Elastic/Perfectly–Plastic Small Scale Yielding at Bi-Material Interfaces

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OVERVIEW

The need for understanding and improving the toughness of advanced composite materials systems has recently rekindled an interest in the analysis of crack tip fields near and on the interfaces separating distinct material phases. In particular, the case of a crack lying along part of such an interface has received much attention.

When each material is isotropic linear elastic, alternative models of near–tip fields can be constructed, depending on whether the crack faces are assumed to be separated and traction free (TF) or contacting and capable of transmitting traction. Solutions of the former type, due to Williams (1959), in general display weakly oscillatory square root singular behavior in the mathematical limit $r \rightarrow 0$, a feature locally associated with oscillatory interpenetration of crack face displacement. For predominant tensile loading, δ, the maximum distance from the crack tip over which interpenetration is predicted, may be several orders of magnitude less than L, a characteristic length scale of the geometry (*e.g.*, crack length). In such cases, simply neglecting the incompatibility of crack face interpenetration can be justified in the sense of "small scale nonlinearity". In cases of combined loading producing appreciable shear traction along the interface plane, the normalized contact length δ/L can become large, even in the presence of applied tension, Comninou and Schmeuser (1979). In such cases, the asymptotic TF solutions become increasingly irrelevant, and crack face contact must be explicitly accounted for, as, *e.g.*, in the closed frictionless (CF) fields of Comninou (1977).

The singular stresses of these elastic solutions cannot be sustained if either material can accommodate limited plastic flow. Recently several models of bi–material crack tip inelastic deformation have appeared. Shih and Asaro (herein denoted 'SA') (1988a,b,c) and Zywicz and Parks (denoted 'ZP') (1988a,b,c) have permitted inelastic deformation in one or both of the adjacent media. SA focussed on power law strain hardening nonlinearity under conditions of open, TF crack faces. ZP considered ideally plastic small scale yielding (SSY) well within the dominant elastic fields of both the TF (1988a, 1988c) and CF (1988b) idealizations. Approaches based on these distinct plastic constitutive models are complementary. The simplification of ideal plasticity readily provides for the

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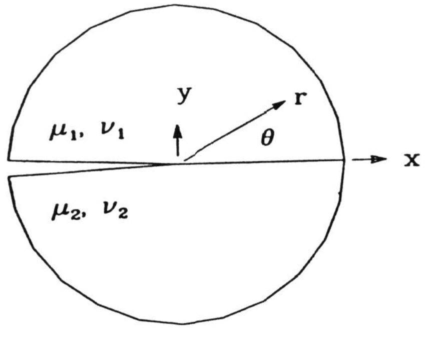

Figure 1: Schematic interfacial crack tip region.

compact and intuitive representation of asymptotic fields in terms of centered fan and constant state slipline regions and elastic wedges. The power law hardening idealization (strain $\propto$ stress^n) leads to asymptotic stress fields which can be well-modeled by the HRR-type form $\sigma \propto \sigma_0 \cdot (J/\sigma_0 r)^{1/(n+1)}$ (SA, 1988a–c). The mixity, or phase, of plastic zones embedded within the TF field can be characterized by parameters independently introduced by SA (1988a) and ZP (1988a). These parameters can also be used to define conditions under which SSY within a dominant TF field takes place — that is, when no significant crack face contact occurs (ZP, 1988a; SA, 1988c).

Here we highlight certain major features emerging from our studies of SSY at bi–material crack tips. The current format precludes detailed discussion, so the interested reader is referred elsewhere for further information.

APPROXIMATE PLASTIC ZONES

Insight regarding SSY plastic zones can be gleaned from the elastic fields in which they are embedded. An approximate method for determining the crack tip plastic zone shape and size consists of equating the elastically-calculated Mises or Tresca equivalent stress with the yield strength of the material. The locus of points satisfying this condition is taken as the (approximate) plastic zone boundary.

ZP (1988a) applied this procedure to the bi–material crack tip region shown in Figure 1. In the respective domains, shear moduli are μ_j $(j = 1, 2)$ and Poisson ratios are ν_j. Remote loading produces an elastic stress field which is locally dominated by the complex bi–material stress intensity factor $\mathbf{K}$ and associated asymptotic interfacial crack-tip stress fields. Hutchinson, *et al.*, (1987) define $\mathbf{K}$ such that, as $r \to 0$ on the interface $\theta = 0$, $\sigma_{yy} + i\sigma_{xy} \to \mathbf{K} r^{i\epsilon}/\sqrt{2\pi r}$. The bi–material constant, ϵ, which modulates the oscillation period, can be defined as $\epsilon = \frac{1}{2\pi}\ln\left[\left(\frac{\kappa_1}{\mu_1} + \frac{1}{\mu_2}\right) / \left(\frac{\kappa_2}{\mu_2} + \frac{1}{\mu_1}\right)\right]$, where $\kappa_j = 3 - 4\nu_j$ for plane strain.

Stress within a planar isotropic elastic solid can be represented by Muskhelishvili potentials: $\sigma_{xx} + \sigma_{yy} = 2\left[\phi' + \bar{\phi}'\right]$,and $\sigma_{yy} - \sigma_{xx} + i2\sigma_{xy} = 2\left[(\bar{z} - z)\phi'' - \phi' + \Omega'\right]$, where ϕ' and Ω' are analytic functions of the complex variable $z = x + iy$, prime denotes derivative, and the overbar denotes complex conjugate. As $z \to 0$ in region "1" of Fig. 1, the dominant

portions of these functions are (Rice, 1988)

$$\phi_1' = \frac{\bar{\mathbf{K}} e^{-2\pi\epsilon}}{2\sqrt{2\pi}\cosh\pi\epsilon} z^{-\frac{1}{2} - i\epsilon}, \quad \text{and} \quad \Omega_1' = \frac{\mathbf{K} e^{2\pi\epsilon}}{2\sqrt{2\pi}\cosh\pi\epsilon} z^{-\frac{1}{2} + i\epsilon}.$$

Using the plane strain result $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$, the Mises equivalent stress, $\bar{\sigma}$, can be calculated in region 1. The locus of points where this measure equals σ_{ys}, the yield strength of material '1', can be expressed in polar repesentation as

$$r_p(\theta) = \frac{3\mathbf{K}\bar{\mathbf{K}}}{\sigma_{ys}^2 8\pi\cosh^2(\pi\epsilon)} \times \left\{ \begin{array}{c} 2\cos(\theta + 2\varsigma(\theta))\left[\left(\frac{4D}{3} - 1\right) e^{2\epsilon(\theta-\pi)} - (2\epsilon\sin\theta + \cos\theta)\right] \\ + e^{2\epsilon(\theta-\pi)}\left[(2\epsilon\sin\theta + \cos\theta)^2 + 2\left(\frac{4D}{3} - 1\right)\right] \\ + e^{2\epsilon(\pi-\theta)} \end{array} \right\}, \quad (1)$$

where $D \equiv \nu_1^2 - \nu_1 + 1$ and $\varsigma(\theta) \equiv \angle\mathbf{K} + \epsilon\ln r_p(\theta)$. Here $\angle\mathbf{K} = \arctan(\Im\mathbf{K}/\Re\mathbf{K})$ is the phase angle of $\mathbf{K}$, defined such that $\pi > |\angle\mathbf{K}|$. In the limiting case of $\epsilon = 0$, (1) provides approximate homogeneous mixed mode plastic zones which are in good qualitative agreement with detailed SSY solutions (Shih, 1974; Dong and Pan, 1988).

Since $\varsigma(\theta)$ appears on the right side of (1), this equation is in general an implicit expression for $r_p(\theta)$. ZP noted, however, that the coefficient of $\cos(\theta + 2\varsigma(\theta))$ vanishes identically at a particular angle, $\theta_0 = f(\epsilon, D)$, thus removing the implicit character of the equation (on the ray $\theta = \theta_0$). Along this ray, plastic zone growth is proportional to $\mathbf{K}\bar{\mathbf{K}}/\sigma_{ys}^2$ but independent of $\angle\mathbf{K}$.

A characteristic size of the plastic zone expression (1) is $r_{p0} = \mathbf{K}\bar{\mathbf{K}}/\pi\sigma_{ys}^2\cosh^2\pi\epsilon$. ZP (1988a) introduced the dimensionless interfacial load phase parameter $\varsigma_0 \equiv \angle\mathbf{K} + \epsilon\ln(r_{p0})$ as characterizing the local mixity by summing the load phase shift, attributable to the change in r_{p0} with increased loading, with that due to the applied loading, $\angle\mathbf{K}$. SA (1988a) independently defined a related load phase parameter, ξ, for elastic-plastic analysis of interface cracks. Under SSY conditions, these parameters are related by $\varsigma_0 = \xi - \ln\left(\pi\cosh^2(\pi\epsilon)\right)$. In view of the weak dependence of their difference on ϵ over the practical range of interface elastic constants, ς_0 and ξ are effectively identical parameterizations of mixity for locally ductile interface cracks.

Figure 2 shows finite element (FE) calculations of plastic zones for a deformation theory Ramberg–Osgood strain hardening material, with strain hardening exponent $n = 10$ adjoint to a rigid material (SA, 1988a), and the approximate plastic zones of (1) for several load levels. The calculations were performed for an interfacial Griffith type crack of length $L = 2a$ with the upper domain having $\nu_1 = 0.3$, leading to $\epsilon = .0935$ and $\theta_0 = 98.2°$. Here σ^∞ represents the remote stress normal to the crack face, and σ_0 is the reference (or yield) stress. The FE plastic zone is defined as the locus of $\bar{\sigma} = \sigma_0$. The overall sizes and shapes are well characterized by the approximation (1). Although the FE plastic zone radii are not *identically* equal at θ_0, the extent of the plastic zone in the vicinity of θ_0 is indeed approximately the same for all loadings (ς_0). Expression (1) is also in modestly good agreement with FE calculations of nonhardening bi–material plastic zones (Zywicz, 1988; ZP, 1988a–c), but the agreement is not as good as shown in Fig. 2 since ideal plasticity formally corresponds to a strain hardening exponent of $n \to \infty$, while the approximation (1) becomes precise as $n \to 1$.

CRACK FACE CONTACT

The asymptotic (relative) crack-face displacement obtained from the TF solution is

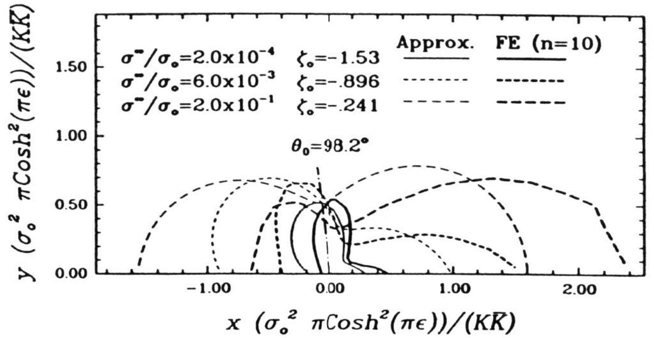

Figure 2: Griffith crack bi-material plastic zone comparisons between (1) and finite element solutions (SA, 1988a) of power law hardening material with $n = 10$. (ZP, 1988a).

(Hutchinson, *et al.*, 1987)

$$\Delta\mathbf{u}(r) \equiv \mathbf{u}(r, \theta = \pi) - \mathbf{u}(r, \theta = -\pi) = \frac{(C_1 + C_2)\mathbf{K}r^{i\epsilon}\sqrt{r}}{2\sqrt{2\pi}(1 + i2\epsilon)\cosh(\pi\epsilon)}, \tag{2}$$

where $\mathbf{u}(r) = u_y(r) + iu_x(r)$, and $C_j = (1 + \kappa_j)/\mu_j$ are elastic compliances. Inadmissible interpenetration is predicted at radii r having $\Delta u_y(r) < 0$ (Rice, 1988), or, equivalently, having $\cos\varsigma_{cf}(r) + 2\epsilon\sin\varsigma_{cf}(r) < 0$ (ZP, 1988a), where $\varsigma_{cf}(r) \equiv \angle\mathbf{K} + \epsilon\ln r$. Conversely, cracks are "elastically open" at radii having the respective inequalities reversed. The $\mathbf{K}$–field dominates the complete elasticity solution up to a distance $r_{\max} \doteq L/10$ from the tip. In order for the plastic zone not to unduly perturb the $\mathbf{K}$–field, its extent, r_{p0}, should not exceed $\sim 1/3 \times L/10$, nor should (elastic) contact occur between r_{p0} and $r_{\max}$. ZP (1988a) combined these requirements to express (in terms of ς_0) approximate $\mathbf{K}$ conditions providing SSY within the TF interface fields. SA (1988c) provided similar limits in terms of their parameter ξ.

ZP (1988b) considered the opposite extreme of SSY within the dominant elastic field of a closed frictionless interface crack and provided an approximate map of remote normal and shear loadings of a Griffith interface crack leading to SSY in both the TF and CF cases. The CF elastic interface stress fields, which depend mildly on ϵ, are $r^{-1/2}$ singular and closely resemble those of homogeneous mode II loading. A small window of loading parameters leads to SSY within a contact zone of length δ, which is itself small compared to $L/10$. In this case, K^c_{II}, the strength of the CF singularity, is $K^c_{II} = \pm\sqrt{\mathbf{K}\bar{\mathbf{K}}/\pi}$, the sign chosen depending on the sign of ϵ so as to assure compressive stress on the crack faces.

The CF SSY field is self-similar, with characteristic size $r^c_p = 3{K^c_{II}}^2/2\sigma^2_{ys}$. For ideal plastic flow (with $\sigma_{ys} = \sigma_0 = \sqrt{3}k$, where k is shear flow strength) adjacent to a rigid substrate, ZP (1988b) give the asymptotic slipline field shown in Figure 3, with characteristic angles $\alpha 1 = 29^o$, $\gamma 1 = 90^o$, $\alpha 2 = 16^o$, and $\gamma 2 = 45^o$. Asymptotic crack face pressure is $0.183\sigma_0$, while on the interface, the shear traction has magnitude k and normal (tensile) stress is $0.131\sigma_0$. The (nonzero) crack tip displacement is pure sliding parallel to the interface, with magnitude $\delta_x = 1.911J/\sigma_0$, where $J = G = \pi(C_1 + C_2){K^c_{II}}^2/16\cosh^2\pi\epsilon$.

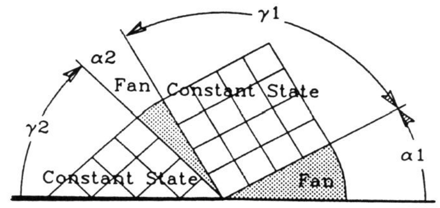

Figure 3: Asymptotic slipline field for SSY within a closed frictionless elastic interface field (ZP, 1988b).

SMALL SCALE YIELDING: TRACTION FREE CRACK FACES

Zywicz (1988) performed an extensive array of plane strain finite element analyses of ideally plastic SSY within TF $\mathbf{K}$–fields. Various combinations of elastic constants (ϵ, ν_1) and applied load phases (ς_0) were considered. Among the critical numerical details were use of the 9–node selective/reduced integration finite element (advocated by SA, 1988a), since all other element formulations examined generated highly irregular hydrostatic stress fields, thus making identification of characteristic near tip fields impossible, and generalization of Sham's (1983) boundary layer formulation to interface cracks.

Figure 4 shows the radial variation of the normalized stress components on the ray $\theta = 3.1^o$ for a material with $\nu = .342$ atop a rigid substrate ($\epsilon = 0.07796$). Proportional increase of $\mathbf{K}$ having $\angle\mathbf{K} = 0$ from zero to a value giving $\varsigma_0 = 30^o$ has taken place. The normalized radius $R \equiv r/r_{p0}$. Very deep within the plastic zone, a centered fan state is obtained, as evidenced by the equality of σ_{rr} and $\sigma_{\theta\theta}$ while $\sigma_{r\theta} = k$.

Based on extensive study of the radial and circumferential variation of stress within the plastic zones, the schematic crack tip field shown in Figure 5 was constructed to describe conditions at $R << 1$. The elastic wedge zones, of extents ξ and $\xi 1$, appear in black. Ahead of the tip a quasi–constant state region with very small (if any) curvature may occur at finite R. Angles ξ and γ are independent of R for $R << 1$, but α, η, and $\xi 1$ do vary with R, suggesting the presence of a cusp. In the limit as $R \to 0$, $\alpha \to 0$ and $\xi 1 \to 0$, suggesting that the *asymptotic* fields should be constructed with $\alpha = \xi 1 = 0$. Because fields are still changing rather significantly at extremely small R values, we emphasize the fields at a small, but *finite* radius, arbitrarily chosen as $R = \gamma_0 = k/\mu$, rather than the asymptotic limit as $R \to 0$, because there the current idealization of linear kinematics breaks down.

Figure 6 shows how the assemblage of various regions matches the finite element stress fields at $R = \gamma_0$ under the conditions of Figure 4. Agreement is good except near the transition between the elastic wedge and the back constant state. As noted by Zywicz (1988), this is likely due to residual out of plane plastic strain accumulated in the "elastic" wedge at smaller loads (and smaller ς_0), since the angles of Fig. 5 all depend on ς_0. This evolution is schematically indicated in Figure 7. For slightly negative ς_0, only a fan of 135^o and a constant state crack face of 45^o obtain, resulting in interface normal traction of $3.22\sigma_0$ *and* shear traction of magnitude k. This state of stress has a hydrostatic part much greater than the limiting Prandtl field of homogeneous fracture mechanics, and it may be expected to be particularly deleterious to the toughness exhibited by the interface.

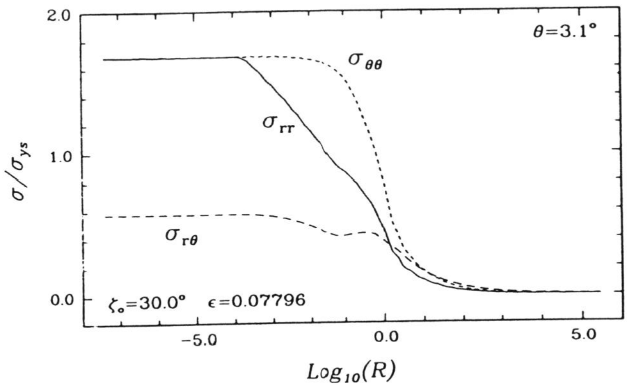

Figure 4: Normalized stress near interface *vs.* normalized radius $R = r/r_{p0}$ for $\epsilon = 0.07796$ and $\zeta_0 = 30°$.

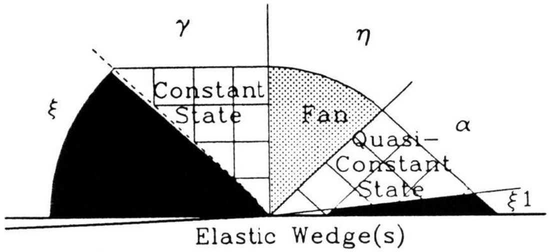

Figure 5: Schematic near tip SSY fields for traction free interface crack faces.

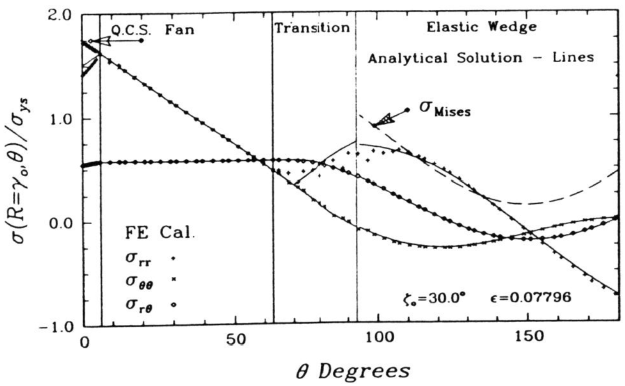

Figure 6: Matching of finite element stresses with a local field of the type shown in Fig. 5 at $R = \gamma_0$.

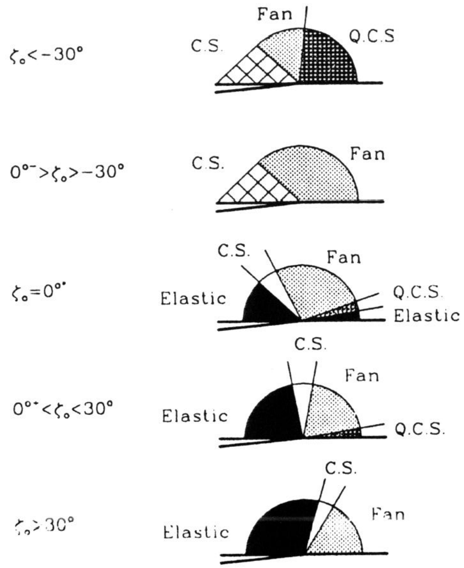

Figure 7: Schematic dependence of near tip fields on ζ_0.

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