Stable Path of Interacting Crack Systems and Micromechanics of Damage

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ABSTRACT

Micromechanics of damage in brittle heterogeneous materials and composites requires analysis of a system of interacting cracks. The response path of a crack system typically exhibits bifurcations such that the states on each post-bifurcation branch can be stable yet only one branch can be reached in a stable manner. Recent results on thermodynamic criteria for stable states and stable response paths of inelastic structures are reviewed and formulated in terms of the incremental internal entropy of the system. The incremental entropy, which can be expressed in terms of second order work, is then calculated for various points on the response paths of some typical symmetric crack systems. It is shown that while the symmetric states may be stable, the path which leads to them is unstable and cannot occur in reality. Generally, nonsymmetry develops at the beginning of softening. The results show that it is insufficient to model distributed cracking only by means of crack systems and linear elastic fracture mechanics. Further aspects, such as material heterogeneity, residual stresses, and cohesive fracture zones for the microcracks, might have to be taken into account.

KEYWORDS

Bifurcation; damage; entropy; fracture mechanics; stability.

INTRODUCTION

Softening damage in brittle heterogeneous materials such as concretes, rocks, and various ceramics or composites, is due chiefly to microcracking. The damage zones may or may not localize, and if they do localize the structure failure is due to macroscopic fracture which is governed by some form of nonlinear fracture mechanics with a softening cohesive zone. The analysis of damage localization and fracture in these materials requires knowledge of the properties of the damage zones containing microcracks, which may be expressed in the form of either stress-displacement relations or stress-strain relations. In view of the experimental difficulties in directly measuring these properties, micromechanics modeling is of tremendous help.

The principal vehicle of estimation of the microcracked material properties has been the study of the effect of crack systems in a material assumed to be homogeneous. At the stage of damage initiation, the cracks may be assumed to be very small and sparsely spaced, in which case their interactions may be neglected. The same is perhaps true of the final stage of damage in which the ligaments remaining between the microcracks are very small compared to the size of the microcracks, i.e., the spacing of the ligaments. However, for the intermediate stage of damage, which is of most interest, interaction between cracks and crack tips is essential. Due to complexity of the problem, some approximate but effective methods have been proposed; e.g., Collins (1962); Mróz (1966); Gross (1982); Chudnovsky et al. (1983); Horii et al. (1985); and Kachanov (1985, 1987). One important aspect, however, has so far escaped attention. It is the fact that, for many interacting crack problems, multiple solutions can be found and the response path exhibits bifurcations. The present paper, in addition to briefly reviewing some recent work on the relation of a microcrack array to a nonlocal continuum model, attempts to shed light on the types of response path bifurcation in a crack system and the method of determining the path which will actually occur.

QUASI-PERIODIC ARRAY OF NONINTERACTING CRACKS AND NONLOCAL CONTINUUM

As a simple model for damage, Bažant (1987a) analyzed a system of small and sparse noninteracting penny-shaped cracks located on a cubic lattice of spacing L, and loaded uniaxially in the direction normal to the crack planes (Fig. 1). To obtain information on the nonlocal aspects, it was necessary to consider changes from one crack plane to another, i.e., the crack array is quasi-periodic, with the crack diameters slightly varying from one layer to the next. The effective macroscopic secant compliance of this system with growing cracks has been calculated explicitly and it has been possible to satisfy exactly the conditions of continuum homogenization, consisting of compatibility of displacements and equality of work with the homogenizing macroscopic continuum.

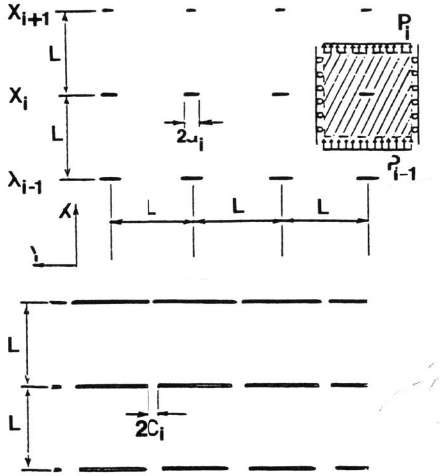

Fig. 1 - Quasi-periodic cubic array of penny shaped cracks analyzed by Bažant (1987a).

The result turned out to be a nonlocal continuum formulation of nonlocal damage type, in which only the cracking strains which cause damage are nonlocal while the elastic response is local. Such a continuum, which had previously been proposed on the basis of arguments of stability and convergence (Pijaudier-Cabot et al., 1987; Bažant and Pijaudier-Cabot, 1987, 1988; Bažant and Lin, 1987), was shown to be amenable to practical application in large finite element systems.

Having a nonlocal continuum model, it is possible to study localization of damage by continuum analysis, however, no information is obtained on the micromechanics of such localizations. To do that, the micromechanics analysis must take into account crack interactions such as the effect of the extension of one crack on the stress intensity factor of another crack. Such interactions generally give rise to multiple solutions, i.e., bifurcations of the response path.

CRITERIA OF STABLE PATH AND STABLE STATE

The existing criteria of bifurcation in inelastic systems (Hill, 1958) are limited to hardening plasticity. Moreover, they do not address the problem of stability of the response path, i.e., do not indicate which path will actualy occur after bifurcation. The existing criteria have been formulated on the basis of the mathematical requirement of uniqueness, how-ever, a more fundamental, physical approach is to use the second law of thermodynamics.

According to the second law, any system, elastic or inelastic, is stable if the internal entropy increment of the system for any kinematically admissible deviation from its initial state is negative. It is unstable if at least one increment produces a positive internal entropy increment. This approach was taken by Bažant (1987), and in a general form by Bažant (1976), who showed that the internal entropy increment of any inelastic structure (with or without softening or cracks) is expressed as:

$$\Delta S_{int} = -\frac{1}{T}\Delta W = -\frac{1}{2T}\sum_i \delta f_i \delta q_i = -\frac{1}{2T}\sum_i\sum_j K_{ij}(\underset{\sim}{\nu})\delta q_i \delta q_j \tag{1}$$

in which T = absolute temperature, ΔW = second order work, q_i = discrete displacements characterizing the state of the structure, f_i = the associated forces, and K_{ij} = the tangential stiffness matrix which, in inelastic systems, depends on the direction vector ν of loading in the n-dimensional space of q_i (i = 1,...n). Eq. 1 is true for either isothermal or isentropic conditions, in which case ΔW must, respectively, be regarded as the Helmholtz free energy or the total energy of the structure-load system, and K_{ij} is expressed on the basis of either isothermal or isentropic material properties. For plasticity, as well as for crack systems, the space of all loading directions ν is subdivided into sectors in which K_{ij} is constant, with discontinuous changes between the sectors. These sectors arise in plasticity from various combinations of loading and unloading in various parts of the structure, and in crack systems they arise through combinations of growing cracks, stationary open subcritical cracks, and closing cracks.

It can be shown that, in inelastic systems, both branches after the bifurcation point may consist of stable states; this is true for example for Shanley's column (Bažant, 1987b). The decision as to which loading path will be followed after the bifurcation point cannot then be answered by the analysis of stability of the states on the loading path. Rather, one must invoke another form of the second law of thermodynamics, which dictates that the path which will actually occur must maximize the increment of internal entropy produced along that path.

Labeling paths n = 1,2...n by superscript (n), the internal entropy increments for equilibrium movements along the path can be expressed, for displacement control (prescribed δq_i), as follows:

$$\Delta S_{int}^{(n)} = - \frac{1}{T} \Delta W^{(n)} = - \frac{1}{2T} \sum_i \delta f_i^{(n)} \delta q_i = - \frac{1}{2T} \sum_i \sum_j K_{ij}(\underset{\sim}{\nu}^{(n)}) \delta q_i \delta q_j \qquad (2)$$

and for load control (prescribed δf_i), as follows:

$$\Delta S_{int}^{(n)} = \frac{1}{T} \Delta W^{(n)} = \frac{1}{2T} \sum_i \delta f_i \, \delta q_i^{(n)} = \frac{1}{2T} \sum_i \sum_j K_{ij}(\underset{\sim}{\nu}^{(n)}) \delta q_i^{(n)} \delta q_j^{(n)} \qquad (3)$$

(Bazant 1987b). The path which actually occurs is that for which

$$\Delta S_{int}^{(n)} = \underset{(n)}{\text{Max}} \qquad (4)$$

where comparisons are now made only among all the possible equilibrium paths emanating from the same bifurcation point, and not among all possible deviations δq_i from the initial equilibrium state, as is the case for the concept of stability of state. From Eqs. 2 and 3 it follows that

$$\Delta W^{(n)} = \text{Min for displacement control}$$
$$\Delta W^{(n)} = \text{Max for load control} \qquad (5)$$

Application to geometrically perfect columns made of a hardened elastoplastic material shows that they are stable up to the reduced modulus load of Engesser and von Karman, but must start to deflect already at Shanley's tangent modulus load. For the purpose of application to interacting cracks. it can be further shown

$$\Delta W^{(n)} = \int_V \frac{1}{2} \Delta \underset{\sim}{\sigma}^{(n)} : \Delta \underset{\sim}{\epsilon}^{(n)} \, dV \qquad (6)$$

in which V = volume of the body, and $\Delta\sigma^{(n)}$ and $\Delta\epsilon^{(n)}$ are the increments of the stress and strain tensors along the loading path.

PATH OF PROPAGATION OF INTERACTING CRACK SYSTEMS

We suppose that the microcracks follow linear elastic fracture mechanics. The second-order incremental work ΔW can then be easily determined by elastic finite element analysis. For those situations for which the dependence of the stress intensity factors on the crack lengths can be found in handbooks, finite element analysis is unnecessary, as shown by Bažant (1987c; also see Appendix). Squaring the stress intensity factor yields the energy release rate of each crack tip, integration along the crack length then yields the total energy release, and its differentiation with respect to the applied load then yields, according to Castigliano's theorem, the additional displacement due to cracks, which may then be used to evaluate ΔW.

Figures 2-4 show the results for some elementary cases with two interacting crack tips. Initially the system is symmetric and the stress intensity factors at the crack tips are equal to the fracture toughness $K_c (K_1 = K_2 = K_c)$. The stress intensity factors used in Fig. 2 were obtained from Murakami's handbook (1987). However, for Figs. 3-4 they were obtained using finite elements. As the basic path (n = 1) we consider the symmetric crack propagation with equal crack extensions ($\delta a_1 = \delta a_2 > 0$ and $\delta K_1 = \delta K_2 = 0$). The bifurcated path (n = 2) corresponds to one crack growing while the other is stationary and its stress intensity factor drops below the critical value ($\delta a_1 > 0, \delta a_2 = 0$, and $\delta K_1 = 0, \delta K_2 < 0$). The third path (n=3) is the unloading path ($\delta a_1 = \delta a_2 = 0$,and $\delta K_1 = \delta K_2 < 0$). In most cases the states on the symmetric response path are unstable ($\Delta W < 0$ for some admissible deviations from the initial state). However, if the displacement is controlled such that

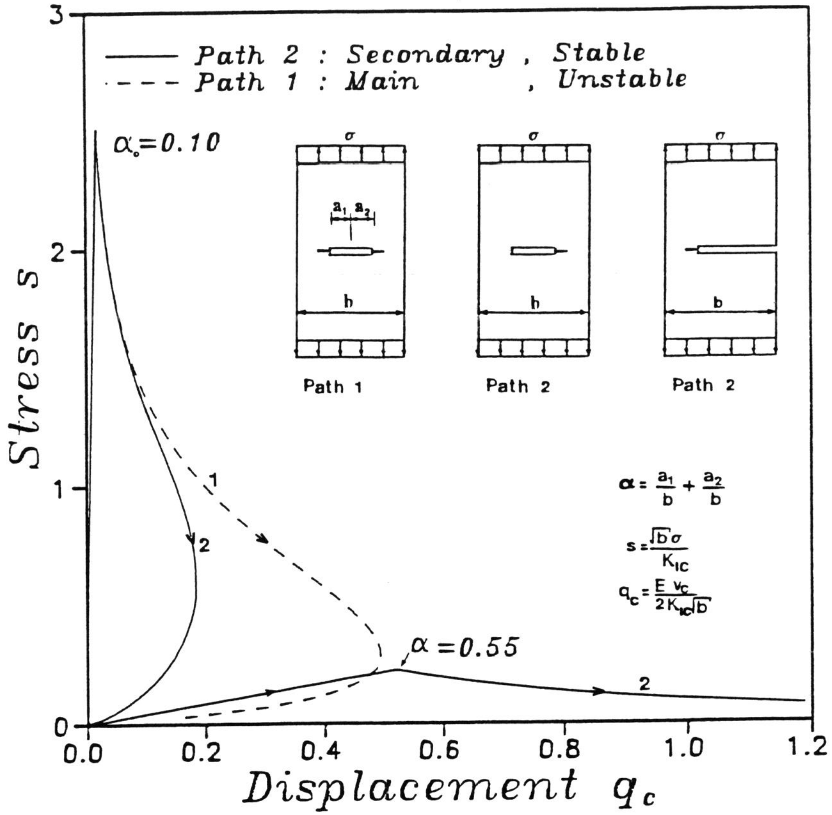

Fig. 2 - Stable (solid curve) and unstable (dashed curve) response path of initially symmetric center crack in a tensioned strip.

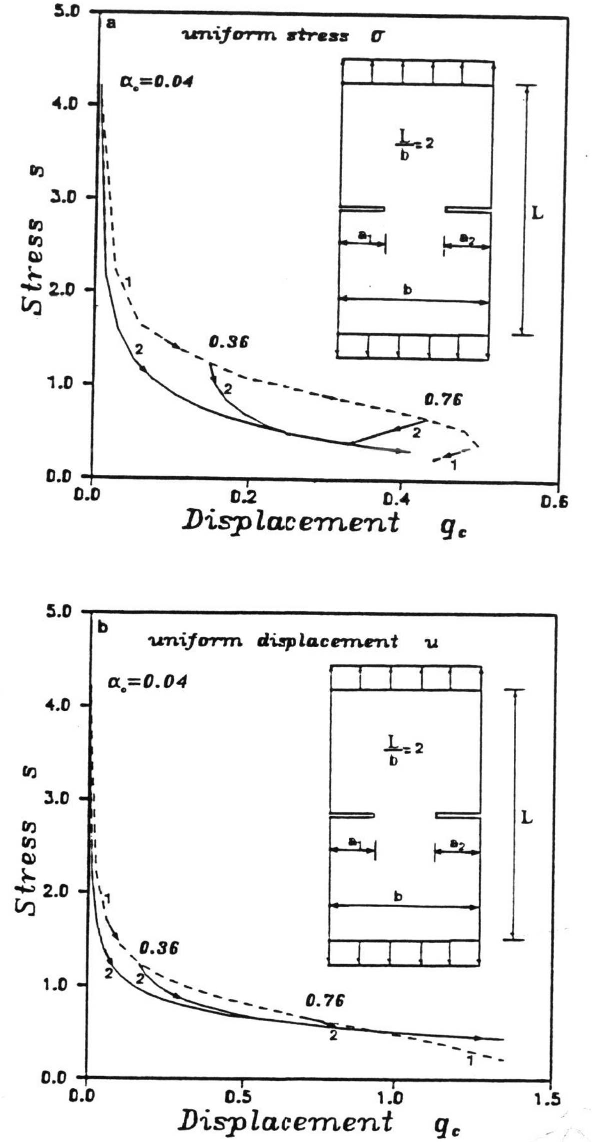

Fig. 3 - (a),(b) Stable (solid curve) and unstable (dashed curve) response path of initially symmetric edge cracks in a tensioned plate (L/b=2).

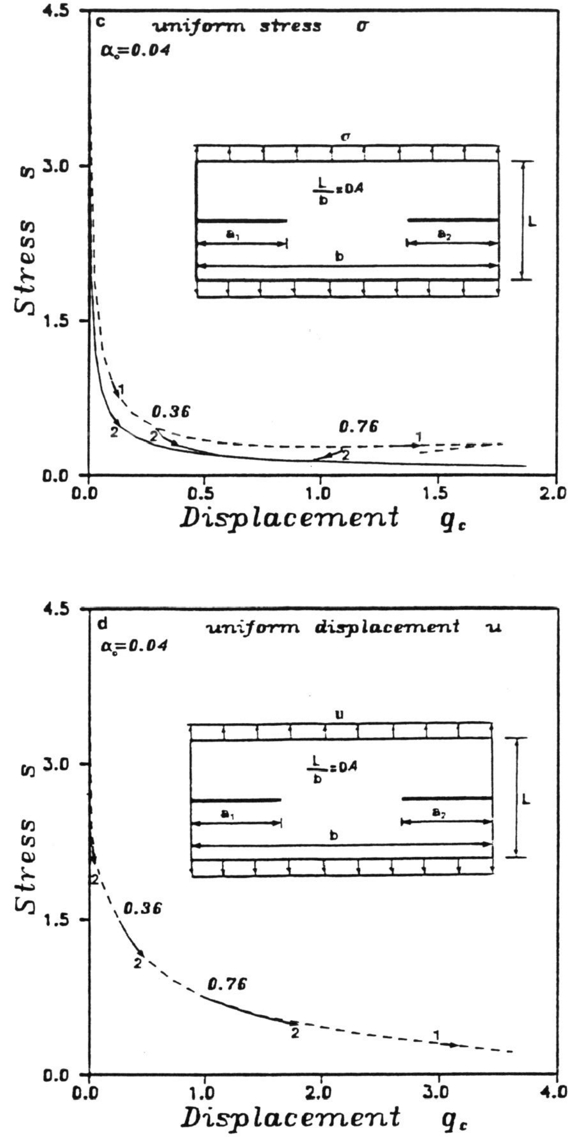

Fig. 3 - (c),(d) Stable (solid curve) and unstable (dashed curve) response path of initially symmetric edge cracks in a tensioned plate (L/b=0.4).

positive deviations are prevented, then the state could be stable. Figure 5 shows the variation of ΔW as a function of $\delta u/b$ and $\delta\theta$, where δu = average displacement variation due to the crack extension and $\delta\theta$ = rotation variation due to the crack extension (Fig. 4a) . Denoting δm and δf as the corresponding moment and force variations for $\delta\theta$ and δu respectively, we therefore have

$$\Delta W = \delta^2 W = \frac{1}{2}\,\delta m \delta\theta + \frac{1}{2}\,\delta f \delta u \qquad (7)$$

The equilibrum paths for δm=0 are shown in Figs. 4b and 5 and are labeled as 1,2, and 3. Depending on the control variable, as indicated in Eq. 5, one of the paths will occur; it is called the stable path. Table 1 shows the stable path for the example presented in Fig. 4. Figure 5b shows that α_0 = 0.44 is the critical state since for δu = 0 the value of ΔW is 0. Furthermore, Fig. 5a shows that for α_0 = 0.20 the state is stable ($\Delta W > 0$). On the other hand, for α_0 = 0.76 in Fig. 5c the state is unstable ($\Delta W < 0$).

Table 1. - Stale Paths in Fig. 4b

Initial Crack	Type of Control	Stable path
$\alpha_0 < 0.44$	$\delta q_c > 0$	2
	$\delta q_c < 0$	3
	$\delta s > 0$	*
	$\delta s < 0$	3

*No equilibrium path (none is stable)

Similar results have been obtained for other crack systems, including cracks on parallel planes in strips or a space, as in Fig. 1. It transpires that while the symmetric crack states exemplified in Figs. 2-4 are stable, under displacement control conditions, the loading path which leads to them is unstable. The states on the unstable path cannot be obtained in a continuous loading process. They can only arise by some other means. It seems that stability of the crack path generally requires localization of crack systems into a single crack with a single crack tip; a row of cracks on a line, or multiple rows of cracks, apparently cannot arise on a stable loading path. If this conclusion is generally true, it would force us to revise the approach to micromechanics of damage.

Multiple cracks on a row and multiple rows of cracks no doubt exist in real materials, as evident from various optical observations as well as measurements of the locations of sound emissions. It seems necessary to conclude that such situations cannot be adequately modeled as cracks behaving according to linear elastic fracture mechanics if a homogeneous continuum is assumed. It might be necessary to include in the analysis of microcrack systems material inhomogeneties (e.g., aggregate pieces whose elastic properties differ from the matrix), and take into account residual stresses due to shrinkage or thermal effects. In this regard, it will be also necessary to clarify physically why systems of symmetric or periodic cracks can apparently grow in a stable fashion in real materials.

CONCLUSION

In the analysis of interacting crack systems, it is important to distinguish between stable states and stable response path. A path consisting of stable states need not be a stable path, and if so it cannot occur in a continuous loading process. The criteria of stable state and stable path can be based on the second law of thermodynamics. These criteria can be reduced to maximization or minimization of the second-order work done on the structure. Numerical studies reveal that various symmetric crack systems which are stable cannot be reached along a stable path. This might require broadening the scope of micromechanics analysis of damage by including on the micro-level consideration of material inhomogeneties, residual stresses and

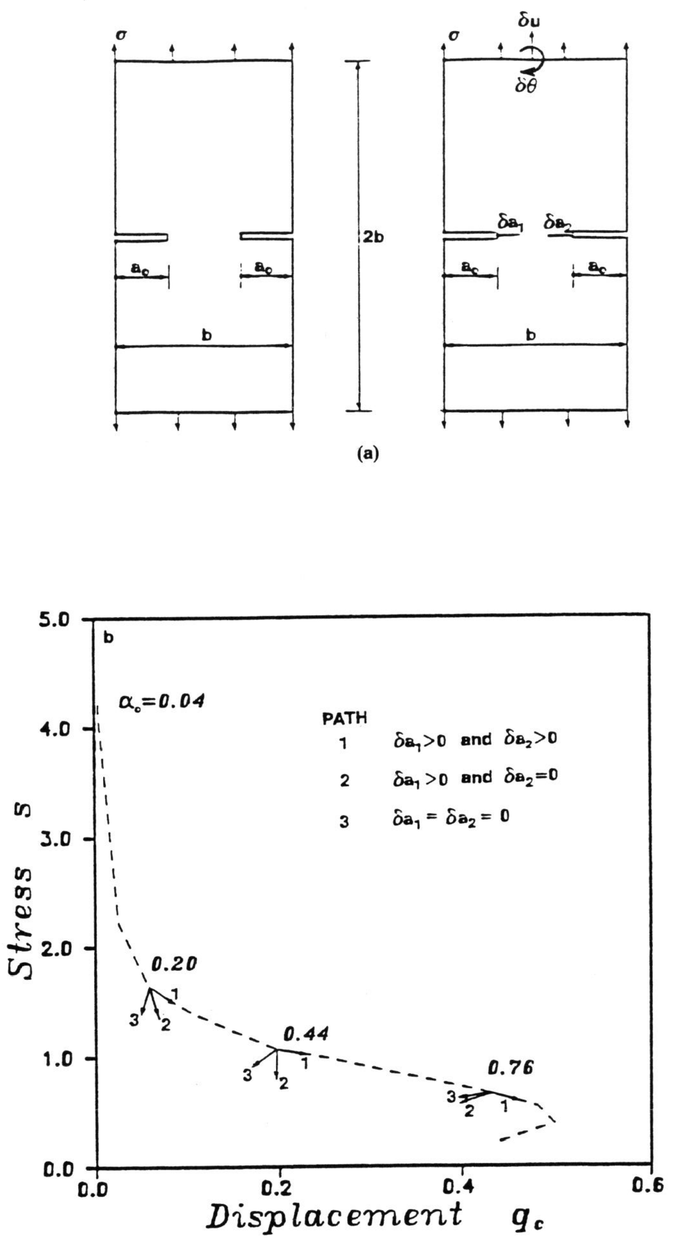

Fig. 4 - (a) Initially symmetric edge cracks in a tensioned plate.(b) Bifurcations from the main equilibrium path at different values of α_0.

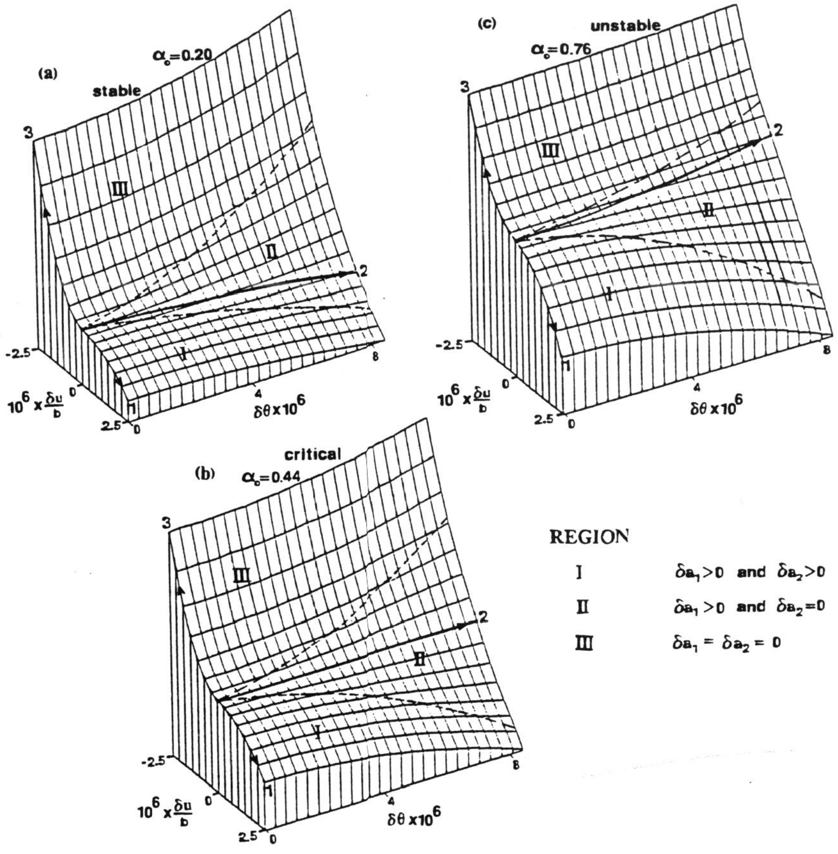

Fig. 5 - Surfaces of second order work (negative internal entropy increment) for initially symmetric edge cracks in a tensioned plate, corresponding to Fig. 4 for: (a) α_0 = 0.20, (b) α_0 = 0.44, and (c) α_0 = 0.76.

existence of cohesive zones at crack tips. However, this does not necessarily require abandoning linear elastic fracture mechanics for the modeling of the microcracks, even though it is certainly inapplicabale on the macrocrack level.

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APPENDIX

Calculation of Displacement Due to Cracks from the Stress Intensity Factor

Based on linear elastic fracture mechanics, the stress intensity factor K_I can be expressed as $K_I = \sigma\sqrt{b}\, f(\alpha_1, \alpha_2)$, where $\alpha_1 = a_1/b$, $\alpha_2 = a_2/b$, and $f(\alpha_1, \alpha_2)$ is a nondimensional function which depends on the geometry of the specimen. The strain energy due to the crack can then be calculated by:

$$U_c = \int_0^{\alpha} \frac{K_I^2}{E}\, b\, d\alpha = \frac{\sigma^2 b^2}{E}\, g(\alpha_1, \alpha_2) \qquad (8)$$

where $g(\alpha_1, \alpha_2) = \int_0^{\alpha} f^2(\alpha_1, \alpha_2)\, d\alpha$ and $\alpha = \alpha_1 + \alpha_2$.

As used by Bažant (1987c), the load-point displacement due to the cracks is according to Castigliano's theorem,

$$v_c = \frac{\partial U_c}{b\,\partial\sigma} = \frac{2\sigma b}{E}\, g(\alpha_1, \alpha_2) \qquad (9)$$

Under the assumption that the fracture toughness K_{Ic} is a constant, crack propagation occurs when at least one stress intensity factor at the crack tip reaches a value equal to K_{Ic}. Thus, the nondimensional stress and displacement for crack propagation are given by the following expressions:

$$s = \frac{\sigma\sqrt{b}}{K_{IC}} = \frac{1}{f(\alpha_1, \alpha_2)}; \qquad q_c = \frac{1}{2}\, \frac{Ev_c}{K_{Ic}\sqrt{b}} = s g(\alpha_1, \alpha_2) \qquad (10)$$