Torsional Vibration of a Penny-shaped Crack in a Transversely Isotropic Material

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ABSTRACT

The dynamic problem of a transversely isotropic medium containing a penny-shaped crack under torsional vibration is solved in terms of an infinite integral that is evaluated through a contour integration to be discontinuous in nature. The dynamic solutions reduce to the associated static solutions when the forcing frequency vanishes for an isotropic material.

An exact expression for the dynamic stress intensity factor is obtained in terms of the frequency factor and the anisotropic material constants. The maximum values of the normalized dynamic stress-intensity factor are shown to occur at different frequencies with the same amplitude of 2.2 times the associated static stress-intensity factor. The distortion of the dynamic crack surface displacement from the associated static displacement depends also on the forcing frequency and the material anisotropy.

KEYWORDS

Penny-shaped crack; torsional vibration; transversely isotropic material; dynamic stress intensity factor; dynamic crack surface displacement.

INTRODUCTION

The scattering of a plane longitudinal wave by a penny-shaped crack in a transversely isotropic material was investigated recently (Tsai, 1987). The wave was harmonic in time, impinging normally on the crack surfaces. The maximum value of the dynamic stress-intensity factor was shown to depend on the wave frequency and the material anisotropy (Tsai, 1987). Many fiber-reinforced composite materials and platelet systems are characterized as being transversely isotropic and have fine elastic constants (Christensen, 1979; Postma, 1955). Hexagonal aeolotropic crystals are also characterized as transversely isotropic media (Elliott, 1948).

The torsional vibration of an isotropic elastic solid containing a penny-shaped crack was investigated with the method of Hankel transforms (Sih and Loeber, 1968). The problem was reduced to the solution of two simultaneous integral equations of the Fredholm type. The processes of solving the integral equations involved numerical calculations of infinite integrals. The dynamic stress field near the crack tip was shown to be singular. The singularity parameter involved was described as essential to a clear understanding of the propagation of cracks through structural components undergoing torsional oscillations (Sih and Loeber, 1968).

The dynamic response of a penny-shaped crack to a torsional oscillatory stress on the crack surface in a transversely isotropic medium is investigated in the present work. The method of Hankel transform is used to solve the equation of motion and satisfy the boundary conditions. The infinite integral involved is evaluated through a contour integration. The results reveal the discontinuous nature of the infinite integral. An exact expression for the dynamic stress intensity factor is obtained in terms of the anisotropic material constants and the loading frequency. The maximum value of the dynamic stress intensity factor is shown to occur at different frequencies for different materials. The dynamic crack surface displacement is shown to deviate significantly from the associated static crack surface. Four anisotropic materials, including composite materials and a metallic substance, are used as example materials in numerical calculations.

FORMAL SOLUTION

The stress-strain relationship in cylindrical coordinates (r, θ, z) for a transversely isotropic medium can be written in the following form (Tsai, 1987; Christensen, 1979; Postma, 1955):

$$\sigma_{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}$$

$$\sigma_{\theta\theta} = c_{12}e_{rr} + c_{11}e_{\theta\theta} + c_{13}e_{zz}$$

$$\sigma_{zz} = c_{13}e_{rr} + c_{13}e_{\theta\theta} + c_{33}e_{zz}$$

$$\sigma_{rz} = c_{44}e_{rz}, \sigma_{\theta z} = c_{44}e_{\theta z}$$

$$\sigma_{r\theta} = \frac{1}{2}(c_{11} - c_{12})e_{r\theta} \tag{1}$$

The z-axis is along the axis of symmetry of the material. A penny-shaped crack with radius a is assumed to locate inside the material, z being normal to the crack surfaces. Torsional oscillatory stresses are assumed to act in opposite directions on the crack surfaces (Sih and Loeber, 1968). The displacement field of the scattered waves generated by the crack can be described as (o, U_θ, o). If the strain-displacement relations and the stress-strain relations in Eq. (1) are used, the equations of motion have only one non-vanishing component (Sih and Loeber, 1968) and can be written in terms of the displacement as follows:

$$\frac{1}{2}(c_{11} - c_{12})\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rU_\theta)\right] + c_{44}\frac{\partial^2 U_\theta}{\partial z^2} = \Delta\frac{\partial^2 U_\theta}{\partial t^2} \tag{2}$$

where Δ is the density of the medium and t is the time variable. In response to the torsional crack surface stresses, the displacement can be written as

$$U_\theta = v\,e^{i\omega t} \tag{3}$$

where ω is the frequency of the applied shear stress. If the first-order Hankel transform is applied to Eq. (2) over the variable r, the transformed equation in terms of the parameter s has the following form:

$$\frac{\partial^2 \hat{v}^1}{\partial z^2} - \delta^2 k^2 \hat{v}^1 = 0 \tag{4}$$

$$k^2 = s^2 - \omega^2/(c_2\delta)^2, c_2 = (c_{44}/\Delta)^{1/2}, \delta^2 = (c_{11} - c_{12})/2c_{44} \tag{5}$$

where $\hat{v}^1$ is the first-order Hankel transform of v.

The boundary conditions for the scattered wavefield at z = 0 can be written as

$$\sigma_{z\theta} = -\tau_o e^{i\omega t} \qquad r \le a \tag{6}$$

$$v = \begin{cases} \phi(r), & r \le a \\ o, & r > a \end{cases} \tag{7}$$

The unknown crack surface function φ is to be determined later. The transformed equation of motion and the boundary conditions can be satisfied if the transformed displacement of the following form is chosen:

$$\hat{v}^1 = A\,e^{-\delta k z} \tag{8}$$

The equation also satisfies the radiation condition. The displacement boundary conditions in Eq. (7) require the constant A to be

$$A = \phi^1 = \int_o^a \phi(r)\,rJ_1(rs)\,dr \tag{9}$$

The stress boundary condition in Eq. (6) will determine the crack surface function φ(r).

CRACK SURFACE DISPLACEMENT

The shear stress in Eq. (1) is calculated in terms of Eqs. (8) and (9) and the result at z = 0 can be written as

$$\sigma_{z\theta} = \sigma^o_{z\theta} - \delta c_{44}\int_o^\infty \hat{\phi}^1 sJ_1(sr)(k - s)ds \tag{10}$$

$$\sigma^o_{z\theta} = -\delta c_{44}\int_o^\infty s^2 J_1(sr)\hat{\phi}^1 ds \tag{11}$$

Equation (10) reduces to Eq. (11) if the frequency tends to zero. The function φ in the reduced, associated, static equation in Eq. (11) is solved and has the following form:

$$\frac{c_{44}\delta\pi}{2}\phi_o = \frac{\partial}{\partial r}\int_r^a \frac{\xi}{\sqrt{\xi^2 - r^2}}\int_\xi^a \frac{1}{n^2}\int_o^n \frac{\partial}{\partial\lambda}\int_o^\lambda \frac{m^2\sigma^o_{\theta z}}{\sqrt{\lambda^2 - m^2}}\,dm\,d\lambda\,dn\,d\xi \tag{12}$$

An integral identity in Watson (1966) is used to obtain the above equation, which is now integrated out for $\sigma_{z\theta}{}^o = -\tau_o$, giving the following associated static crack surface function

$$\phi_o = \tau_o a\bar{\phi}_o/2\delta c_{44}, \bar{\phi}_o = \rho\,\ell n\left[(1 + \sqrt{1 - \rho^2})/\rho\right] \tag{13}$$

where ρ = r/a. The above crack shape function agrees with the associated static function in Sneddon and Lowengrub (1969).

The operations on the right-hand side of Eq. (12) are applied to Eq. (10). An integral equation for the dynamic crack surface function is established as follows:

$$\phi = \phi_o + 2r/\pi\int_r^a (\xi^2 - r^2)^{-1/2}G(\xi)d\xi \tag{14}$$

$$G(\xi) = \int_o^a \phi(\lambda)\frac{2}{\pi}\int_o^\lambda \sqrt{\lambda^2 - t^2}\,\frac{\partial}{\partial\xi}\frac{1}{\xi}M(t,\xi)dt\,d\lambda \tag{15}$$

$$M = \int_o^\infty \cos(st)\sin(s\xi)(k - s)ds \tag{16}$$

The above infinite integral is evaluated by the techniques of contour integration (Tsai, 1987; Lamb, 1904; Tsai and Kolsky, 1967) to have the following discontinuous nature:

$$M = \begin{cases} -\int_o^\beta \sqrt{\beta^2 - s^2}\, e^{-i\xi s} \cos(ts)ds, \ \xi > t \\ i \int_o^\beta \sqrt{\beta^2 - s^2}\, e^{-its} \sin(\xi s)ds, \ \xi < t \end{cases} \quad (17)$$

where $\beta = \omega/c_2\delta$. The integrations in Eq. (14) are all of finite range. In view of Eq. (17), the crack surface function is a complex-valued function. If the order of integrations between ξ and λ are exchanged, Eq. (14) becomes the Fredholm integral equation of the second kind. If the transformation $s = \omega\eta/c_2$ is used and theother integration variables are normalized by the crack radius a, Eq. (14) is reduced to an integral equation to solve for the normalized crack shape function $\phi = \phi_1 + i\phi_2 = \phi 2c_{44}/\tau_o a$ in terms of the frequency factor $\bar{k} = a\omega/c_2$. The nondimensionalized integral equation is solved numerically for the example materials, which are described in the next section.

DYNAMIC STRESS INTENSITY FACTOR

The shear stress around the crack tip is singular. To reveal this singular nature, the transform in Eq. (9) is calculated in terms of Eq. (14) (Watson, 1966) and has the following form:

$$s\hat{\phi}^1 = 2/\pi\delta \int_o^a \left| \int_o^\xi \cos(sm)dm - \xi \cos(s\xi) \right| [\pi\tau_o/4c_{44} + G(\xi)]d\xi \quad (18)$$

The shear stress in Eq. (10) is found to recover its prescribed value for $r \leq a$. For $r > a$, integrations of Eq. (11) in terms of Eq. (18) yield

$$\sigma^o_{z\theta}/\tau_o = a^2(r^2 - a^2)^{-1/2}[1 + \bar{G}]/2r - 1$$

$$+ (1 - a^2/r^2)^{1/2}[1 + \bar{G}/2]$$

$$- 2c_{44}/\pi\tau_o r \int_o^a (r^2 - \xi^2)^{-1/2}[\xi G(\xi) + r^2 G'(\xi)]d\xi \quad (19)$$

where the nondimensional quantity $\bar{G}$ is equal to $4G(a)/\pi c_{44}\tau_o$. $G'(\xi)$ is the derivative of $G(\xi)$. The second term on the right-hand side of Eq. (10) is also integrated. In terms of Eq. (19), the shear stress for $r > a$ has the following form:

$$\sigma_{z\theta} = \sigma^o_{z\theta} + 2c_{44}/\pi r \int_o^r (r^2 - \xi^2)^{-1/2}[\xi G(\xi) + r^2 G'(\xi)]d\xi \quad (20)$$

The shear stress is found to be vanishing when r tends to infinity. The singular term of the shear stress is the first term on the right-hand side of Eq. (19). From this singular term, the dynamic stress intensity factor is determined as follows:

$$K_D = K_{III}(1 + \bar{G}), K_{III} = \tau_o\sqrt{\pi a}/2 \quad (21)$$

The nondimensional function $\bar{G}$ depends on the frequency ratio $\bar{k}$ and the ratio δ of the material constants. For an isotropic material, δ reduces to unity. When $\bar{k}$ is equal to zero for an isotropic material, K_D reduces to the associated static stress intensity factor K_{III} (Sneddon and Lowengrub, 1969). It is also noticed that $\sigma_{\theta z}$ and ϕ reduce, respectively, to the associated static stress and crack shape function for an isotropic material at $\bar{k} = 0$ (Sneddon and Lowengrub, 1969).

Both composite and metallic materials are used as example materials for the study of the combined effects of the material anisotropy and theforcing frequency. Graphite/epoxy and E glass/epoxy composites have been described as transversely isotropic materials (Behrens, 1971). The material constants for graphite/epoxy composite are $c_{11} = 0.828$, $c_{33} = 8.68$, $c_{13} = 0.0285$, $c_{12} = 0.2767$, and $c_{44} = 0.4147$; for E glass/epoxy compositethey are $c_{11} = 1.493$, $c_{33} = 4.727$, $c_{13} = 0.5244$, $c_{12} = 0.6567$, and $c_{44} = 0.4745$, all in the unit of 10^4 MN/m^2 The material constants for magnesium also in 10^4 MN/m^2 are $c_{11} = 5.97$, $c_{33} = 6.17$, $c_{13} = 2.17$, $c_{12} = 2.62$, and $c_{44} = 1.64$ (Elliott, 1948). To have a wider range of real material constants for comparison, the limestone/sandstone layered system

is also used as an example material. The constants for the system in 10^4 MN/m^2 are $c_{11} = 6.25$, $c_{33} = 4.57$, $c_{13} = 1.74$, $c_{12} = 2.19$, and $c_{44} = 1.4$ (Postma, 1955).

The normalized crack shape function for all the above example materials and an isotropic material are solved from the Fredholm integral equation in Eq. (13) by using the numerical procedures devised in Baker *et al.* (1964). Typical crack surface displacements are shown in Fig. 1 for the graphite/epoxy composite and an isotropic material at the frequency factors of 0, 0.6, and 1.2. The dynamic stress intensity factors calculated from Eq. (20) are shown in Fig. 2 for a wide range of the frequency factor to reveal their maximum values. The value of the ratio δ is also given in Fig. 2.

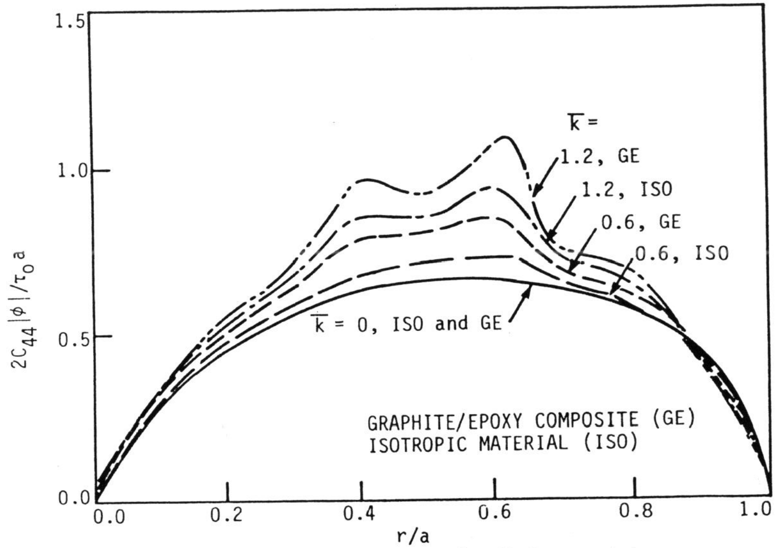

Fig. 1. Normalized crack surface displacement.

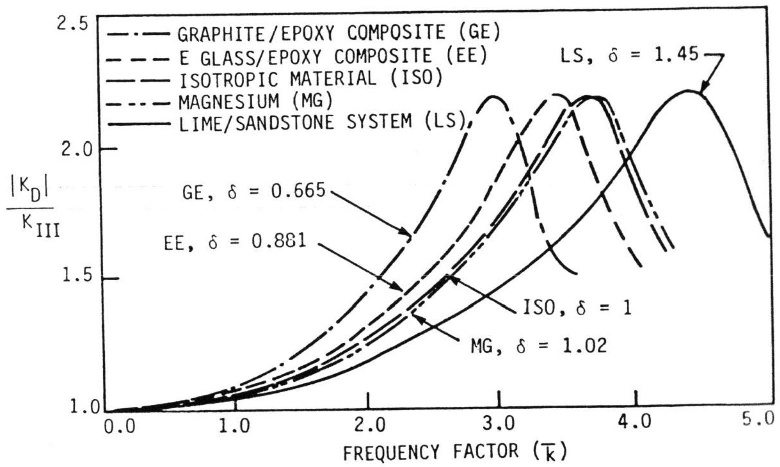

Fig. 2. Normalized stress intensity factor.

DISCUSSION AND CONCLUSIONS

The torsional vibrations of a transversely isotropic medium containing a penny-shaped crack is investigated with the method of Hankel transform. An infinite integral involved in the process of solution is evaluated through a contour integral and shown to be discontinuous in nature. The dynamic solutions reduce to the corresponding static solutions when the forcing frequency vanishes for an isotropic material.

An exact expression for the dynamic stress-intensity factor is obtained in terms of the frequency factor $\bar{k}$ and the material constant ratio $\delta = (c_{11} - c_{12})/2c_{44}$. The maximum values of the normalized dynamic stress-intensity factor are shown in Fig. 2 to be 2.2 times the corresponding static stress-intensity factors for all the example composite and metallic materials considered. The value of the frequency factor at which the maximum dynamic stress-intensity factor occurs increases if the value of δ increases.

The distortion of the dynamic crack surface displacement from the associated static displacement depends also on the values of $\bar{k}$ and δ. At the value of $\bar{k} = 1.2$, the normalized crack surface displacements for both the graphite/epoxy composite and the isotropic material are seen in Fig. 1 to have wavy forms with different amplitudes.

ACKNOWLEDGMENT

This research is supported by the Engineering Research Institute of Iowa State University.

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