can be calculated through $\mu = \sqrt{3}\alpha/(3+\alpha)$. The experimental curves given by Carapellucci and Yee (1986) show that the μ value for glassy bisphenol A-polycarbonate is about 0.14. For ZrO_2-containing ceramics, Chen and Reyes Morel (1986) reported that the constant α may approach to 2.0 which corresponds to $\mu = 0.69$. Note that the pressure sensitivity factor μ generally depends on the current stress and deformation state. In this study a constant μ is presumed for a given material for simplicity to explore the major features of the effects of pressure-sensitivity on crack-tip fields.

In this analysis the material shear response is specified by the Ramberg-Osgood relation

$$\gamma/\gamma_0 = \tau/\tau_0 + \alpha(\tau/\tau_0)^n \tag{2}$$

where γ is the shear strain, τ is the shear stress, n is the strain hardening exponent, α is a material constant, and τ_0 and γ_0 are the reference shear stress and the reference shear strain. Within the context of deformation theory of plasticity, the relation between shear stress and plastic shear strain (the second term on the right-hand side of (2)) is generalized to multiaxial state by assuming that the yield surface expands isotropically and the plastic strains obey the normality flow rule. The resulting relation between stress σ_{ij} and plastic strain ϵ_{ij}^p is

$$\frac{\epsilon_{ij}^p}{\gamma_0} = \alpha\left(\frac{\tau_{ge}}{\tau_0}\right)^n \left(\frac{s_{ij}}{2\tau_e} + \frac{\mu}{3}\delta_{ij}\right) \tag{3}$$

The stress-strain relation (3) is based on the deformation theory of plasticity. Incremental constitutive equations accounting for pressure-sensitivity and plastic dilatancy can be found, for example, in Rudnicki and Rice (1975) and Needleman and Rice (1978).

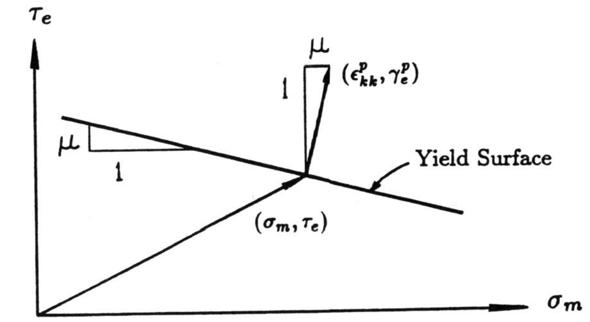

Fig. 1. Sketch of a yield surface illustrating the geometric interpretation of the pressure sensitivity factor μ. For normality plastic flow, the factor μ also serves as the plastic dilatancy factor.

PLANE-STRAIN MODE I CRACK-TIP FIELDS

We condider a planar crack problem depicted in Fig. 2, where the Cartesian coordinates (x_1,x_2) and the associated polar coordinates (r,θ) are centered at the crack tip, with the x_3 axis being perpendicular to the x_1-x_2 plane. When the crack tip is approached, the elastic strains are negligible compared to the plastic strains. Invoking the plane strain condition leads to the equation $s_{33}/(2\tau_e) + \mu/3 = 0$. By solving this equation for the stress component σ_{33}, the mean stress σ_m and the generalized effective stress τ_{ge} (or σ_{ge}) can be expressed in terms of the three in-plane components σ_{11}, σ_{22} and σ_{12}. Substituting these expressions into (3) gives the following stress-strain relations in plane strain

$$\begin{aligned}
\frac{\epsilon_{11}}{\gamma_0} &= \frac{1}{2}\alpha\left(\frac{\tau_{ge}}{\tau_0}\right)^n \left[\frac{\frac{1}{2}(\sigma_{11}-\sigma_{22})}{\tau_e} + \mu\right] \\
\frac{\epsilon_{22}}{\gamma_0} &= \frac{1}{2}\alpha\left(\frac{\tau_{ge}}{\tau_0}\right)^n \left[\frac{\frac{1}{2}(\sigma_{22}-\sigma_{11})}{\tau_e} + \mu\right] \\
\frac{\epsilon_{12}}{\gamma_0} &= \frac{1}{2}\alpha\left(\frac{\tau_{ge}}{\tau_0}\right)^n \frac{\sigma_{12}}{\tau_e}
\end{aligned} \tag{4}$$

where $\tau_{ge} = \left[1 - \frac{1}{3}\mu^2\right]^{1/2} \left[\frac{1}{4}(\sigma_{11}-\sigma_{22})^2 + \sigma_{12}^2\right]^{1/2} + \frac{\mu}{2}(\sigma_{11}+\sigma_{22})$.

It is more convenient to describe the crack-tip fields in terms of the effective tensile stress σ_e and the generalized effective tensile stress σ_{ge} (rather than τ_e and τ_{ge}) for a crack under mode I loading. With $\sigma_0 = \sqrt{3}\,\tau_0$ and $\epsilon_0 = \gamma_0/\sqrt{3}$, an alternative expression of (4) can be obtained. We follow the solution procedures of Hutchinson (1968) for Mises materials to obtain the dominant asymptotic crack-tip fields for pressure-sensitive dilatant materials. The outline of the procedures is as follows. An Airy stress function of seperable form of r and θ is employed to satisfy the equilibrium equations. The strains are expressed in terms of the stress function using the constitutive law (4) and then are inserted into the compatibility equation. A fourth order nonlinear ordinary differential equation with θ as the independent variable is obtained. The traction-free condition on the crack faces and the symmetry or antisymmetry condition (depending on the remote loading, mode I or II) provide the necessary boundary conditions for the differential equation. A combined fourth-fifth order Runge-Kutta integration scheme is employed and a shooting method is used to generate our solution.

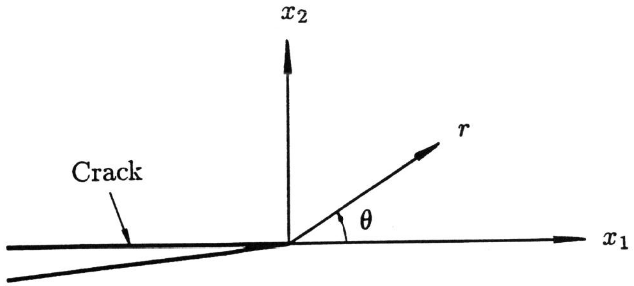

Fig. 2. Conventions at the crack tip.

Following the argument given by Rice and Rosengren (1968), the singular crack-tip stress, strain and displacement fields can be written as

$$\sigma_{ij} = \sigma_0 \left[\frac{J}{\alpha\sigma_0\epsilon_0 I(n,\mu) r}\right]^{1/(n+1)} \tilde{\sigma}_{ij}(\theta; n, \mu)$$

$$\epsilon_{ij} = \alpha\epsilon_0 \left[\frac{J}{\alpha\sigma_0\epsilon_0 I(n,\mu) r}\right]^{n/(n+1)} \tilde{\epsilon}_{ij}(\theta; n, \mu) \qquad (5)$$

$$u_i = \alpha\epsilon_0 r \left[\frac{J}{\alpha\sigma_0\epsilon_0 I(n,\mu) r}\right]^{n/(n+1)} \tilde{u}_i(\theta; n, \mu).$$

The dimensionless constant I and the θ-variations of the dimensionless functions, $\tilde{\sigma}_{ij}$, $\tilde{\epsilon}_{ij}$ and $\tilde{u}_i$, depend on the strain hardening exponent n and the pressure sensitivity factor μ. These angular functions are normalized by setting the maximum value of the generalized effective tensile stress $\tilde{\sigma}_{ge}$ to unity. With the normalization in (5), J represents the magnitude of the singularity amplitude of the crack-tip stress and strain fields. Note that J can not be determined by the asymptotic analysis since it depends on the geometry and loading of the cracked body.

We restrict our attention on mode I loading which is symmetric with respect to the crack. The asymptotic crack-tip fields, written in the seperable form as in (5), when $\mu = 0$, reduce exactly to the mode I HRR solutions. In this study, the angular functions, $\tilde{\sigma}_{ij}$, $\tilde{\epsilon}_{ij}$ and $\tilde{u}_i$, are obtained numerically for n=2 to 100 and for $\mu < \mu_{\rm lim}$. Here $\mu_{\rm lim}$ depends on the material hardening exponent n. Our $\mu = 0$ solutions for all n's are exactly the same as the tabulated values of the HRR solutions given by Shih (1983). We found that the values of the constant I decreases monotonically with an increasing n or μ, and a large $\mu_{\rm lim}$ corresponds to a large n (low hardening).

We present the solution of the crack-tip stress and strain fields for $n = 5$ (intermediate hardening) as a representative case. The θ-variations of the normalized stresses, $\tilde{\sigma}_{ij}$ and $\tilde{\sigma}_{ge}$, for $\mu = 0$, 0.1 and 0.2 are plotted in Fig. 3 to show the effects of pressure sensitivity on the singular stress fields. The solutions for $\mu = 0$ give exactly the HRR crack-tip stresses. The hoop stress $\tilde{\sigma}_{\theta\theta}$ at $\theta = 0$ decreases much faster than the radial stress $\tilde{\sigma}_{rr}$ does as μ increases. Consequently, a large μ gives a small difference between the hoop stress $\tilde{\sigma}_{\theta\theta}$ and the radial stress $\tilde{\sigma}_{rr}$ at $\theta = 0$. The mean stress $\tilde{\sigma}_m$ at $\theta = 0$ (note that $\tilde{\sigma}_m \neq (\tilde{\sigma}_{rr} + \tilde{\sigma}_{\theta\theta})/2$ when $\mu \neq 0$) decreases as μ increases.

The generalized effective tensile stress $\tilde{\sigma}_{ge}$ (reducing to $\tilde{\sigma}_e$ when $\mu = 0$) is found to peak at somewhere between 90° and 100° (this is true for all values of n and $\mu < \mu_{\rm lim}$). The peak value of the shear stress $\tilde{\sigma}_{r\theta}$ occurs at about 90° for $\mu = 0$ and at a larger angle for a larger μ.

The θ-variations of the normalized strains $\tilde{\epsilon}_{ij}$ and $\tilde{\epsilon}_{kk}$ for $n = 5$ and $\mu = 0$, 0.1 and 0.2 are plotted in Fig. 4. The volumetric plastic strain $\tilde{\epsilon}_{kk}$ is about 40 percents of the maximum shear strain $\tilde{\epsilon}_{r\theta}$ for $\mu = 0.2$. In all the cases we have studied, the strain $\tilde{\epsilon}_{kk}$ is found to increase with an increasing μ and to peak at about $\theta = 90°$.

PERFECTLY PLASTIC SOLUTION

Plane-strain crack-tip stress solutions for the corresponding pressure-sensitive perfectly plastic materials can be found by solving the two equilibrium equations together with the yield condition for three unknown stress components, σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$. Introducing the angular parameter ϕ, defined as $\sin\phi = \mu/[1-(1/3)\mu^2]^{1/2}$, we write the yield condition, $\sigma_{ge} = \sigma_0$, as follows

$$\left[\left(\frac{\sigma_{rr}-\sigma_{\theta\theta}}{2}\right)^2 + \sigma_{r\theta}^2\right]^{1/2} + \sin\phi\,\frac{\sigma_{rr}+\sigma_{\theta\theta}}{2} = \frac{(1+\frac{1}{3}\sin^2\phi)^{1/2}}{\sqrt{3}}\sigma_0 \qquad (6)$$

With the yield condition written in the above form, crack-tip stress solutions are obtained in a similar manner as in the Prandtl punch problem for soils and concretes using the Coulomb yield criterion. A summary of the results is as follows (for details, see Li and Pan 1988).

The crack-tip fields consist of two constant stress sectors and a centered fan sector. The θ-variations of the stress components and the angular spans of these sectors depend on the pressure sensitivity factor μ only through the parameter ϕ. The perfectly plastic solution for $\mu = 0$ is exactly the same as that of Rice (1968). The low-hardening solutions of the crack-tip stress fields are found to agree with the corresponding perfectly plastic solutions.

It should be noted that the perfect plasticity solution is valid for $\phi < \pi/2$, which is equivalent to the requirement, $\mu < \sqrt{3}/2$. As mentioned in the previous section, there exists a limit value $\mu_{\rm lim}$ corresponding to each n so that no solution of the form (5) is found so far when $\mu > \mu_{\rm lim}$. The values of $\mu_{\rm lim}$ for $n = 5$, 10, 20 and 100 are 0.2, 0.32, 0.45 and 0.6, respectively. It seems that as n increases, the value of $\mu_{\rm lim}$ increases and approaches to the number $\sqrt{3}/2 = 0.866$ which is the $\mu_{\rm lim}$ for the perfectly plastic solutions.

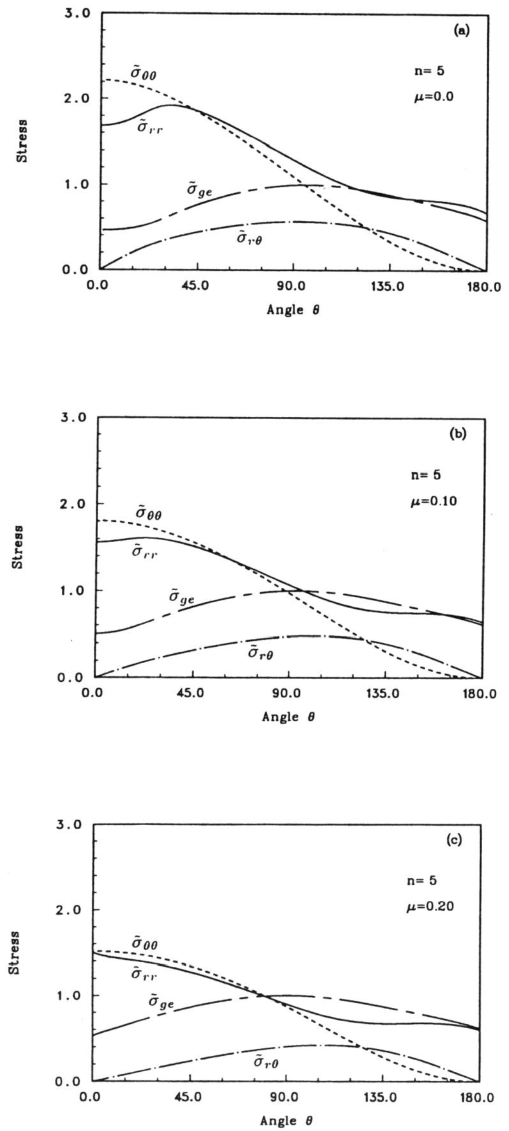

Fig. 3. Angular distributions of the normalized stresses for $n = 5$, (a) $\mu = 0.0$, (b) $\mu = 0.10$, (c) $\mu = 0.20$.

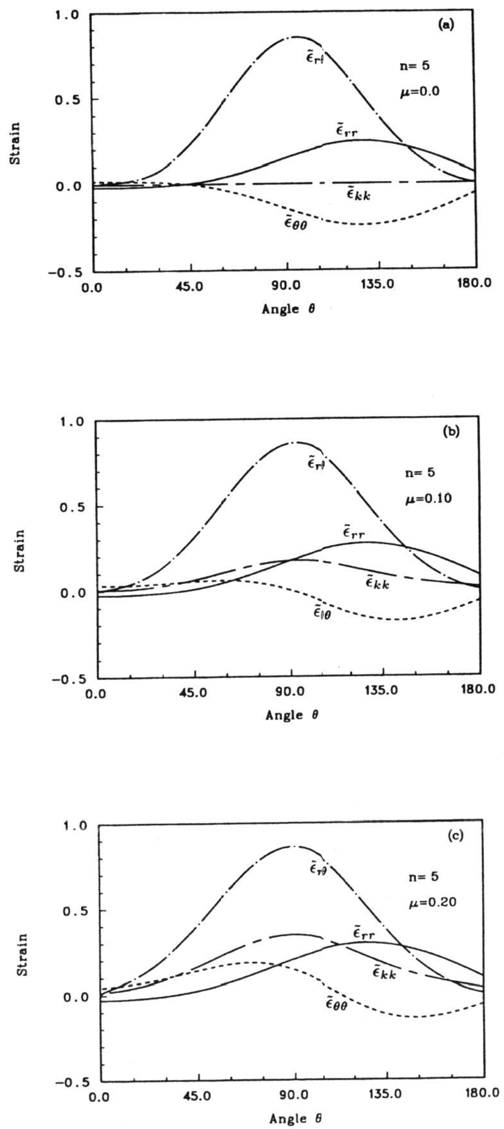

Fig. 4. Angular distributions of the normalized strains for $n = 5$, (a) $\mu = 0.0$, (b) $\mu = 0.10$, (c) $\mu = 0.20$.

CONCLUDING REMARKS

It is clear from our hardening solution (5) that J can be regarded as a measure of the intensity of the singular crack-tip fields. If the finite deformation zone and the fracture process zone are well contained within the zone of dominance of the singular field, J can be used as a characterizing parameter to correlate the initiation of crack growth in the pressure-sensitive dilatant materials. Furthermore, under small-scale yielding conditions J can be related to the elastic intensity factor K, and be inferred from the geometry and remote loading. In other words, with the existence of the solution (5) it is possible to use the so-called one-characterizing-parameter approach to predict damage in polymers and ceramics within the context of nonlinear fracture mechanics.

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