On Some Recent Advances in Computational Methods in the Mechanics of Fracture

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Summary

This paper provides an overview of some of the recent (1984-1988) developments in analytical/computational methods in the mechanics of fracture: (i) solution methods for the traction boundary integral equations for arbitrary-shaped cracks embedded in infinite solids, the crack-faces being subjected to arbitrary tractions; (ii) analytical solutions for elliptical or circular cracks embedded in isotropic or transversely isotropic solids (the crack-plane being at an arbitrary angle to the axis of transverse isotropy), with crack-faces being subjected to arbitrary tractions; (iii) finite-element or boundary-element alternating methods for two-dimensional crack-problems, as well as for three-dimensional problems of surface flaws in finite-dimensional structures of linear elastic solids; (iv) domain-integral methods in elastic-plastic or inelastic crack mechanics; and (v) methods for generation of weight-functions in 2 and 3-D linear elastic crack problems.

Introduction

The starting point for this overview is the monograph on "Computational Methods in the Mechanics of Fracture" edited by Atluri (1986), with contributions by several noted researchers. The various articles in that monograph were prepared by individual authors in the 1984-85 time frame, with the material being current mostly as of 1984. In the present paper, some recent advances in computational fracture mechanics in the intervening years are summarized. The coverage of topics is limited to those listed in the "Summary" above; and furthermore the scope of the article is limited by the authors' own interests. Furthermore, space limitations have precluded the treatment of computational methods in viscoplastic dynamic crack propagation analysis, a subject of current interest in general, as well as to the authors.

In each of the topics listed in the "Summary", a reasonably self-contained account of the generic issues, and the progress made in addressing them, is provided below.

1 Embedded Cracks in an Infinite Linear Elastic Solid

An important class of problems in hydraulic fracturing and in earth-quake control, involves the analysis of flat cracks of *arbitrary shapes* (rectangular, elliptical, and circular, being special cases), *embedded* in a linear elastic solid of infinite dimensions. Here, the generic problem of interest is that of arbitrary normal as well as shear tractions on the crack-faces themselves. Let the crack-surfaces (upper and lower) be in the planes $\xi_3 = \pm 0$, in a coordinate system with ξ_3 normal to the plane of the crack, and the boundary of the crack being described by an arbitrary curve in the ξ_1, ξ_2 plane. The tractions on the crack faces are t_j^+ and t_j^-, respectively (t_3 the opening mode and t_α, $\alpha = 1,2$ the shear modes), such that $t_j^+ + t_j^- = 0$. Let A be the area of the crack.

The problem for an *arbitrary-shaped* crack in an infinite solid is most conveniently formulated in the form of traction-boundary-integral equations [Cruse (1975); Bui (1977); Weaver (1977); Clifton and Abou-Sayed (1981); Ioakimidis (1982)]. However, the numerical solution of these integral equations has, until recently, not been very satisfactory. Some important contributions to resolve these numerical difficulties have been made recently [Polch, Cruse, and Huang (1987); Gu and Yew (1988); and Okada, Rajiyah, and Atluri (1988)]. A brief discussion of this state-of-the-science is given below.

The well-known integral representation for displacements in a 3-D linear elastic solid [see Cruse (1969), for instance] is:

$$u_i(\boldsymbol{x}) = \int_{\partial\Omega} [t_j(\boldsymbol{\xi})u_{ij}^*(\boldsymbol{\xi},\boldsymbol{x}) - u_j(\boldsymbol{\xi})t_{ij}^*(\boldsymbol{\xi},\boldsymbol{x})]\,\mathrm{d}A(\boldsymbol{\xi}) \tag{1}$$

where $\boldsymbol{x}$ is a point inside the 3-D domain, and $\boldsymbol{\xi}$ is a point at the boundary $\partial\Omega$, and $\mathrm{d}A(\boldsymbol{\xi})$ is a differential area in $\partial\Omega$ centered at $\boldsymbol{\xi}$. In the case of the infinite solid with a crack, with the only applied tractions being on the crack faces, it is seen that $\partial\Omega = A^+ + A^-$ where $(+)$ and $(-)$ refer to the upper and lower sides of the crack, respectively. In (1), u_{ij}^* and t_{ij}^* are the displacement and traction kernels respectively, from the fundamental solution [Kelvin's] of the problem of a point load in an infinite solid [Cruse (1969)]. When (1) is differentiated with respect to x_m, one has:

$$u_{i,m}(\boldsymbol{x}) = \int_{\partial\Omega} [t_j(\boldsymbol{\xi})u_{ij,m}^*(\boldsymbol{\xi},\boldsymbol{x}) - u_j(\boldsymbol{\xi})t_{ij,m}^*(\boldsymbol{\xi},\boldsymbol{x})]\,\mathrm{d}A(\boldsymbol{\xi}) \tag{2}$$

where $(\ )_{,m} = \partial(\ )/\partial x_m$. When $\boldsymbol{x} \to \boldsymbol{\xi}$ (where $\boldsymbol{\xi}$ is on $\partial\Omega$), Eq. (2) involves a *hyper singular* kernel $t_{ij,m}^*(\boldsymbol{\xi},\boldsymbol{x})$ (in the limit as $\boldsymbol{x} \to \boldsymbol{\xi}$). This is the major source of numerical difficulties in the traction boundary integral equation method.

Since $t_j^+ + t_j^- = 0$ on the crack faces $A^\pm$, and also since the kernels u_{ij}^* and t_{ij}^* have the properties $u_{ij}^{*+} = u_{ij}^{*-}$; and $t_{ij}^{*+} = -t_{ij}^{*-}$, Weaver (1977) has noted that Eq. (1) and Eq. (2) may be simplified as:

$$u_i(\boldsymbol{x}) = -\int_{A^+} \Delta u_j(\boldsymbol{\xi})t_{ij}^{*+}(\boldsymbol{\xi},\boldsymbol{x})\,\mathrm{d}A(\boldsymbol{\xi}) \tag{3}$$

and

$$u_{i,m}(\boldsymbol{x}) = -\int_{A^+} \Delta u_j(\boldsymbol{\xi})t_{ij,m}^{*+}(\boldsymbol{\xi},\boldsymbol{x})\,\mathrm{d}A(\boldsymbol{\xi}) \tag{4}$$

where $\Delta u_j = u_j^+ - u_j^-$. Also, since $\partial(t_{ij}^*)/\partial x_m \equiv -\partial(t_{ij}^*)/\partial\xi_m$, as noted by Weaver (1977), one may write (4) as:

$$\begin{aligned} u_{i,m}(\boldsymbol{x}) &= +\int_{A^+} \Delta u_j(\boldsymbol{\xi})\frac{\partial t_{ij}^{*+}}{\partial \xi_m}\,\mathrm{d}A(\boldsymbol{\xi}) \\ &= -\int_{A^+} \frac{\partial \Delta u_j}{\partial \xi_m} t_{ij}^{*+}\,\mathrm{d}A(\boldsymbol{\xi}) \end{aligned} \tag{5}$$

since Δu_j is zero at ∂A^+. The limit as $\boldsymbol{x} \to \boldsymbol{\xi}$ can be taken in (5b) in the Cauchy principal value sense. By using the (stress)-(displacement-gradient) relations, and taking the limit as $\boldsymbol{x}$ falls on A^+, Weaver (1977) has derived the following traction integral equations *on the crack-face* [$\boldsymbol{x}$ at A^+ and $\boldsymbol{\xi}$ at A^+]

$$\sigma_{33}(\boldsymbol{x}) = \frac{\mu}{4\pi(1-\nu)}\int_{A^+} \frac{\partial r}{\partial x_\beta}\frac{\partial \Delta u_3}{\partial \xi_\beta}\frac{1}{r^2}\,\mathrm{d}A(\boldsymbol{\xi}) \tag{6}$$

$$\begin{aligned} \sigma_{13}(\boldsymbol{x}) = \frac{\mu}{4\pi(1-\nu)}\int_{A^+} \Bigg[&\frac{1}{r^2}\frac{\partial r}{\partial x_1}\frac{\partial \Delta u_\beta}{\partial \xi_\beta} \\ &+\frac{(1-\nu)}{r^2}\left(\frac{\partial \Delta u_1}{\partial \xi_2} - \frac{\partial \Delta u_2}{\partial \xi_1}\right)\frac{\partial r}{\partial x_2}\Bigg]\,\mathrm{d}A(\boldsymbol{\xi}) \end{aligned} \tag{7}$$

$$\begin{aligned} \sigma_{23}(\boldsymbol{x}) = \frac{\mu}{4\pi(1-\nu)}\int_{A^+} \Bigg[&\frac{1}{r^2}\frac{\partial r}{\partial x_2}\frac{\partial \Delta u_\beta}{\partial \xi_\beta} \\ &+\frac{(1-\nu)}{r^2}\left(\frac{\partial \Delta u_2}{\partial \xi_1} - \frac{\partial \Delta u_1}{\partial \xi_2}\right)\frac{\partial r}{\partial x_2}\Bigg]\,\mathrm{d}A(\boldsymbol{\xi}) \end{aligned} \tag{8}$$

where $r^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2$.

When $\sigma_{j3}(\boldsymbol{x})$, $j = 1,2,3$ are specified on the crack-faces, Eqs. (6), (7), (8) represent singular integral equations of the first-kind for the *gradients* of the crack-surface displacements, i.e., $\partial u_i/\partial\xi_\alpha$ [$i = 1,2,3$; $\alpha = 1,2$] at the crack-surface, A^+.

A direct numerical attack on Eqs. (6–8) was attempted by Polch, Cruse, and Huang (1987). The germane issues in such a numerical method are: (i) the presence of *derivatives* of the trial functions u_j in ξ_α directions at A^+; (ii) the modeling of the singular behaviour of the trial field for the gradients $\partial u_i/\partial\xi_\alpha$ [which may vary as $\sqrt{1/R}$ where R is the distance normal to the crack-front, in the crack-plane]; and (iii) the treatment of the singular kernels,

$$\frac{1}{r^2}\frac{\partial r}{\partial x_\beta} = -\frac{\partial}{\partial x_\beta}\left[\frac{1}{r}\right] \tag{9}$$

in the limit as $x_\beta \to \xi_\beta$.

In the direct numerical treatment of Polch, Cruse and Huang (1987), a proper handling of the principal values of integrals involving the kernels of the type (9) requires a *continuous interpolation* of direct *trial functions for the 6 gradients* $u_{i,\alpha}$ at A^+. Thus, by discretizing A^+ into finite elements, these authors introduce C^0 type shape functions directly for $\Delta u_{i,\alpha}$ at A^+. However, if $\Delta u_{i,\alpha}$ is viewed as an independent variable at A^+, *at each node at* A^+ there are 6 unknown variables, and only 3 given quantities (σ_{i3}). To overcome this inconsistency, these authors introduce two sets of shape functions for

$\Delta u_{i,\alpha}$: (i) one an independent C^0 interpolation in terms of nodal values of $\Delta u_{i,\alpha}$ (6 per node); and (ii) an independent C^0 interpolation only for Δu_i (in terms of nodal values of Δu_i, 3 per node) from which the derivatives $\Delta u_{i,\alpha}$ are derived by differentiation. However, $\Delta u_{i,\alpha}$ derived in this fashion is discontinuous at inter-element boundaries in A^+. In elements near the crack front, the shape functions for u_i have $\sqrt{R}$ variation, and those for $u_{i,\alpha}$ have $(1/\sqrt{R})$ variation. The six independent values of $\Delta u_{i,\alpha}$ at each node are sought to be related to the 3 independent values of Δu_i at each node, through a least-squares technique, which seeks to minimize the integral of square of the error in $\Delta u_{i,\alpha}$ over A^+ (which is defined as the difference between $\Delta u_{i,\alpha}$ values determined from the two interpolation schemes described above), with respect to the nodal values of $(\Delta u_{i,\alpha})$ in interpolation (i). With the nodal values of Δu_i alone as the ultimate independent variables, the usual boundary element collocation procedure can continue. If interpolation (i) is written as:

$$\Delta u_{i,\alpha}(\xi) = M^j(\xi)\Delta u^j_{i,\alpha}; \qquad \xi \in A^+ \tag{10}$$

where $j = 1, \ldots n$ are nodes at A^+; and $M^j(\xi)$ are the usual finite element (local) shape functions written as equivalent globally valid functions; and $\Delta u^j_{i,\alpha}$ are nodal values of $\Delta u_{i,\alpha}$ at the node $j(1, \ldots m)$. When (10) is introduced in (6–8), a generic singular integral that arises, is of the form:

$$\int_{A^+} M^j(\xi)\frac{1}{r^2}\frac{\partial r}{\partial x_\beta}\,\mathrm{d}A(\xi) \tag{11}$$

$$\equiv \int_{\Gamma_j} M^j(\xi)\frac{1}{r^2(x,\xi)}\frac{\partial r(\xi,x)}{\partial x_\beta}\,\mathrm{d}A(\xi) \tag{12}$$

where Γ_j is a patch of elements over the node j, and (12) follows from (11) since in the usual finite element sense, $M^j(\xi)$ is non-zero over only Γ_j. For an arbitrary-shaped patch Γ_j, the integral (12) may be written as equal to:

$$\int_{\Gamma_j}\frac{M^j(\xi)-M^j(x)}{r^2(x,\xi)}\frac{\partial r(\xi,x)}{\partial x_\beta}\,\mathrm{d}A(\xi) + M^j(x)\int_{\Gamma_j}\frac{1}{r^2(x,\xi)}\frac{\partial r(r,x)}{\partial x_\beta}\,\mathrm{d}A(\xi). \tag{13}$$

In the limit as $x \to \xi$, in the first term in the integral (13), the quantity $M^j(\xi) - M^j(x)$ is of $O(r)$. Thus the integral of the first term is $O(1/r)$, which can be evaluated numerically. The second term in (13) contains a non-integrable singularity, but can be converted to a line integral through the divergence theorem:

$$\int_{\Gamma_j}\frac{1}{r^2}\frac{\partial r}{\partial x_\beta}\,\mathrm{d}A = -\int_{\Gamma_j}\frac{\partial}{\partial x_\beta}\left(\frac{1}{r}\right)\mathrm{d}A = -\int_{\partial\Gamma_j} n_\beta\frac{1}{r}\,\mathrm{d}\ell \tag{14}$$

where $\partial\Gamma_j$ is the curve bounding Γ_j; and n_β are components of a unit normal to $\partial\Gamma_j$ in the plane A^+, and $\mathrm{d}\ell$ is the differential arc-length along $\partial\Gamma_j$. Recall that the special treatment as in Eqs. (13) and (14) is needed when the source point x is inside the patch Γ_j at node j. Polch et al. [1987] subdivide $\partial\Gamma_j$ into a large number (up to 100) straight line segments. Thus, while a precise convergence study of the various numerical approximations is lacking, it is nevertheless commendable that the rather complicated

procedures employed have been shown to work in a few simple buried crack problems [Polch et al. (1987)].

A remarkable simplification results if one drops the idea of satisfying the boundary-integral equation at a few points at A^+ in the sense of collocation, and use, instead, a direct *weak solution* of Eqs. (6–8). This procedure, which mitigates the difficulties with singular integrals, has been suggested and numerically implemented by Gu and Yew (1988). The penalty, however, is that *double integrals* over A^+ should be evaluated; and the procedure is inherently limited to the case of an *infinite solid* with an *embedded crack* with the only applied tractions being on the crack-faces. Consider, for instance, the mode I problem of Eq. (6). Let Δu_3 be the *trial solution* that is assumed at A^+. Rather than satisfying (6) through *point-collocation*, Gu and Yew (1988) try to satisfy it in a weak (or integrated average) sense, by introducing *test functions* Δv_3 at A^+. Thus, the weak form of (6) becomes:

$$\int_{A^+}\sigma_{33}(x)\Delta v_3(x)\,\mathrm{d}A(x) = \frac{\mu}{4\pi(1-\nu)}\int_{A^+}\left[\int_{A^+}\frac{\partial r}{\partial x_\beta}\frac{\partial\Delta u_3}{\partial\xi_\beta}\frac{1}{r^2}\,\mathrm{d}A(\xi)\right]\Delta v_3(x)\,\mathrm{d}A(x) \tag{15}$$

The test function Δv_3 is chosen to have the same nature as the trial function Δu_3, i.e., it is continuous in A^+, and *vanishes* at ∂A^+. The *double-integral* on the right-hand-side of (15) involves the generic term (by interchanging the order of integration):

$$\int_{A^+}\Delta v_3(x)\frac{\partial r(x,\xi)}{\partial x_\beta}\frac{1}{r^2}\,\mathrm{d}A(x) = -\int_{A^+}\Delta v_3(x)\frac{\partial}{\partial x_\beta}\left[\frac{1}{r(x,\xi)}\right]\mathrm{d}A(x) \tag{16}$$

By a careful analysis of the singular integral on the r.h.s. of (16), Gu and Yew (1988) show that:

$$-\int_{A^+}\Delta v_3(x)\frac{\partial}{\partial x_\beta}\left[\frac{1}{r(x,\xi)}\right]\mathrm{d}A(x) = \int_{A^+}\frac{1}{r(x,\xi)}\frac{\partial\Delta v_3(x)}{\partial x_\beta}\,\mathrm{d}A(x) \tag{17}$$

Thus, the differentiation of the singular term $(1/r)$ has been transferred to the test function Δv_3. The use of (16) and (17) in (15) leads to:

$$\int_{A^+}\sigma_{33}(x)\Delta v_3(x)\,\mathrm{d}A(x) = \frac{\mu}{4\pi(1-\nu)}\int_{A^+}\int_{A^+}\frac{1}{r(x,\xi)}\frac{\partial\Delta v_3}{\partial x_\beta}\frac{\partial\Delta u_3}{\partial\xi_\beta}\,\mathrm{d}A(\xi)\,\mathrm{d}A(x) \tag{18}$$

The inner-integral on the right hand side of (18) is an improper integral with a removable singularity. When transformed to polar coordinates and the origin of coordinates is placed at the singular point, this inner integral is simply transformed to a regular integral — and thus is easily amenable to a numerical integration.

For analyzing arbitrary-shaped, embedded, or surface cracks in *finite-sized structural* components, the Schwartz-Neumann alternating method, based on the principle of linear superposition, may be used. Two generic solutions are needed in this alternating technique: (i) a boundary-element method for the stress analysis of the finite structure subjected to the given external loading, *but without the crack*; (ii) the solution for embedded crack in an infinite solid, the crack faces being subjected to tractions that are equal and opposite to those at the location of the crack in an otherwise uncracked structure, as determined from solution (i). The solution (ii) would lead to residual tractions at the location of the finite boundaries of the given structure, in an otherwise infinite solid. It is thus seen that an iterative use of solutions (i) and (ii) alternatively, leads, through a linear superposition principle, to the desired solution.

For generating solutions (ii) in the above sketched alternating method, the traction B.I.E. formulation of Eqs. (6, 7, 8) are useful. However, for the solution of problem (i), the traction B.I.E. method based on equation (2) is not convenient, as it leads to hypersingular integrals at the boundary. It is in this context that the recent work by Okada, Rajiyah, and Atluri (1988a,b), in developing new integral equations for displacement-gradients directly, is useful. For linear elasticity, these authors have developed (1988a) the following representation directly for $u_{i,k}$:

$$\begin{aligned} u_{i,k}(\boldsymbol{x}) &= \int_{\partial\Omega} [n_k(\boldsymbol{\xi}) E_{ijpq} u_{p,q}(\boldsymbol{\xi}) u^{\star}_{j\ell,i}(\boldsymbol{x},\boldsymbol{\xi}) \\ &\quad - t_j(\boldsymbol{\xi}) u^{\star}_{j\ell,k}(\boldsymbol{x},\boldsymbol{\xi}) - u_{p,k}(\boldsymbol{\xi}) t^{\star}_{p\ell}(\boldsymbol{x},\boldsymbol{\xi})]\, \mathrm{d}A(\boldsymbol{\xi}) \end{aligned} \tag{19}$$

Unlike the usual representation given in Eq. (2), the new representation in (19) involves the kernels $u^{\star}_{j\ell,i}$ and $t^{\star}_{p\ell}$ which have the same order of singularity (which is also *less than* the order of singularity in $t^{\star}_{ij,m}$ in Eq. (2)) and are quite tractable from a numerical point of view. Okada, Rajiyah, and Atluri (1988a) have demonstrated the superior accuracy in the computed stresses, as well as the ease of computation itself, using the displacement gradient representation in Eq. (19). Similar velocity-gradient representations for small as well as large-strain elasto-plasticity were given in Okada, Rajiyah, and Atluri (1987), and implemented by them [Okada et al. (1988b)].

Another noteworthy development of integral equation methods of relevance in fracture mechanics is due to Benitez and Rosakis (1988). It is a specialization of the integral equation method for 3-D elasticity, for problems involving *cylindrical regions* i.e., bodies with a generator and identical cross-section at any location along the generator. This formulation uses the fundamental solution for an *infinite elastic plate of uniform thickness*. They show that the integral identities, corresponding to the class of problems which involve cylindrical regions, and with the cross-sections at the *ends* of the generator being *traction-free*, contain only integrals evaluated over the lateral surfaces of the cylindrical region. For instance in the analysis of surface cracks or through cracks in *plates of uniform thickness*, only the lateral surfaces of the cylindrical region need be discretized through boundary elements, with no elements being needed for the parallel end cross-sections.

2 Analytical Solutions for Elliptical or Circular Cracks in Isotropic or Transversely Isotropic Solids with Arbitrary Crack-Face Tractions

In practice, the actual flaws in three-dimensional structural components are often approximated by elliptical cracks. For this reason, the problem of an embedded elliptical crack in an infinite solid has been the focus of a considerable number of studies.

Vijayakumar and Atluri (1981) have presented a general solution procedure for an embedded elliptical crack in an isotropic infinite solid, subject to arbitrary crack-face tractions. Later, Nishioka and Atluri (1983) have refined and completed this solution, deriving (i) alternative non-singular forms for linear algebraic equations relating crack-face tractions and potential functions, (ii) a general procedure for the evaluation of the required elliptic integrals, and (iii) a systematic procedure for the evaluation of the partial derivatives of the potential functions. These derivations made it possible

to extract the analytical close-form solution for any polynomial order of crack-face tractions. For later convenience, we cite this general solution altogether (Vijaykumar and Atluri, Nishioka and Atluri) as VNA solution. The VNA solution represents a generalization, hitherto thought to be unachievable, of the potential representation of Segedin (1967) and Shah and Kobayashi (1971).

In the following, we present a brief summary of the VNA solution. Further details of the VNA solution can be found in the cited original papers.

2.1 The VNA Solution (An Elliptical Crack in an Isotropic Solid)

Suppose that x_1 and x_2 are Cartesian coordinates in the plane of the elliptical crack and x_3 is normal to the crack-plane such that:

$$(x_1/a_1)^2 + (x_2/a_2)^2 = 1, \qquad a_1 > a_2 \tag{20}$$

describes the border of the elliptical crack of aspect ratio (a_1/a_2). The necessary ellipsoidal coordinates ξ_α $(\alpha = 1,2,3)$ are defined as the roots of the cubic equation

$$\omega(s) = 1 - \left(\frac{x_1^2}{a_1^2+s}\right) - \left(\frac{x_2^2}{a_2^2+s}\right) - \left(\frac{x_3^2}{s}\right) = 0 \tag{21}$$

where

$$\infty > \xi_3 \ge 0 \ge \xi_2 \ge -a_2^2 \ge \xi_1 \ge -a_1^2$$

so that the interior of the ellipse is given by $\xi_3 = 0$ and its boundary by $\xi_2 = \xi_3 = 0$.

Let the tractions along the crack-surface be expressed in the form

$$\sigma^{(0)}_{3\alpha} = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{m=0}^{M}\sum_{n=0}^{m} A^{(i,j)}_{\alpha,m-n,n} x_1^{2m-2n+i} x_2^{2n+j}; \qquad (\alpha = 1,2,3) \tag{22}$$

so that the values of (i,j) specify the symmetries of the load with respect to the axes of the ellipse. M is an arbitrary integer which is related to the order of the polynomial. The solution corresponding to the load expressed by Eq. (22) can be assumed in terms of the potential functions

$$f_\alpha = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{M}\sum_{\ell=0}^{k} C^{(i,j)}_{\alpha,k-\ell,\ell} F_{2k-2\ell+i,2\ell+j}; \qquad (\alpha = 1,2,3) \tag{23}$$

where

$$F_{2k-2\ell+i,2\ell+j} = \frac{\partial^{2k+i+j}}{\partial x_1^{2k-2\ell+i}\partial x_2^{2\ell+j}} \int_{\xi_3}^{\infty} [\omega(s)]^{2k+i+j+1} \frac{\mathrm{d}s}{\sqrt{Q(s)}} \tag{24}$$

and $Q(s) = s(s+a_1^2)(s+a_2^2)$. The components of displacement u_i and stress σ_{ij} in terms of f_α $(\alpha = 1,2,3)$ are given by

$$\begin{aligned} u_1 &= (1-2\nu)(f_{1,3}+f_{3,1}) - (3-4\nu)f_{1,3} + x_3(\nabla\cdot\hat{\boldsymbol{f}})_{,1} \\ u_2 &= (1-2\nu)(f_{2,3}+f_{3,2}) - (3-4\nu)f_{2,3} + x_3(\nabla\cdot\hat{\boldsymbol{f}})_{,2} \\ u_3 &= -(1-2\nu)(f_{1,1}+f_{2,2}) - 2(1-\nu)f_{3,3} + x_3(\nabla\cdot\hat{\boldsymbol{f}})_{,3} \end{aligned} \tag{25}$$

and

$$
\begin{aligned}
\sigma_{11} &= 2\mu[f_{3,11} + 2\nu f_{3,22} - 2f_{1,31} - 2\nu f_{2,32} + x_3(\nabla\cdot\hat{f})_{,11}] \\
\sigma_{22} &= 2\mu[f_{3,22} + 2\nu f_{3,11} - 2f_{2,32} - 2\nu f_{1,31} + x_3(\nabla\cdot\hat{f})_{,22}] \\
\sigma_{12} &= 2\mu[(1-2\nu)f_{3,12} - (1-\nu)(f_{1,23} + f_{2,13}) + x_3(\nabla\cdot\hat{f})_{,12}] \\
\sigma_{33} &= 2\mu[-f_{3,33} + x_3(\nabla\cdot\hat{f})_{,33}] \\
\sigma_{31} &= 2\mu[-(1-\nu)f_{1,33} + \nu(f_{1,11} + f_{2,21}) + x_3(\nabla\cdot\hat{f})_{,13}] \\
\sigma_{32} &= 2\mu[-(1-\nu)f_{2,23} + \nu(f_{1,12} + f_{2,22}) + x_3(\nabla\cdot\hat{f})_{,23}]
\end{aligned} \tag{26}
$$

and

$$\nabla\cdot\hat{f} = f_{1,1} + f_{2,2} + f_{3,3} \tag{27}$$

where μ and ν are the shear modulus and Poisson's ratio.

By successive differentiation, it can be shown from (24) that, since $\omega(\xi_3) = 0$,

$$F_{k\ell} = \int_{\xi_3}^{\infty} \frac{\partial^{k+\ell}\omega^{k+\ell+1}}{\partial x_1^k \partial x_2^\ell} \frac{ds}{[Q(s)]^{1/2}} \equiv \int_{\xi_3}^{\infty} \partial_1^{k_1}\partial_2^{\ell_1}\partial_3^{m_1}\omega^{k+\ell+1} \frac{ds}{[Q(s)]^{1/2}} \tag{28}$$

wherein $(2k - 2\ell + i)$ and $(2\ell + j)$ in Eq. (24) are replaced by k and ℓ in the above equation and

$$k_1 = k, \qquad \ell_1 = \ell, \qquad m_1 = 0$$

In (28) we have used the additional notation that ∂_α^j implies the jth partial derivative with respect to x_α. Similarly, the first-order partial derivatives of $F_{k\ell}$ with respect to x_β $(\beta = 1, 2, 3)$ can be expressed by

$$F_{k\ell,\beta} = \int_{\xi_3}^{\infty} \partial_1^{k_1}\partial_2^{\ell_1}\partial_3^{m_1}\omega^{k+\ell+1} \frac{ds}{[Q(s)]^{1/2}} \tag{29}$$

where

$$k_1 = k + \delta_{1\beta}, \qquad \ell_1 = \ell + \delta_{2\beta}, \qquad m_1 = \delta_{3\beta}$$

and $\delta_{1\beta}$, etc., are the well-known Kronecker deltas.

In the case of the second- and third-order partial derivatives, we derive:

$$F_{k\ell,\beta\gamma} = \int_{\xi_3}^{\infty} \partial_1^{k_1}\partial_2^{\ell_1}\partial_3^{m_1}\omega^{k+\ell+1} \frac{ds}{[Q(s)]^{1/2}} + F^0_{k\ell\beta\gamma} \tag{30}$$

where

$$F^0_{k\ell\beta\gamma} = (k+\ell+1)! \frac{x_1^{k_1}x_2^{\ell_1}x_3^{m_1}}{(\xi_3-\xi_1)(\xi_3-\xi_2)} \{\rho_1^{k_1}\rho_2^{\ell_1}\rho_3^{m_1}[Q(s)]^{1/2}\}_{s=\xi_3} \tag{31}$$

$$k_1 = k + \delta_{1\beta} + \delta_{1\gamma}, \quad \ell_1 = \ell + \delta_{2\beta} + \delta_{2\gamma}, \quad m_1 = \delta_{3\beta} + \delta_{3\gamma} \tag{32}$$

$$\rho_\alpha = \frac{\partial^2\omega}{\partial x_\alpha^2} = -2/(a_\alpha^2 + s), \qquad (\alpha = 1, 2, 3) \tag{33}$$

in which $a_3 \equiv 0$, and

$$F_{k\ell,\beta\gamma\delta} = \int_{\xi_3}^{\infty} \partial_1^{k_1}\partial_2^{\ell_1}\partial_3^{m_1}\omega^{k+\ell+1} \frac{ds}{[Q(s)]^{1/2}} + \frac{\partial F^0_{k\ell\beta\gamma}}{\partial x_\delta} + G^0 \tag{34}$$

where

$$
\begin{aligned}
G^0 &= \frac{(k+\ell+1)![Q(s)]^{1/2}}{(\xi_3-\xi_1)(\xi_3-\xi_2)} \rho_1^{k_1}\rho_2^{\ell_1}\rho_3^{m_1}x_1^{k_1}x_2^{\ell_1}x_3^{m_1} \\
&\quad \times \left[\frac{k_0(k_0-1)}{2\rho_1 x_1^2} + \frac{\ell_0(\ell_0-1)}{2\rho_2 x_2^2} + \frac{m_0(m_0-1)}{2\rho_3 x_3^2}\right]_{s=\xi_3}
\end{aligned}
$$

$$
\begin{aligned}
k_0 &= k + \delta_{1\beta} + \delta_{1\gamma}; \qquad \ell_0 = \ell + \delta_{2\beta} + \delta_{2\gamma}; \qquad m_0 = \delta_{3\beta} + \delta_{3\gamma} \\
k_1 &= k_0 + \delta_{1\delta}; \qquad \ell_1 = \ell_0 + \delta_{2\delta}; \qquad m_1 = m_0 + \delta_{3\delta}
\end{aligned} \tag{35}
$$

A systematic procedure for the evaluation of the partial derivatives of $F^0_{k\ell\beta\alpha}$ in (34) is given in Nishioka and Atluri (1983a). It is noted that these derivatives are needed (i) in satisfying the boundary conditions on the crack-face and (ii) in evaluating the far-field stresses in the solid containing the elliptical crack which is subject to arbitrary tractions.

It is now seen from (28)–(34) that one needs to evaluate a generic integral of the type:

$$\int_{\xi_3}^{\infty} \partial_1^{k_1}\partial_2^{\ell_1}\partial_3^{m_1}\omega^{k+\ell+1} \frac{ds}{[Q(s)]^{1/2}} \tag{36}$$

To accomplish this, we expand $\omega^{k+\ell+1}$ in terms of x_α^2 and carry out the indicated differentiations term by term. Thus, one obtains:

$$
\begin{aligned}
\int_{\xi_3}^{\infty} \partial_1^{k_1}\partial_2^{\ell_1}\partial_3^{m_1}\omega^{k+\ell+1} \frac{ds}{[Q(s)]^{1/2}} &= (k+\ell+1)! \sum_{p=0}^{k+\ell+1}\sum_{q=0}^{p}\sum_{r=0}^{q} \frac{(-1)^p}{(k+\ell+1-p)!} \\
&\quad \times \frac{(2p-2q)!}{(p-q)!}\frac{(2p-2r)!}{(q-r)!}\frac{(2r)!}{(r)!}\frac{x_1^{2p-2q-k_1}}{(2p-2q-k_1)!} \\
&\quad \times \frac{x_2^{2q-2r-\ell_1}}{(2q-2r-\ell_1)!}\frac{x_3^{2r-m_1}}{(2r-m_1)!} J_{p-q,q-r,r}(\xi_3)
\end{aligned} \tag{37}
$$

where

$$J_{p-q,q-r,r}(\xi_3) = \int_{\xi_3}^{\infty} \frac{ds}{(s+a_1^2)^{p-q}(s+a_2^2)^{q-r}(s+a_3^2)^r[Q(s)]^{1/2}} \tag{38}$$

In general, the integral indicated in Eq. (38), for a given set of parameters p, q, r, can be evaluated in terms of incomplete elliptic integrals of the first and second kinds, and Jacobian elliptic functions. The derivation of the closed-form expressions involves exorbitant work even for relatively lower-order components of $J_{p-q,q-r,r}$ (see Shah and Kobayashi, 1972). Therefore, the derivation of a systematic generic procedure for the evaluation of the elliptic integrals $J_{p-q,q-r,r}$ was important in the development as well as in the numerical implementation of the VNA solution. A procedure for this has been developed by Nishioka and Atluri (1983a) and is summarized as follows.

Eq. (38) can be rewritten in terms of Jacobian elliptic functions, as:

$$
\begin{aligned}
J_{p-q,q-r,r} &= \frac{2}{a_1^{2p+1}} \int_0^{u_1} (\mathrm{sn}^{2p}u)(\mathrm{nd}^{2q-2r}u)(\mathrm{nc}^{2r}u)du \\
&\equiv \frac{2}{a_1^{2p+1}} L_{p,q-r,r}
\end{aligned} \tag{39}
$$

where

$$\mathrm{sn}^2 u_1 = a_1^2/(a_1^2 + \xi_3)$$

The following identities for Jacobian elliptic functions are used:

$$
\begin{aligned}
&\mathrm{sn}^2 u_1 + \mathrm{cn}^2 u = 1; && k^2\mathrm{sn}^2 u + \mathrm{dn}^2 u = 1 \\
&\mathrm{dn}^2 u - k^2\mathrm{cn}^2 u = k'^2; && k'^2\mathrm{sn}^2 u + \mathrm{cn}^2 u = \mathrm{dn}^2 u \\
&\mathrm{tn}u = \mathrm{sn}u/\mathrm{cn}u; \quad && \mathrm{dc}u = \mathrm{dn}u/\mathrm{cn}u; \quad && \mathrm{cd}u = \mathrm{cn}u/\mathrm{dn}u \\
&\mathrm{nd}u = 1/\mathrm{dn}u; && \mathrm{nc}u = 1/\mathrm{cn}u; && \mathrm{sd}u = \mathrm{sn}u/\mathrm{dn}u
\end{aligned} \tag{40}
$$

where

$$
k^2 = (a_1^2 - a_2^2)/a_1^2; \qquad k'^2 = 1 - k^2 \tag{41}
$$

By using integration by parts in (39), one sees that:

$$
\begin{aligned}
L_{p,q-r,r} = {} & \frac{1}{(2r-1)k'^2}\{(\mathrm{sn}^{2p+1}u)(\mathrm{nc}^{2r-1}u)(\mathrm{nd}^{2q-2r-1}u)|_0^{u_1} \\
& + [2(-p+r-1) + 2(p-q-r+2)k^2]L_{p,q-r,r-1} \\
& + k^2(-2p+2q-3)L_{p,q-r,r-2}\}
\end{aligned} \tag{42}
$$

Thus, one needs the starting values of $L_{p,q-r,r-1}$ and $L_{p,q-r,r-2}$ to evaluate $L_{p,q-r,r}$. The lowest-order starting values are:

$$
\begin{aligned}
L_{p,q,-1} &= \int_0^{u_1} \mathrm{sn}^{2p}u\,\mathrm{nd}^{2q}u\,\mathrm{nc}^{-2}u\,du \\
L_{p,q,-2} &= \int_0^{u_1} \mathrm{sn}^{2p}u\,\mathrm{nd}^{2q}u\,\mathrm{nc}^{-4}u\,du
\end{aligned} \tag{43}
$$

The above integrals can be reduced to the forms:

$$
\begin{aligned}
L_{p,q,-1} &= \frac{1}{k^{2p+2}}\sum_{j=0}^{p}\sum_{\gamma=0}^{1}\frac{(-1)^{j+\gamma+1}k'^{2(1-\gamma)}p!}{(p-j)!j!(1-\gamma)!\gamma!}I_{2(q-j-\gamma)} \\
L_{p,q,-2} &= \frac{1}{k^{2p+4}}\sum_{j=0}^{p}\sum_{\gamma=0}^{2}\frac{(-1)^{j+\gamma+2}k'^{2(2-\gamma)}p!2}{(p-j)!j!(2-\gamma)!\gamma!}I_{2(q-j-\gamma)}
\end{aligned} \tag{44}
$$

where

$$
I_{2m} = \int_0^{u_1} \mathrm{nd}^m u\,du \tag{45}
$$

$$
I_{2m+2} = \frac{2m(2-k^2)I_{2m} + (1-2m)I_{2m-2} - k^2\mathrm{sn}u_1\mathrm{cn}u_1\mathrm{nd}^{2m+1}u_1}{(2m+1)k'^2} \tag{46}
$$

For $2(p-j-\gamma) < 0$ in (44), we find $I_{-2m} = G_{2m}$, where

$$
G_{2m+2} = \frac{k^2\mathrm{dn}^{2m-1}u_1\mathrm{sn}u_1\mathrm{cn}u_1 + (1-2m)k'^2G_{2m-2} + 2m(2-k^2)G_{2m}}{(2m+1)} \tag{47}
$$

Thus, finally we see that one needs the following starting values for evaluating the general terms of I_{2m+2} and G_{2m+2}:

$$
\begin{aligned}
I_0 &= G_0 = F(u_1) = u_1 \\
I_2 &= (1/k'^2)[E(u_1) - k^2\mathrm{sn}u_1\mathrm{cd}u_1] \\
G_2 &= E(u_1)
\end{aligned} \tag{48}
$$

where $F(u_1)$ and $E(u_1)$ are incomplete elliptic integrals of the first and second kinds, respectively.

Now, the boundary conditions on the crack-face ($\sigma_{3\alpha} = \sigma^0_{3\alpha}$) can be expressed in terms of the potential functions, as follows:

$$
\begin{aligned}
\sigma^{(0)}_{33} &= -2\mu f_{3,33} \\
\sigma^{(0)}_{3\alpha} &= -2\mu[(1-\nu)f_{\alpha,33} - \nu(f_{1,1\alpha} + f_{2,2\alpha})]\alpha = 1,2
\end{aligned} \tag{49}
$$

in which the boundary condition for f_3 is uncoupled from f_1 and f_2. However, if Eqs. (49) are used directly as in Vijaykumar and Atluri (1981), finite parts of the singular terms in the equations relating the coefficients C of (23) to coefficients A of Eq. (22) have to be considered. Alternative non-singular forms for the boundary conditions may also be used. Since f_α ($\alpha = 1,2,3$) are harmonic functions, it is seen that:

$$
f_{\alpha,33} = -f_{\alpha,11} - f_{\alpha,22} \qquad (\alpha = 1,2,3) \tag{50}
$$

Then, (49a, b) can be rewritten as follows:

$$
\begin{aligned}
\sigma^{(0)}_{33} &= 2\mu(f_{3,11} + f_{3,22}) \\
\sigma^{(0)}_{3\alpha} &= 2\mu[(1-\nu)(f_{\alpha,11} + f_{\alpha,22}) + \nu(f_{1,1\alpha} + f_{2,2\alpha})]
\end{aligned} \tag{51}
$$

Substituting (22) and (23) into (51a, b), we obtain the following linear algebraic equations, upon comparing coefficients of like powers in the polynomial series. The relation between the parameters A and parameters C can be summarized in a matrix form:

$$
\underset{N\times 1}{\{A\}} = \underset{N\times N}{[B]}\ \underset{N\times 1}{\{C\}} \tag{52}
$$

where N is the total number of coefficients A or C. For a complete polynomial expressed by (22), the maximum degree of the polynomial M_c and the number of coefficients N can be expressed, respectively, as $M_c = 2M + 1$ and $N = (M+1)(2M+3) \times 3$. For an incomplete polynomial, the maximum degree of polynomial and the number of coefficients depend not only on the parameter M but also on the parameters i and j in (22). Detailed expressions of the components of matrix $[B]$ are given in Nishioka and Atluri (1983a) for Mode I and mixed modes of II and III. A more convenient form for the mixed modes of II and III also can be found in Simon, O'Donoghue and Atluri (1987).

Once the coefficients C are determined by solving (52) for given loadings on the crack surface, the stress-intensity factors corresponding to these loads are computed from the following equation (Vijayakumar and Atluri, 1981).

For the Mode I problem,

$$
\begin{aligned}
K_I = {} & 8\mu\left(\frac{\pi}{a_1a_2}\right)^{1/2}A^{1/4}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{M}\sum_{\ell=0}^{k}(-2)^{2k+i+j}(2k+i+j+1)! \\
& \times\frac{1}{a_1a_2}\left(\frac{\cos\theta}{a_1}\right)^{2k-2\ell+i}\left(\frac{\sin\theta}{a_2}\right)^{2\ell+j}C^{(i,j)}_{3,k-\ell,\ell}
\end{aligned} \tag{53}
$$

where θ is the elliptic angle and

$$
A = a_1^2\sin^2\theta + a_2^2\cos^2\theta \tag{54}
$$

For the mixed-mode problem of Modes II and III

$$K_{II} = 8\mu\left(\frac{\pi}{a_1a_2}\right)^{1/2} A^{-1/4}\frac{1}{a_1a_2}[H_1a_2\cos\theta + H_2a_1\sin\theta] \tag{55}$$

$$K_{III} = 8\mu\left(\frac{\pi}{a_1a_2}\right)^{1/2} A^{-1/4}\frac{(1-\nu)}{a_1a_2}[H_2a_2\cos\theta - H_1a_1\sin\theta] \tag{56}$$

in which

$$\begin{aligned} H_1 &= \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{M}\sum_{\ell=0}^{k}(-2)^{2k+i+j}(2k+i+j+1)! \\ &\times \left(\frac{\cos\theta}{a_1}\right)^{2k-2\ell+i}\left(\frac{\sin\theta}{a_2}\right)^{2\ell+j} C_{1,k-\ell,\ell}^{(i,j)} \end{aligned} \tag{57}$$

$$\begin{aligned} H_2 &= \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{M}\sum_{\ell=0}^{k}(-2)^{2k+2-i-j}(2k+3-i-j)! \\ &\times \left(\frac{\cos\theta}{a_1}\right)^{2k-2\ell+1-i}\left(\frac{\sin\theta}{a_2}\right)^{2\ell+1-j} C_{2,k-\ell,\ell}^{(1-i,1-j)} \end{aligned} \tag{58}$$

The VNA solution has been implemented by Nishioka and Atluri (1981, 1983a) in a new finite-element alternating method for the solution of problems of embedded or surface flaws in complex structural geometries.

2.2 An Elliptical Crack in a Transversely Isotropic Solid

Recently, Rajiyah, Zhong and Atluri (1988) have extended the VNA solution procedure to a transversely isotropic case, with arbitrary tractions on the face of an elliptical crack in a transversely isotropic solid, with the crack plane being at an arbitrary angle to the axis of elastic symmetry.

A material is said to be transversely isotropic when it possesses an axis of elastic symmetry such that the material is isotropic in the planes normal to this axis. Let $\bar{z}$ be the direction of elastic symmetry. Then the stress-strain relations in the $(\bar{x},\bar{y},\bar{z})$ system could be written as,

$$\begin{aligned} \sigma_{\bar{x}} &= C_{11}\frac{\partial u_{\bar{x}}}{\partial \bar{x}} + C_{12}\frac{\partial u_{\bar{y}}}{\partial \bar{y}} + C_{13}\frac{\partial u_{\bar{z}}}{\partial \bar{z}} \\ \sigma_{\bar{y}} &= C_{12}\frac{\partial u_{\bar{x}}}{\partial \bar{x}} + C_{11}\frac{\partial u_{\bar{y}}}{\partial \bar{y}} + C_{13}\frac{\partial u_{\bar{z}}}{\partial \bar{z}} \\ \sigma_{\bar{z}} &= C_{13}\left(\frac{\partial u_{\bar{x}}}{\partial \bar{x}} + \frac{\partial u_{\bar{y}}}{\partial \bar{y}}\right) + C_{33}\frac{\partial u_{\bar{z}}}{\partial \bar{z}} \\ \tau_{\bar{y}\bar{z}} &= C_{44}\left(\frac{\partial u_{\bar{z}}}{\partial \bar{y}} + \frac{\partial u_{\bar{y}}}{\partial \bar{z}}\right) \\ \tau_{\bar{z}\bar{x}} &= C_{44}\left(\frac{\partial u_{\bar{x}}}{\partial \bar{z}} + \frac{\partial u_{\bar{z}}}{\partial \bar{x}}\right) \\ \tau_{\bar{x}\bar{y}} &= \frac{1}{2}(C_{11}-C_{12})\left(\frac{\partial u_{\bar{x}}}{\partial \bar{y}} + \frac{\partial u_{\bar{y}}}{\partial \bar{x}}\right) \end{aligned} \tag{59}$$

$(\sigma_{\bar{x}},\sigma_{\bar{y}},\sigma_{\bar{z}},\tau_{\bar{y}\bar{z}},\tau_{\bar{z}\bar{x}},\tau_{\bar{x}\bar{y}})$ and $(u_{\bar{x}},u_{\bar{y}},u_{\bar{z}})$ are stresses and displacement components in the $(\bar{x},\bar{y},\bar{z})$ system and C_{ij} are elastic constants, as discussed in Lekhnitski (1981).

The displacement field $u_{\bar{x}}$, $u_{\bar{y}}$, and $u_{\bar{z}}$ is represented in terms of potential functions ϕ_j $(j=1,2,3)$, such that it satisfies the equilibrium equations expressed in terms of displacements, identically, as follows:

$$\begin{aligned} u_{\bar{x}} &= \frac{\partial}{\partial \bar{x}}(\phi_1+\phi_2) - \frac{\partial \phi_3}{\partial \bar{y}} \\ u_{\bar{y}} &= \frac{\partial}{\partial \bar{y}}(\phi_1+\phi_2) + \frac{\partial \phi_3}{\partial \bar{x}} \\ u_{\bar{z}} &= \frac{\partial}{\partial \bar{z}}(m_1\phi_1+m_2\phi_2) \end{aligned} \tag{60}$$

$$m_j = \frac{C_{11}n_j - C_{44}}{C_{13}+C_{44}} = \frac{(C_{13}+C_{44})n_j}{C_{33}-C_{44}n_j}, \qquad j=1,2 \tag{61}$$

The quantities n_1 and n_2 are the roots of the quadratic equation in n

$$C_{11}C_{44}n^2 + [C_{13}(C_{13}+2C_{44}) - C_{11}C_{33}]n + C_{33}C_{44} = 0 \tag{62}$$

and n_3 is defined as:

$$n_3 = \frac{2C_{44}}{(C_{11}-C_{12})} \tag{63}$$

Introduction of the following modified coordinate systems (x_j,y_j,z_j) $(j=1,2,3)$

$$\begin{aligned} x_j &= \bar{x} \\ y_j &= \bar{y}\cos\theta_j + \frac{\bar{z}}{\sqrt{n_j}}\sin\theta_j \\ z_j &= -\bar{y}\sin\theta_j + \frac{\bar{z}}{\sqrt{n_j}}\cos\theta_j \end{aligned} \tag{64}$$

yields the expressions of the elliptical crack in each modified coordinate system

$$\frac{x_j^2}{a_j^2} + \frac{y_j^2}{b_j^2} = 0 \qquad \text{in} \quad z_j = 0 \quad (j = 1,2,3\,\text{no sum}) \tag{65}$$

where

$$\begin{aligned} a_j &= a \\ b_j &= b\left(\cos\bar{\theta}\cos\theta_j + \frac{1}{\sqrt{n_j}}\sin\bar{\theta}\sin\theta_j\right) \end{aligned} \tag{66}$$

and

$$\tan\theta_j = \frac{1}{\sqrt{n_j}}\tan\bar{\theta} \tag{67}$$

Here, $\bar{\theta}$ is the angle between the physical crack axis (z) and the material axis $(\bar{z})$.

Now, each of the potentials ϕ_j $(j=1,2,3)$ can be expressed by the harmonic equations in a set of coordinate (x_j,y_j,z_j) $(j=1,2,3)$, as:

$$\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z_j^2}\right)\phi_j = 0 \qquad (j = 1,2,3\,\text{no sum on}\,j) \tag{68}$$

Thus, to solve the problem on hand, appropriate potential functions ϕ_j $(j = 1,2,3)$, each in a different set of coordinates (x_j, y_j, z_j) $(j = 1,2,3)$ will now be assumed. The necessary ellipsoidal coordinates ξ_1^j, ξ_2^j, ξ_3^j $(j = 1,2,3)$ for a point in the x_j, y_j, z_j $(j = 1,2,3)$ coordinate system are given by the roots of the cubic equation

$$\omega_j(\xi^j) = 0 \qquad (j = 1,2,3 \text{ no sum on } j) \tag{69}$$

where

$$\omega_j(\xi^j) = 1 - \frac{x_j^2}{a^2 + \xi^j} - \frac{y_j^2}{b_j^2 + \xi^j} - \frac{z_j}{\xi^j} \tag{70}$$

and

$$-a^2 \leq \xi_1^j \leq -b_j^2 \leq \xi_2^j \leq 0 \leq \xi_3^j < \infty \tag{71}$$

Appropriate expressions of the potentials ϕ_j for the present case of transverse isotropy as discussed above, are assumed as:

$$\phi_j = \sum_{\ell=0}^{M} \sum_{k=0}^{\ell} B_{\ell-k,k}^j F_{\ell-k,k}^j \tag{72}$$

where, $B_{\ell-k,k}^j$ are unknown coefficients to be determined, M is the highest order polynomial terms considered and $F_{\ell-k,k}^j$ is defined as:

$$F_{k\ell}^j = \frac{\partial^{k+\ell}}{\partial x_j^k \partial y_j^\ell} \int_{\xi_3^j}^{\infty} [\omega_j(s)]^{k+\ell+1} \frac{ds}{\sqrt{Q_j(s)}} \tag{73}$$

As can be easily noticed, the above equation is basically the same as Eq. (24) or Eq. (29) in the VNA solution procedure for real n_1 and n_2 Eq. (63). Therefore, the VNA solution procedure can be used to obtain the complete general solution for an transversely isotropic material. However, for complex n_1 and n_2 analytic continuation from the real axis to the complex plane has to be carried out. This solution is given by Rajiyah, Zhong, and Atluri (1988) in more detail.

2.3 A Circular Crack in an Isotropic Solid

The analytical general solution for a circular crack in an infinite isotropic elastic solid, subject to arbitrary crack-face tractions, is briefly summarized here. The solution was revisited by Liao and Atluri (1988), based on the Fourier-Hankel transform technique developed by Sneddon (1951), which was later generalized by Kassir and Sih (1975). Although Kassir and Sih (1975) have derived the general solution for this type of problem, certain portions of the mixed-mode solution were lacking in their final results. Thus, the complete form of the general solution has been recently rederived by Liao and Atluri (1988), as follows:

For a penny-shaped crack embedded in an infinite 3-D elastic body, we need to solve the following mixed boundary value problems.

Mode I

$$\begin{array}{lll} \sigma_{rz}(r,\theta,0) = \sigma_{\theta z}(r,\theta,0) = 0 & r \geq 0; & 0 \leq \theta \leq 2\pi \\ \sigma_{zz}(r,\theta,0) = p_1(r,\theta) & 0 \leq r < a; & 0 \leq \theta \leq 2\pi \\ u_z(r,\theta,0) = 0 & r > a; & 0 \leq \theta \leq 2\pi \end{array} \tag{74}$$

(a is the crack radius)

Modes II and III

$$\begin{array}{lll} \sigma_{zz}(r,\theta,0) = 0 & r \geq 0 & 0 \leq \theta \leq 2\pi \\ \sigma_{rz}(r,\theta,0) = p_2(r,\theta) & 0 \leq r < a & 0 \leq \theta \leq 2\pi \\ \sigma_{\theta z}(r,\theta,0) = p_3(r,\theta) & 0 \leq r < a & 0 \leq \theta \leq 2\pi \\ u_r(r,\theta,,0) = u_\theta(r,\theta,0) = 0 & r > a & 0 \leq \theta \leq 2\pi \end{array} \tag{75}$$

where $p_\alpha(r,\theta)$ $(\alpha = 1,2,3)$ are given functions describing the distribution of the loads applied to the crack surface.

An appropriate solution for this boundary-value problem can be obtained by expressing the displacement components in terms of three harmonic functions ϕ_α $(\alpha = 1,2,3)$, as

$$\begin{aligned} u_r &= (1-2\nu)\frac{\partial \phi_1}{\partial r} + z\frac{\partial^2 \phi_1}{\partial r \partial z} + \frac{2}{r}\frac{\partial \phi_2}{\partial \theta} + 2(1-\nu)\frac{\partial \phi_3}{\partial r} + z\frac{\partial^2 \phi_3}{\partial r \partial \theta} \\ u_\theta &= (1-2\nu)\frac{1}{r}\frac{\partial \phi_1}{\partial \theta} + \frac{z}{r}\frac{\partial^2 \phi_1}{\partial \theta \partial z} - 2\frac{\partial \phi_2}{\partial r} + 2(1-\nu)\frac{1}{r}\frac{\partial \phi_3}{\partial \theta} + \frac{z}{r}\frac{\partial^2 \phi_3}{\partial \theta \partial z} \\ u_z &= -2(1-\nu)\frac{\partial \phi_1}{\partial z} + z\frac{\partial^2 \phi_1}{\partial z^2} - (1-2\nu)\frac{\partial \phi_3}{\partial z} + z\frac{\partial^2 \phi_3}{\partial z^2} \end{aligned} \tag{76}$$

The corresponding stress components in terms of the potential functions can be easily obtained by the above equations through the use of the strain-displacement and the stress-strain relations. From the above equations, it is seen that ϕ_1 is related with mode I, and ϕ_2 and ϕ_3 are related with the mixed mode of II and III.

In order to express general loadings, the applied loads $p_\alpha(r,\theta)$ are expressed by Fourier series as follows:

$$p_\alpha(r,\theta) = \sum_{n=0}^{\infty} \begin{array}{c} \cos n\theta \cdot A_{\alpha n}(r) \\ \sin n\theta \cdot B_{\alpha n}(r) \end{array} \qquad (\alpha = 1,2,3) \tag{77}$$

In order to solve the proposed problem, the potential functions are represented by the Fourier-Hankel transform:

$$\phi_\alpha(r,\theta,z) = \sum_{n=0}^{\infty} \begin{array}{c} \cos n\theta \\ \sin n\theta \end{array} \int_0^\infty \begin{array}{c} C_{\alpha n}(s) \\ D_{\alpha n}(s) \end{array} \frac{1}{s} J_n(rs) e^{-sz}\, ds \qquad (\alpha = 1,2,3) \tag{78}$$

The substitution of Eqs. (77) and (78) into Eqs. (74), (75) and (76) yields the following relations:

Mode I

$$\begin{aligned} C_{1n}(s) &= -\frac{1}{\mu}\sqrt{\frac{s}{2\pi}} \int_0^a J_{n+1/2}(st) \frac{dt}{t^{n-1/2}} \int_0^t \frac{r^{n+1} A_{1n}(r)\, dr}{(t^2 - r^2)^{1/2}} \\ D_{1n}(s) &= -\frac{1}{\mu}\sqrt{\frac{s}{2\pi}} \int_0^a J_{n+1/2}(st) \frac{dt}{t^{n-1/2}} \int_0^t \frac{B_{1n}(r)\, dr}{(t^2 - r^2)^{1/2}} \end{aligned} \tag{79}$$

Mode II and III

$$\begin{aligned} C_{20}(s) &= -\frac{1}{\mu}\sqrt{\frac{s}{2\pi}} \int_0^a J_{3/2}(st) \frac{dt}{t^{1/2}} \int_0^t \frac{r^2 A_{30}(r)}{(t^2 - r^2)^{1/2}}\, dr \\ C_{30}(s) &= \frac{1}{\mu}\sqrt{\frac{s}{2\pi}} \int_0^a J_{3/2}(st) \frac{dt}{t^{1/2}} \int_0^t \frac{r^2 A_{20}(r)}{(t^2 - r^2)^{1/2}}\, dr \end{aligned} \tag{80}$$

$$
\begin{aligned}
C_{2n}(s) &= \sqrt{s}\int_0^a [(\nu-1)\Phi_1^*(t)J_{n-1/2}(st) + \Phi_2^*(t)J_{n+3/2}(st)]\,dt, && n \geq 1 \\
C_{3n}(s) &= \sqrt{s}\int_0^a [\Phi_1(t)J_{n-1/2}(st) + \Phi_2(t)J_{n+3/2}(st)]\,dt && n \geq 1 \\
D_{2n}(s) &= \sqrt{s}\int_0^a [(1-\nu)\Phi_1(t)J_{n-1/2}(st) - \Phi_2(t)J_{n+3/2}(st)]\,dt, && n \geq 1 \\
D_{3n}(s) &= \sqrt{s}\int_0^a [\Phi^*(t)J_{n-1/2}(st) + \Phi_2^*(t)J_{n+3/2}(st)]\,dt && n \geq 1
\end{aligned}
\tag{81}
$$

where,

$$
\begin{aligned}
\Phi_1(t) &= \frac{-t^{-n+3/2}}{(2-\nu)\mu\sqrt{2\pi}}\int_0^t \frac{r^n[A_{2n}(r) - B_{3n}(r)]}{(t^2-r^2)^{1/2}}\,dr \\
\Phi_1^*(t) &= \frac{-t^{-n+3/2}}{(2-\nu)\mu\sqrt{2\pi}}\int_0^t \frac{r^n[A_{3n}(r) + B_{2n}(r)]}{(t^2-r^2)^{1/2}}\,dr \\
\Phi_2(t) &= \frac{\nu}{2}\Phi_1(t) + \frac{t^{-n-1/2}}{2\mu\sqrt{2\pi}}\left\{\frac{(1+2n)\nu}{2-\nu}\int_0^t r^n[A_{2n}(r) - B_{3n}(r)]\right. \\
&\quad \left. \times\,(t^2-r^2)^{1/2}\,dr + \int_0^t \frac{r^{n+2}[A_{2n}(r) + B_{3n}(r)]}{(t^2-r^2)^{1/2}}\,dr\right\} \\
\Phi_2^*(t) &= \frac{\nu}{2}\Phi_1^*(t) + \frac{t^{-n-1/2}}{2\mu\sqrt{2\pi}}\left\{\frac{(1+2n)\nu}{2-\nu}\int_0^t r^n[A_{3n}(r) + B_{2n}(r)]\right. \\
&\quad \left. \times\,(t^2-r^2)^{1/2}\,dr + \int_0^t \frac{r^{n+2}[B_{2n}(r) - A_{3n}(r)]}{(t^2-r^2)^{1/2}}\,dr\right\}
\end{aligned}
\tag{82}
$$

Without going into details, the stress intensity factors for all the modes are given by

$$
\begin{aligned}
K_I &= -\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty} \begin{matrix}\cos n\theta \\ \sin n\theta\end{matrix}\,\frac{1}{a^{n+1/2}}\int_0^a \begin{matrix}A_{1n}(r) \\ B_{1n}(r)\end{matrix}\,\frac{r^{n+1}}{(a^2-r^2)^{1/2}}\,dr \\
K_{II} &= -\frac{2}{\sqrt{\pi}a^{3/2}}\int_0^a \frac{r^2 A_{20}(r)}{(a^2-r^2)^{1/2}}\,dr + \frac{2\mu\sqrt{2}}{a}\sum_{n=1}^{\infty}[\Phi_1(a) - \Phi_2(a)]\cos n\theta \\
&\quad - \frac{2\mu\sqrt{2}}{a}\sum_{n=1}^{\infty}[(1-\nu)\Phi_1^*(a) + \Phi_2^*(a)]\sin n\theta \\
K_{III} &= -\frac{2\mu\sqrt{2}}{a}\sum_{n=1}^{\infty}[(1-\nu)\Phi_1(a) + \Phi_2(a)]\sin n\theta \\
&\quad - \frac{2}{\sqrt{\pi}a^{3/2}}\int_0^a \frac{r^2 A_{30}(r)}{(a^2-r^2)^{1/2}}\,dr + \frac{2\mu\sqrt{2}}{a}\sum_{n=1}^{\infty}[\Phi_1^*(a) - \Phi_2^*(a)]\cos n\theta
\end{aligned}
\tag{83}
$$

3 Finite-Element and Boundary-Element Alternating Methods for 2D and 3D Crack Problems

3.1 Concepts of Schwartz-Neumann Alternating Method

In the alternating method for an embedded crack in a finite solid, two types of solution are required, as follows:

Solution 1: A general solution for an embedded crack in an infinite body, with the crack-faces being subjected to arbitrary tractions.

Solution 2: A general numerical solution technique such as the finite element method or boundary element method to solve for the stresses at the location of the postulated crack in an otherwise uncracked structure.

Henceforth we assume, for convenience, that a finite-element method is used. The steps involved in the finite-element alternating method for an embedded crack in a finite body are described below:

1. Solve the *uncracked finite body* under the prescribed external loads by using the finite element method. The uncracked body has the same geometry as the given problem except for the crack.

2. Using the finite element solution, compute the stresses at the location of the crack.

3. Compare the residual stresses calculated in step 2 with a permissible stress magnitude.

4. To create traction-free crack faces as in the given problem, reverse the residual stress at the location of the crack as computed in Step 2 and "least-squares fit" them to polynomials.

5. Obtain the analytical solution to the infinite body with the crack subject to the polynomial loading as in Step 4.

6. Calculate the stress intensity factors for the current iteration, using the above analytical solution.

7. Calculate the residual stresses on external surfaces at the body due to the applied loads on crack-faces, as in step 4. To satisfy the given traction boundary conditions, at the external boundaries, reverse the residual stresses on the external surfaces of the body, and calculate the equivalent nodal forces.

8. Consider the nodal forces in step 7 as externally applied loads acting on the uncracked body.

Repeat all steps in the iteration process until the residual stress on the crack surface becomes negligible. To obtain the final solution, add the stress intensity factors for all iterations.

Since the alternating method is iterative in nature, the finite-element equations may, in general, have to be solved repeatedly for different applied loads, while keeping the stiffness matrix the same. To save computational time, special computational techniques were implemented by Nishioka and Atluri (1983a). These are explained below.

As seen from above, for the finite-element alternating method, we need to solve the following type of finite-element equations:

$$
[K][q^0, q^1, \ldots, q^n] = [Q^0, Q^1, \ldots, Q^n] \tag{84}
$$

and

$$
Q^i = Q^i(q^{i-1}); \qquad i = 1, 2, \ldots, n \tag{85}
$$

in which the superscript denotes the cycle of iteration, $[K]$ is the global (assembled) stiffness matrix of the *uncracked* body, and remains the same during the iteration process, and q^i is the nodal displacement vector for ith iteration. Q^i is the nodal force vector for the ith iteration and depends on the solution for the previous iteration q^{i-1} as expressed by Eq. (85).

An efficient equation solver OPTBLOK developed by Mondkar and Powell (1974) may be used to save computational time in solving Eq. (84). The solution algorithm is divided into three parts, i.e. (i) reduction of stiffness matrix, (ii) reduction of load vector, and (iii) back substitution. In OPTBLOK the reduction of stiffness matrix is done only once, although the reduction of load vector and back substitution may be repeated for any number of load cases. Thus, denoting the CPU time for each part by T_1, T_2, and T_3, respectively, the total CPU time T in solving Eq. (84) using OPTBLOK can be expressed as

$$T = T_1 + (n+1)(T_2 + T_3) = (T_1 + T_2 + T_3) + n(T_2 + T_3) \tag{86}$$

where n is the total number of iterations. Since T_1 is much larger than $(T_2 + T_3)$, a substantial reduction in computational time, compared with the case in which Eq. (84) is solved for each iteration [i.e. $T^* = (n+1)(T_1 + T_2 + T_3)$], may be expected. To illustrate this situation, we consider the example of a set of linear equations with the number of equations of 1960, and half bandwidth of 200, wherein the CPU time for reduction of load vector and back substitution was about 5.6% of the total CPU time ($T_2 + T_3 \simeq 0.056T$). Since, for a typical problem, the present alternating method needs three iterations ($n = 3$), the additional cost in this case is only about 16.8%, which is considerably smaller than the 300% in the case when Eq. (84) is solved for each iteration.

An efficient procedure was also devised for the calculation of the nodal forces required in step 7 (see also Eq. (85)). In general, the stress field in a general solution can be expressed by

$$\boldsymbol{\sigma} = \mathbf{P}C \tag{87}$$

where $\mathbf{P}$ is the basis function matrix for stresses, and C is the vector of unknown coefficients in the general solution which will be determined in step 5. Then, the equivalent nodal forces in step 7 can be computed through:

$$\boldsymbol{Q}_m = -\mathbf{G}_m C \tag{88}$$

and

$$\mathbf{G}_m = \int_{S_m} \mathbf{N}^t \mathbf{n} \mathbf{P} \, dS \tag{89}$$

where m denotes the number for a finite element, $\boldsymbol{Q}_m$ are nodal forces, $\mathbf{N}$ is the matrix of the element shape functions, $\mathbf{n}$ is the matrix of the normal direction cosines. Although the matrix $\mathbf{P}$ has the singularity of order $1/\sqrt{r}$ at the crack-front, the functions in $\mathbf{P}$ decay very rapidly with the distance from the crack-front. Thus, the matrices $\mathbf{G}_m$ are calculated only at the external boundary-surface elements which satisfy the condition $r_{\min} < 5a_1$, where $r_{\min}$ is the distance of the closest nodal point of each external boundary-surface element from the center of the ellipse and a_1 is the semi-major axis of the ellipse.

To save computation time, the $\mathbf{G}_m$ matrix can be calculated only once prior to the start of the iteration process. Thus, the equivalent nodal forces $\boldsymbol{Q}^i$ in each iteration can be evaluated without integration.

3.2 3D Alternating Techniques for Part-Elliptical Surface Cracks

The VNA solution given in 2.1 serves as Solution 1 required in the alternating technique.

Now, some comments concerning the solution of *surface flaw* problems in finite bodies, through the present procedure, are in order. Since the analytical solution for an elliptical crack in an infinite solid is implemented as solution (1), it is necessary to define the residual stresses over the entire crack plane including the fictitious portion of the crack which lies outside of the finite body. Moreover, it is well known that the accuracy of the "least-squares" function interpolation inside the interpolated region can be increased with the number of polynomial terms; however, the interpolating curve may change drastically outside the region of interpolation. For these reasons, in Nishioka and Atluri (1983a) numerical experimentation was carried out to arrive at an optimum pressure distribution on the crack surface extended into the fictitious region. For a semi-elliptical crack which lies in the region of $-a_1 \le x_1 \le a_1$ and $0 \le x_2 \le a_2$, it was concluded that the fictitious pressure, which, for the region of $-a_2 \le x_2 \le 0$, remains constant in the x_2 direction but varies in the x_1 direction, gives the best result among the several numerical experiments performed in Nishioka and Atluri (1983a), even though the results for other types of assumed pressure in the fictitious region differed only slightly (±2%).

This procedure of fictitious pressure distribution for a semi-elliptical surface crack was successfully used on the analyses of surface cracks, in finite-thickness plates subject to remote tension as well as remote bending [Nishioka and Atluri (1983a)] and in pressure vessels [Nishioka and Atluri (1982)].

Based on the studies in Nishioka and Atluri (1983b), the following "fictitious" stress distribution is recommended for quarter-elliptical surface cracks. For the first quadrant $(x_1, x_2 \ge 0)$ (namely, the actual surface crack), the residual stress can be calculated by the finite element method and is a function of the coordinates x_1 and x_2. For the other quadrants, the fictitious residual stress is defined as

$$\sigma_{33}^R = \begin{cases} \sigma_{33}^R(0, x_2) & \text{for the second quadrant} \quad (x_1 \le 0, x_2 \ge 0) \\ \sigma_{33}^R(0,0) & \text{for the third quadrant} \quad (x_1 \le 0, x_2 \le 0) \\ \sigma_{33}^R(x_1, 0) & \text{for the fourth quadrant} \quad (x_1 \ge 0, x_2 \le 0) \end{cases} \tag{90}$$

The above alternating method has been successfully applied to the problem of semi-elliptical surface flaws in plates subjected to tension and bending [Nishioka and Atluri (1981, 1983a)], semi-elliptical surface flaws in the meridonal direction at the outer and inner surfaces of pressurized thick and thin cylindrical vessels [Nishioka and Atluri (1982)], quarter-elliptical surface flaws emanating from pin-holes in attachment lugs [Nishioka and Atluri (1983b)], *multiple* coplanar embedded elliptical flaws in an infinite solid subject to arbitrary crack-face tractions [O'Donoghue, Nishioka, and Atluri (1985)], and *multiple* semi-elliptical surface flaws in the meridonal as well as circumferential directions in cylindrical pressure vessels [O'Donoghue, Nishioka, and Atluri (1984a, 1984b)].

The nature of singularity at the point where the crack-front interests the free-surface is still not yet completely understood. The consensus emerging from the literature of a weaker singuarity (than $1/\sqrt{r}$) at a normal crack/surface interaction has been corraborated recently, by Burton, Sinclair, Solecki, and Swedlow (1984). These authors present two independent numerical analysis techniques for the investigation of some global crack/surface interaction problems. They summarize their findings, thus: "the decays in the energy release rates found as the free surface is approached in the various problems treated are probably not significant from a fracture toughness testing point of view and not of major consequence in cyclic life calculations, although there are some indications that this may not be the case if near-surface residual stress fields are present; and that these variations in energy release rate can be compensated for by relatively minor perturbations in crack-front profiles".

Thus the results obtained by the above finite element alternating method based on the VNA solution may be thought of as being of adequate acuraccy for most engineering applications. Recently, Nishioka and Furutani (1987) have developed a more efficient alternating method for the analysis of a group of interacting multiple elliptical cracks, by taking account of geometrical symmetries of crack shapes and location in conjunction with the symmetry of the VNA solution.

Intensive studies of the performance of the finite-element alternating method have been made by Raju, Newman, and Atluri (1987) for small surface and corner cracks, and by Raju and Atluri (1988) for a part-elliptical surface crack in a cylinder. From the performance studies of the finite-element alternating method, Raju and Atluri (1988) summarized the attractive features of the alternating technique as follows:

(i) The method models only the uncracked solid with finite elements: hence, no special modeling of the crack front is required. In addition, the finite element mesh at the location of the crack, in the uncracked solid, can be completely arbitrary in geometry.

(ii) The method uses the closed-form solution for a crack in an infinite solid which can accommadate arbitrary tractions on the crack surfaces and, therefore, can handle complex loading conditions.

(iii) The stress-intensity factors, including the individual modes, are obtained as part of the solution, in an analytical form, and, hence, post-processing of the output data, as is usually done in the finite-element method, is not needed.

(iv) Several crack configurations could be analysed with a single arbitrary mesh idealization of the uncracked solid, whereas the conventional finite-element method requires a different mesh idealization of the cracked structure for each crack configuration. Thus, this method can efficiently generate very accurate stress-intensity factor weight functions or influence functions, for a variety of crack aspect ratios, in a single computer run.

The above applications of the alternating technique were limited to mode I cases. Recently, a *mixed-mode* alternating finite-element technique in conjunction with the VNA solution (with further improvements in algebraic details), has been developed by Simon, O'Donoghue, and Atluri (1987). They evaluated the polynomial influence functions for an infinite solid with an elliptical crack subject to shear loading, and for a cantilever beam with a semi-elliptical surface crack subject to end load.

Applications of the finite-element alternating method have been made by Nishioka, Rhee, and Atluri (1986), O'Donoghue, Atluri, and Rhee (1986), and Rhee (1986), for fracture mechanics analyses of various offshore structural components, such as stiffened plate and shells, tethers, or risers. A recent literature survey [Rhee and Kanninen (1988)] pointed out that the alternating method is most efficient for stress intensity factor analyses of *planar* surface or embedded flaws in complex geometries such as intersecting tubular structures, etc.

3.3 2D Alternating Techniques for Line Cracks

As mentioned earlier, the general solution for a crack subject to arbitrary crack-face tractions i.e. Solution 1 is required. The general solution for an infinite 2D anisotropic body, developed along the lines of Gladwell and England (1977), is given below. Following the solution procedure in Sih and Liebowitz (1968), the stress and displacement field can be expressed in terms of two potential ϕ and ψ as follows:

$$\begin{aligned}
\tau_{xx} &= 2\,\mathrm{Re}[s_1^2\phi'(z_1) + s_2^2\psi'(z_2)] \\
\tau_{yy} &= 2\,\mathrm{Re}[\phi'(z_1) + \psi'(z_2)] \\
\tau_{xy} &= -2\,\mathrm{Re}[s_1\phi'(z_1) + s_2\psi'(z_2)] \\
u &= 2\,\mathrm{Re}[p_1\phi(z_1) + p_2\psi(z_2)] \\
v &= 2\,\mathrm{Re}[q_1\phi(z_1) + q_2\psi(z_2)]
\end{aligned} \tag{91}$$

where

$$\begin{gathered}
s_1 = \mu_1 = \alpha_1 + i\beta_1, \qquad s_2 = \mu_2 = \alpha_2 + i\beta_2 \\
\mu_3 = \bar{\mu}_1 \qquad \mu_4 = \bar{\mu}_2 \\
z_1 = x + s_1 y \qquad z_2 = x + s_2 y \qquad s_1 \neq s_2
\end{gathered} \tag{92}$$

where α_j β_j $(j = 1,2)$ are real constants. μ_j $(j = 1,2,\cdots 4)$ are the roots of the characteristic equation

$$a_{11}\mu_j^4 - 2a_{16}\mu_j^3 + (2a_{12} + a_{66})\mu_j^2 - 2a_{26}\mu_j + a_{22} = 0 \tag{93}$$

where a_{ij} $(i,j = 1,2,\cdots,6)$ are the material constants of generalized Hooke's law

$$\begin{aligned}
\varepsilon_x &= a_{11}\tau_{xx} + a_{12}\tau_{yy} + a_{16}\tau_{xy} \\
\varepsilon_y &= a_{12}\tau_{xx} + a_{22}\tau_{yy} + a_{26}\tau_{xy} \\
\gamma_{xy} &= a_{16}\tau_{xx} + a_{26}\tau_{yy} + a_{66}\tau_{xy}
\end{aligned} \tag{94}$$

The other constants in Eq. (91) are defined as:

$$\begin{aligned}
p_1 &= a_{11}s_1^2 + a_{12} - a_{16}s_1 & p_2 &= a_{11}s_2^2 + a_{12} - a_{16}s_2 \\
q_1 &= \frac{a_{12}s_1^2 + a_{22} - a_{26}s_1}{s_1} & q_2 &= \frac{a_{12}s_2^2 + a_{22} - a_{26}s_2}{s_2}
\end{aligned} \tag{95}$$

Suppose a line crack on $y = 0$ $|x| \leq a$ in an infinite plane is inflated by equal and opposite tractions, over the faces of the crack, given by

$$\tau_{yy} - i\tau_{xy} = -[p(t) + is(t)], \qquad |t| \leq a \tag{96}$$

with zero tractions at infinity. Then the potential functions can be written as below:

$$\begin{aligned}
\phi'(z_1) &= \phi_1'(z_1) + \phi_2'(z_1) \\
\psi'(z_2) &= \psi_1'(z_2) + \psi_2'(z_2)
\end{aligned} \tag{97}$$

where

$$\begin{aligned}
\left(\frac{s_2 - s_1}{s_2}\right)\phi_1'(z_1) &= -\frac{X(z_1)}{2\pi i}\int_{-a}^{a}\frac{p(t)\mathrm{d}t}{[X(t)]^+(t - z_1)} \\
\left(\frac{s_1 - s_2}{s_1}\right)\psi_1'(z_2) &= -\frac{Y(z_2)}{2\pi i}\int_{-a}^{a}\frac{p(t)\mathrm{d}t}{[Y(t)]^+(t - z_2)}
\end{aligned} \tag{98}$$

where

$$\begin{aligned} X(z_1) &= (z_1+a)^{-1/2}(z_1-a)^{-1/2} \\ Y(z_2) &= (z_2+a)^{-1/2}(z_2-a)^{-1/2} \end{aligned} \tag{99}$$

and

$$\begin{aligned} (s_1-s_2)\phi_2'(z_1) &= -\frac{X(z_1)}{2\pi i}\int_{-a}^{a}\frac{s(t)dt}{[X(t)]^+(t-z_1)} \\ (s_2-s_1)\psi_2'(z_2) &= -\frac{Y(z_2)}{2\pi i}\int_{-a}^{a}\frac{s(t)dt}{[Y(t)]^+(t-z_2)} \end{aligned} \tag{100}$$

We approximate the applied crack-face tractions in the form [Gladwell and England (1977)]

$$p(t)+is(t) = -\sum_{n=1}^{N} b_n U_{n-1}(t) \qquad |t|\le a \tag{101}$$

where $U_{n-1}(t)$ is the Chebyshev polynomials of the second kind and is defined as

$$U_n = \sin[(n+1)\theta]/\sin\theta \qquad t = a\cos\theta \tag{102}$$

It could be easily shown that:

$$\begin{aligned} 2\phi'(z_1) &= \left(\frac{s_2}{s_2-s_1}\right)\sum_{n=1}^{N} c_n G_{n-1}(z_1) + \left(\frac{1}{s_1-s_2}\right)\sum_{n=1}^{N} d_n G_{n-1}(z_1) \\ 2\psi'(z_2) &= \left(\frac{s_1}{s_1-s_2}\right)\sum_{n=1}^{N} c_n G_{n-1}(z_2) + \left(\frac{1}{s_2-s_1}\right)\sum_{n=1}^{N} d_n G_{n-1}(z_2) \end{aligned} \tag{103}$$

where

$$\begin{aligned} c_n &= \text{real}\,(b_n) \\ d_n &= i(\text{imag}\,(b_n)) \end{aligned}$$

$$\begin{aligned} 2\phi(z_1) &= \left(\frac{s_2}{s_2-s_1}\right)\sum_{n=1}^{N} c_n \frac{R_n(z_1)}{n} + \left(\frac{1}{s_1-s_2}\right)\sum_{n=1}^{N} d_n \frac{R_n(z_1)}{n} \\ 2\psi(z_2) &= \left(\frac{s_1}{s_1-s_2}\right)\sum_{n=1}^{N} c_n \frac{R_n(z_2)}{n} + \left(\frac{1}{s_2-s_1}\right)\sum_{n=1}^{N} d_n \frac{R_n(z_2)}{n} \end{aligned} \tag{104}$$

and

$$G_{n-1}(z_\alpha) = -(z_\alpha^2-a^2)^{-1/2}R_n(z_\alpha) \qquad (\alpha=1,2) \tag{105}$$

$$R_n(z_\alpha) = a\{z_\alpha/a - (z_\alpha^2/a^2-1)^{1/2}\}^n \qquad (\alpha=1,2) \tag{106}$$

The stress intensity factors K_j $(j=I,II)$ are defined in a manner consistent with those for isotropic materials.

$$\begin{aligned} K_I &= 2\sqrt{2\pi}\left(\frac{s_2-s_1}{s_2}\right)\lim_{z_1\to a}(z_1-a)^{1/2}\phi_1'(z_1) \\ K_{II} &= 2\sqrt{2\pi}(s_2-s_1)\lim_{z_1\to a}(z_1-a)^{1/2}\phi_2'(z_1) \end{aligned} \tag{107}$$

It is easy to show that

$$\begin{aligned} K_I - iK_{II} &= -\sqrt{\pi a}\sum_{n=1}^{N} b_n \qquad \text{for} \quad x=a \\ K_I - iK_{II} &= \sqrt{\pi a}\sum_{n=1}^{N}(-1)^n b_n \qquad \text{for} \quad x=-a \end{aligned} \tag{108}$$

The general solution for an infinite isotropic 2D solid with a crack subject to arbitrary Chebyshev polynomial loadings can be found in Rajiyah and Atluri (1988).

In the infinite solution, the crack face tractions are defined on the entire embedded crack. It is thus necessary, in edge crack problems, for tractions to be defined over the entire crack plane, including the fictitious portion of the crack which lies outside the finite body. The actual distribution chosen for the fictitious tractions on the crack face defined outside the finite body will only affect the character of convergence. The solution procedure for an edge crack is the same as the one described for an embedded crack except for minor differences. Now a mirror image of the half crack and the tractions acting on the crack face are extended to obtain the full crack length including the fictitious portion of the crack. Using the analytical solution for an embedded crack in an infinite domain, the stresses can be evaluated at the boundary of the cracked specimen. The rest of the algorithm remains the same as the case of an embedded crack.

Using the above-mentioned 2D alternating technique the mixed-mode weight functions have been cost-effectively determined for isotropic plates (Rajiyah and Atluri, 1988a,b) and for orthotropic plates (Chen and Atluri, 1988a,b). For isotropic plates, Rajiyah and Atluri (1988a,b) have used the boundary element alternating method. In this case, the boundary element method is better suited, since, a point-wise evaluation of stresses at the location of the crack in the uncracked body is more accurate, and more simple, once the tractions and displacement on the boundary are determined through the standard BEM. It can be expected that the above method would yield highly accurate results for this class of problems, in the least expensive way even compared to the finite element alternating method.

On the other hand, for anisotropic plates, since the boundary element method itself becomes cumbersome, the finite element alternating method is more suited, as has been used by Chen and Atluri (1988a,b).

4 Domain Integral Methods for Computing Fracture Parameters in 3-Dimensional Crack Problems, Under Arbitrary Histories of Loading

It is well-known that energetic methods, such as based on the J-integral and other crack-tip integral parameters, play an important role in elastic-plastic and inelastic fracture mechanics, under arbitrary histories of loading [see, for instance, the monograph edited by Atluri (1986)]. For elastic crack problems, in three-dimensions, the evaluation of the J-integral (in Mode I problems), the stiffness derivative method [Parks (1974, 1977)] and the virtual crack extension method [Hellen (1975)] proved to be quite useful. Various extensions to, and a certain variety of improvements of, these methods were recently presented by Delorenzi (1985); Li, Shih, and Needleman (1985); Shih, Moran, and Nakamura (1986); Nakamura, Shih, and Freund (1986); Nikishkov and Atluri (1987a,b and 1988). These later methods are currently labeled as the "Domain Integral Methods" for the computation of crack-tip integral parameters in non-elastic fracture under arbitrary load histories. In the following, a brief description of these domain-integral methods, as applicable to the analysis of mixed-mode behaviour of arbitrary shaped cracks (surfaces of discontinuity) in 3-D structures, with elastic-plastic or inelastic material behaviour, and under arbitrary loading, is given.

For *elastic* problems, the energy release rate per unit crack-extension (in a self-similar fashion) is given by:

$$J_1 = \mathcal{G} = \int_{S_t} t_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \mathrm{d}S + \int_V f_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}a} \int_V W \,\mathrm{d}V \tag{109}$$

(assuming that S_u, where non-zero u_i are prescribed, is zero). Eq. (109) is valid for mixed-mode loadings, and in 3-D problems, it is understood that $(\mathrm{d}u_i/\mathrm{d}a)$ represents a first-order change in u_i due to a *local* perturbation in the crack-front, and J_1 is the *local* energy-release rate. In a finite element model, Eq. (109) may be written as:

$$J_1 = \boldsymbol{Q}\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}a} - \frac{\mathrm{d}}{\mathrm{d}a}(\frac{1}{2}\boldsymbol{qKq}) \tag{110}$$

$$= -\frac{1}{2}\boldsymbol{q}\frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}a}\boldsymbol{q} \tag{111}$$

$$= -\frac{1}{2}\boldsymbol{q}_o\frac{\mathrm{d}\boldsymbol{K}_o}{\mathrm{d}a}\boldsymbol{q}_o \tag{112}$$

Eq. (111) follows from (110) since $\boldsymbol{Q} = \boldsymbol{qK}$ at equilibrium and Eq. (112) follows from (111) since a change in crack length can be seen to affect the stiffness matrix of only a small-core of material, in the domain V_o, near the crack front. The evaluation of $\mathrm{d}\boldsymbol{K}_o/\mathrm{d}a$ is usually accomplished through a finite difference method [Parks (1974), and Hellen (1975)]. The definitions of J_2 and J_3, which involve the more mathematical concepts of translation of the crack, in x_2 and x_3 directions [see Atluri (1986)], are not, in general, amenable to calculation through the above stiffness-derivative methods.

The basic advantages of the domain-integral method can be seen from the following, to be: (i) they allow simple computation of all the 3 components J_k of the vector J integral; and (ii) in the case of J_1, the need for a finite difference evaluation of $(\mathrm{d}\boldsymbol{K}_o/\mathrm{d}a)$ is obviated.

In a general 3-D problem, for arbitrary material behaviour and loading, one may define the J-integral vector components as:

$$J_k = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \left[W n_k - \sigma_{ij} \frac{\partial u_i}{\partial x_k} n_j \right] \mathrm{d}\Gamma \tag{113}$$

where W is the density of *stress-work* in arbitrary loading and arbitrary material behaviour; σ_{ij} are stresses, u_i are displacements, and n_k are components of the unit normal vector to the surface of the tube at points on contour Γ_ε. In principle it is possible to define J_k in any coordinate system, but for the purposes of prediction of crack behaviour, it is more convenient to have a local crack-front coordinate system x_1, x_2, x_3: x_1 is normal to the crack front and lies in the plane of the crack surface, x_2 is orthogonal to x_1 and the crack surface, and x_3 is tangential to the crack-front and in

the crack-plane. We introduce the equivalent definition of the near-tip J-integral along the surface of the tube, as:

$$J_k \Delta = \lim_{\substack{\varepsilon/\Delta \to 0 \\ \Delta \to 0}} \int_{A_\varepsilon} \left[W n_k - \sigma_{ij} \frac{\partial u_i}{\partial x_k} n_j \right] \mathrm{d}A, \qquad k = 1, 2 \tag{114}$$

where A_ε is the surface of a cylinder, with the centerline along the crack-front, its radius being ε, and the length of the generator being Δ along the crack-front. This definition is more convenient for numerical applications. Likewise, one may be define the energy-release components for symmetric and anti-symmetric deformation modes, as:

$$G_I \Delta = \lim_{\substack{\varepsilon/\Delta \to 0 \\ \Delta \to 0}} \int_{A_\varepsilon} \left(W^I n_1 - \sigma_{ij}^I \frac{\partial u_i^I}{\partial x_1} n_j \right) \mathrm{d}A \tag{115}$$

$$G_{II} \Delta = \lim_{\substack{\varepsilon/\Delta \to 0 \\ \Delta \to 0}} \int_{A_\varepsilon} \left(W^{II} n_1 - \sigma_{ij}^{II} \frac{\partial u_i^{II}}{\partial x_1} n_j \right) \mathrm{d}A \tag{116}$$

and

$$G_{III} \Delta = \lim_{\substack{\varepsilon/\Delta \to 0 \\ \Delta \to 0}} \int_{A_\varepsilon} \left(W^{III} n_1 - \sigma_{3j} \frac{\partial u_3}{\partial x_1} n_j \right) \mathrm{d}A \tag{117}$$

where the deformation field is decomposed, locally near each differential segment of the crack-front, into symmetrical and skew-symmetric parts about the crack-plane locally, as follows:

$$\{u\} = \{u^I\} + \{u^{II}\} + \{u^{III}\} = \frac{1}{2}\left\{ \begin{array}{c} u_1 + u_1' \\ u_2 - u_2' \\ u_3 + u_3' \end{array} \right\} + \frac{1}{2}\left\{ \begin{array}{c} u_1 - u_1' \\ u_2 + u_2' \\ 0 \end{array} \right\} + \frac{1}{2}\left\{ \begin{array}{c} 0 \\ 0 \\ u_3 - u_3' \end{array} \right\} \tag{118}$$

$$\{\sigma\} = \{\sigma^I\} + \{\sigma^{II}\} + \{\sigma^{III}\} = \frac{1}{2}\left[\begin{array}{c} \sigma_{11} + \sigma_{11}' \\ \sigma_{22} + \sigma_{22}' \\ \sigma_{33} + \sigma_{33}' \\ \sigma_{12} - \sigma_{12}' \\ \sigma_{23} - \sigma_{23}' \\ \sigma_{31} - \sigma_{31}' \end{array} \right] + \frac{1}{2}\left[\begin{array}{c} \sigma_{11} - \sigma_{11}' \\ \sigma_{22} - \sigma_{22}' \\ 0 \\ \sigma_{12} + \sigma_{12}' \\ 0 \\ 0 \end{array} \right] + \frac{1}{2}\left[\begin{array}{c} 0 \\ 0 \\ \sigma_{33} - \sigma_{33}' \\ 0 \\ \sigma_{23} + \sigma_{23}' \\ \sigma_{31} + \sigma_{31}' \end{array} \right] \tag{119}$$

$$u_i'(x_1, x_2, x_3) = u_i(x_1, -x_2, x_3) \tag{120}$$

$$\sigma_{ij}'(x_1, x_2, x_3) = \sigma_{ij}(x_1, -x_2, x_3) \tag{121}$$

4.1 Transformation of Displacements, Strains, and Stresses to the Crack Front Coordinate System

We can simplify many developments if this transformation is performed prior to the calculation of J- and G components. Let X_1, X_2, X_3 be a global Cartesian coordinate system; and x_1, x_2, x_3 be the crack front coordinate system for a particular point along the crack front. For the definition of a crack front coordinate system at any point, it is sufficient to have the direction cosines for a unit vector along x_1

$$X_p = \{X_{p1}, X_{p2}, X_{p3}\} \tag{122}$$

and for a unit vector along x_3

$$Z_p = \{Z_{p1}, Z_{p2}, Z_{p3}\} \tag{123}$$

Then it is easy to define the orientation of x_2 as

$$\begin{aligned} Y_p &= Z_p X X_p & (124) \\ Y_{p1} &= Z_{p2}X_{p3} - Z_{p3}X_{p2} \\ Y_{p2} &= Z_{p3}X_{p1} - Z_{p1}X_{p3} \\ Y_{p3} &= X_{p1}X_{p2} - Z_{p2}X_{p1} & (125) \end{aligned}$$

We define the coefficients of a transformation matrix a_{ij} as:

$$\begin{array}{lll} a_{11} = X_{p1} & a_{12} = X_{p2} & a_{13} = X_{p3} \\ a_{21} = Y_{p1} & a_{22} = Y_{p2} & a_{23} = Y_{p3} \\ a_{31} = Z_{p1} & a_{32} = Z_{p2} & a_{33} = Z_{p3} \end{array} \tag{126}$$

The transformation of coordinates, displacements, strains, and stresses can be done as follows:

$$\begin{aligned} x_i &= a_{ij}X_j & (127) \\ u_i &= a_{ij}u_i^g & (128) \\ \varepsilon_{ij} &= a_{ip}a_{jq}\varepsilon_{pq}^g & (129) \\ \sigma_{ij} &= a_{ip}a_{jq}\sigma_{pq}^g & (130) \end{aligned}$$

Here the superscript g stands for the values in global coordinate system.

4.2 EDI-Technique for J_1, J_2, and G Calculation

After a point by point coordinate transformation (127), the crack front is straight. Let us consider the segment of crack front and the volume around this segment inside a *larger* cylindrical domain V. V is the volume of the larger cylinder, V_ε is the volume of small cylinder of radius ε around the crack front segment, A is the cylindrical surface of V; A_ε is the cylindrical surface of V_ε and A_1, A_2 are side surfaces of V. Note that at any differential segment along the crack-front, the considered domain V is still much smaller than the overall dimensions of the structure.

Then, in general, we can redefine the near-tip parameters J_k and G_{III} as:

$$J_k f = -\int_{A-A_\varepsilon} \left(W n_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k} n_j \right) s\, \mathrm{d}A \tag{131}$$

$$G_{III} f = -\int_{A-A_\varepsilon} \left(W^{III} n_1 - \sigma_{3j}\frac{\partial u_3}{\partial x_1} n_j \right) s\, \mathrm{d}A \tag{132}$$

Here, $s = s(x_1, x_2, x_3)$ is an arbitrary but continous function which is equal to zero on A; and nonzero (equal to 1) on A_ε; and f is the area under the s-function curve along the segment of the crack-front under consideration.

Using the divergence theorem, we have the following representation of J_k.

$$\begin{aligned} J_k f = & -\int_{V-V_\varepsilon} \left(W\frac{\partial s}{\partial x_k} - \sigma_{ij}\frac{\partial u_i}{\partial x_k}\frac{\partial s}{\partial x_j} \right) \mathrm{d}V - \int_{V-V_\varepsilon} \left[\frac{\partial W}{\partial x_k} - \frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k} \right) \right] s\, \mathrm{d}V \\ & + \int_{A_1+A_2} \left(W n_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k} n_j \right) s\, \mathrm{d}A \qquad k = 1, 2 \end{aligned} \tag{133}$$

This expression represents a further variant of the virtual crack extension method, but the elimination of the actual process of virtual crack extension during the development of (133) allows us to use any s-function for the calculation of J_k. Thus, we have a new and computationally more appealing interpretation of the VCE approach.

In the case of the presence of nonelastic (thermal and plastic) deformations we can define W as

$$W = \int \sigma_{ij}\, \mathrm{d}\varepsilon_{ij} \tag{134}$$

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p + \varepsilon_{ij}^t \tag{135}$$

where ε_{ij}^e, ε_{ij}^p, and ε_{ij}^t are elastic, plastic, and thermal parts of strains. Assuming that the stresses have an elastic potential, i.e.,

$$\sigma_{ij} = \frac{\partial W^e}{\partial \varepsilon_{ij}^e} \tag{136}$$

the second term of (133) can have the form

$$\begin{aligned} (J_k f)_2 &= -\int_{V-V_\varepsilon} \left(\frac{\partial W}{\partial x_k} - \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_k} \right) s\, \mathrm{d}V \\ &= -\int_{V-V_\varepsilon} \left(\frac{\partial W^p}{\partial x_k} - \sigma_{ij}\frac{\partial \varepsilon_{ij}^{pt}}{\partial x_k} \right) s\, \mathrm{d}V \end{aligned} \tag{137}$$

Here we used equilibrium equations (in the absence of body forces), and introduced the definitions:

$$W^p = \int \sigma_{ij}\, \mathrm{d}\varepsilon_{ij}^{pt} \tag{138}$$

$$\varepsilon_{ij}^{pt} = \varepsilon_{ij}^p + \varepsilon_{ij}^t \tag{139}$$

It is evident that in the absence of nonelastic strains the second term of (133) is equal to zero. If the s function is equal to zero on faces A_1 and A_2, then the third term of (133) will be equal to zero as well.

Considering G_{III} for the linear elastic case (in the absence of body forces), we can have from equation (132),

$$\begin{aligned} G_{III} f = & -\int_{V-V_\varepsilon} \left(W^{III}\frac{\partial s}{\partial x_1} - \sigma_{3j}\frac{\partial u_3}{\partial x_1}\frac{\partial s}{\partial x_j} \right) \mathrm{d}V - \int_{V-V_\varepsilon} \sigma_{3j}\frac{\partial \varepsilon_{ij}^t}{\partial x_1} s\, \mathrm{d}V \\ & + \int_{A_1+A_2} \left(W^{III} n_1 - \sigma_{3j}\frac{\partial u_3}{\partial x_1} n_j \right) s\, \mathrm{d}A \end{aligned} \tag{140}$$

The third terms of (133) and (140) can be simplified if the faces A_1 and A_2 are orthogonal to the crack front ($n_1 = n_l = 0$ on A_1 and A_2).

Again, in Eq. (140), the second term is equal to zero if $\varepsilon^t = 0$ and the third term is absent if $s = 0$ on A_1 and A_2. We note that the domain integral algorithms analogous to those in (133) can be developed directly for the energy-release-rate quantities G_I and G_{II} as defined in Eqs. (115) and (116) also.

It is now easy to see the advantage of the domain-integral method over the "stiffness derivative" method for the computation of the first component of J, i.e., J_1, for *elasto-static problems.* If in Eq. (133), the function s is taken to be zero at $(A_1 + A_2)$; and if body-forces are zero, in *elasto-static* problems the second and third terms on the r.h.s. in (133) vanish identically, and J_1 can be written as:

$$J_1 = -\lim_{\varepsilon\to 0}\int_{V-V_\varepsilon}\left[W\frac{\partial s}{\partial x_1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1}\frac{\partial s}{\partial x_j}\right] dV \tag{141}$$

We compare Eq. (112) of the stiffness derivative method to the above Eq. (141). Without loss of generality, we assume that the domain V_o of Eq. (112) to be the same as the domain $(V - V_\varepsilon)$ of (141). In (112), it is clear that the integral is quadratic in $\boldsymbol{q}_o$, say of the form $\boldsymbol{q}_o \cdot \mathbf{A}_o \cdot \boldsymbol{q}_o$. Thus the domain integral method gives *directly* the matrix $\boldsymbol{A}_o$ [which is equivalent to $d\boldsymbol{K}_o/da$] without using the finite difference method to evaluate $(d\boldsymbol{K}_o/da)$ as in (112) of the virtual crack extension or the stiffness derivative method.

4.3 Choice of s-Function

It is natural to use a parametric representation of function s inside any element as:

$$s = N^I s^I \qquad (I = 1\ldots 20) \tag{142}$$

where $N^I = N^I(\xi,\eta,\varsigma)$ — quadratic shape functions, I is the node number. We suppose summation over repeated indices. Then the s-function should be defined in terms of (142) by using 1 or 2 elements in x_3 direction for the crack front disk. Usually it is not useful to have a s-function more complicated than a linear function in radial direction. Several simple functions s are discussed below:

(a) The disk has two elements each in x_1 in x_3 directions respectively. The function s can be defined on the small tube of radius ε, for both the elements along x_3 (with ς being the natural coordinate along the crack front segment), as:

$$\text{El. 1} \quad : \quad s = \frac{1}{2}(1+\varsigma)$$
$$\text{El. 2} \quad : \quad s = \frac{1}{2}(1-\varsigma)$$

The area under the s-function curve along the meridian of the surface of the small tube or on the crack-front ($\varepsilon = 0$) is equal to

$$f = \frac{1}{2}(\Delta_1 + \Delta_2)$$

(b)

$$\text{El. 1} \quad : \quad s = \frac{1}{2}(\varsigma^2+\varsigma)$$
$$\text{El. 2} \quad : \quad s = \frac{1}{2}(\varsigma^2-\varsigma)$$
$$f = \frac{1}{3} + (\Delta_1 + \Delta_2)$$

(c)

$$s = (1-\varsigma^2)$$
$$f = \frac{2}{3}\Delta$$

(d)

$$s = 1$$
$$f = \Delta$$

Assuming a s-function that is linear in the r-direction we have its value for a particular point:

$$s = s_o\frac{r - r_\varepsilon}{r_f - r_\varepsilon}$$

where s_o – the value of s function at the point $r = r_\varepsilon$, r is the distance of the point in question from the surface of small tube, r_ε is the radius of the small tube, r_f – the outer radius of crack front disk. In practice it is often useful to have one element in the r-direction for the disk and to employ degenerate quarter-point singular elements around the crack tip.

4.4 First Term of J_k

Using the parametric representation of displacements

$$u_i = N^J u_i^J \tag{143}$$

where i is the direction of crack front coordinate system and the superscript J is the node number, it is possible to have such an expression for the calculation of the first term of J-integral of Eq. (133):

$$(J_k f)_1 = -\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(W\frac{\partial N^L}{\partial x_k}s^L - \sigma_{ij}\frac{\partial N^M}{\partial x_k}\frac{\partial N^L}{\partial x_j}u_i^M s^L\right)\det(j)\,d\xi\,d\eta\,d\varsigma \tag{144}$$

where $\det(j)$ is the determinant of Jacobi matrix. An effective procedure of computing J_k with several types of s-functions consists of a separate $2\times 2\times 2$ integration of the expression

$$R_k^L = -\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(W\frac{\partial N^L}{\partial x_k} - \sigma_{ij}\frac{\partial N^M}{\partial x_k}\frac{\partial N^L}{\partial x_j}u_i^M\right)\det(j)\,d\xi\,d\eta\,d\varsigma \tag{145}$$

and defining a scalar product:

$$(J_k f)_1 = R_k^L s^L \tag{146}$$

4.5 The Second Term of J_k

The main difficulty in integrating (137) arises from the fact that we know precise enough values of strain energy density, and strain, only at the $2\times2\times2$ Gauss Integration points. A possible way of integrating the derivative of such functions is to obtain the derivative at the center of the element and to perform a one-point integration.

Consider a 20-node element in local coordinate system ξ_i

$$\xi_1 = \xi, \quad \xi_2 = \eta, \quad \xi_3 = \varsigma$$

Let's assume that we know that values of the function F only at the integration points as $F^{(I)}$. Using a parametric representation, it is possible to write

$$F^{(J)} = L^{I(J)} F^I \tag{147}$$

where F^I are unknown values of F at corner nodes $1\ldots8$, L^I are linear shape functions for corner nodes, $L^{I(J)}$ are values of shape functions at integration points (J).

The inversion of (147) gives

$$F^I = (L^{I(J)})^{-1} F^{(J)} \tag{148}$$

The coefficients of the extrapolation matrix $(L^{I(J)})^{-1}$ are:

$$(L^{I(J)})^{-1} = \begin{bmatrix} A & B & C & B & B & C & D & C \\ & A & B & C & C & B & C & D \\ & & A & B & D & C & B & C \\ & & & A & C & D & C & B \\ & & & & A & B & C & B \\ \text{sym.} & & & & & A & B & C \\ & & & & & & A & B \\ & & & & & & & A \end{bmatrix} \tag{149}$$

$$A = \frac{5+3\sqrt{3}}{4} \quad B = -\frac{\sqrt{3}+1}{4} \quad C = \frac{\sqrt{3}-1}{4} \quad D = \frac{5-3\sqrt{3}}{4} \tag{150}$$

Now, from (148) we can calculate the derivative at the center of the element:

$$\frac{\partial F}{\partial x_k} = \frac{\partial L^I}{\partial x_k} (L^{I(J)})^{-1} F^{(J)} \tag{151}$$

where, the derivatives $\partial L^I/\partial x_k$ should be calculated for $\xi = \eta = \varsigma = 0$.

4.6 Third Term of J_k of Eq. (133)

For simplicity consider the disk with A_1 and A_2 being orthogonal to the crack front segment. Then

$$n_1 = n_2 = 0, \quad n_3 = 1 \quad \text{on } A_1$$
$$n_1 = n_2 = 0, \quad n_3 = -1 \quad \text{on } A_2$$

$$(J_k f)_3 = -\int_{A_1+A_2} \sigma_{i3} \frac{\partial u_i}{\partial x_k} n_3 s \, \mathrm{d}A \tag{152}$$

If $A_1 = A_2$

$$(J_k f)_3 = -\int_{A_1} \Delta\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right) s \, \mathrm{d}A \tag{153}$$

where $\Delta(F) = (F)_{A_1} - (F)_{A_2}$.

Assuming that every function is linear in the x_3 direction and using the values of functions at integration points $\varsigma = \pm(1/\sqrt{3})$, it is possible to define ΔF as

$$\Delta F = \sqrt{3}\left[F\left(\varsigma = \frac{1}{\sqrt{3}}\right) - F\left(\varsigma = -\frac{1}{\sqrt{3}}\right)\right]$$

Then

$$(J_k f)_3 = -\int_{-1}^{1}\int_{-1}^{1} \sqrt{3}\Delta\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right)^{(T)} s \det(J) \, \mathrm{d}\xi \, \mathrm{d}\eta \tag{154}$$

where $\Delta F^{(T)} = F(\varsigma = \frac{1}{\sqrt{3}}) - F(\varsigma = -\frac{1}{\sqrt{3}})$.

Examples of three-dimensional J_1 computations using the domain-integral methods may be found in Nikishkov and Atluri (1987b), and Nakamura, Shih, and Freund (1986). Application of the "domain-integral" type evaluation of the crack-tip integral parameters in viscoplastic dynamic crack propagation at fast speeds, has been discussed in Yoshimura, Yagawa, and Atluri (1988).

5 Weight-Functions for 2 and 3-D Elastic Crack Problems

The concept of weight functions for elastic crack problems dates back to the work of Bueckner (1971) and Rice (1972) [see also Bortman and Banks-Sills (1983)]. The "weight function" may generally be viewed as the appropriately normalized rate of change of displacements (at the surface where tractions are applied, or in the domain where body forces are applied) due to a unit change in the crack length for a *reference state* of loading. The practical importance of the concept of the weight functions lies in the fact that, when the weight functions are evaluated from a (*perhaps simple*) *reference state of loading*, then the stress-intensity factors for any arbitrary state of loading can be computed by using an integral of the worklike product between the applied tractions at a point on the surface in the arbitrary state of loading and the weight function for the reference state at the same point.

The energy-release due to a unit crack-extension in a cracked elastic body, subject to a system of surface tractions, (We assume, for simplicity, that the surface S_u where non-zero displacements are prescribed, is zero. *One can easily generalize the ensuing discussion to the situation when S_u is nonzero.*), and body forces, is given by:

$$\mathcal{G} = \int_{S_t} t_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \mathrm{d}S + \int_V f_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}a}\int_V W \, \mathrm{d}V \tag{155}$$

where t_i are tractions applied at the surface S_t; f_i are body forces in the domain V; u_i are displacements, and W is the strain-energy density (internal energy in mechanical work). Eq. (155) may be written as:

$$\mathcal{G} + \int_{S_t} \frac{\mathrm{d}t_i}{\mathrm{d}a} u_i \, \mathrm{d}S + \int_V \frac{\mathrm{d}f_i}{\mathrm{d}a} u_i \, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}a}\{\int_{S_t} t_i u_i \, \mathrm{d}S + \int_V f_i u_i \, \mathrm{d}V - \int_V W \, \mathrm{d}V\} \tag{156}$$

or

$$\mathcal{G}\,\mathrm{d}a + \int_{S_t} \mathrm{d}t_i u_i \,\mathrm{d}S + \int_V \mathrm{d}f_i u_i \,\mathrm{d}V = -\mathrm{d}\pi \tag{156}$$

Let the reference load state be characterized by a parameter λ. Thus,

$$\mathcal{G}\,\mathrm{d}a + \mathrm{d}\lambda q(\lambda) = -\mathrm{d}\pi \tag{157}$$

where

$$q(\lambda) = \int_{S_t} \hat{t}_i u_i \,\mathrm{d}S + \int_V \hat{f}_i u_i \,\mathrm{d}V \tag{158}$$

where, $\mathrm{d}t_i = \mathrm{d}\lambda \hat{t}_i$; $\mathrm{d}f_i = \mathrm{d}\lambda \hat{f}_i$; and, in general, in a nonlinear elastic problem, the generalized displacement q is a nonlinear function of λ. Equation (157) implies that:

$$\left(\frac{\partial \mathcal{G}}{\partial \lambda}\right)_a = \left(\frac{\partial q}{\partial a}\right)_\lambda \tag{159}$$

Consider a linear-elastic homogeneous solid, that is in general *anisotropic*, and consider the case when the crack is at an arbitrary angle to the material directions, and under a general mixed mode loading. The energy release rate, $\mathcal{G}$, for a mixed-mode crack in a monoclinic anisotropic solid may be written as:

$$\mathcal{G} = AK_I^2 + BK_{II}^2 + CK_I K_{II} \tag{160}$$

where

$$\begin{aligned} A &= -\frac{\pi}{2} a_{12} \,\mathrm{Im}\left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2}\right) \\ B &= \frac{\pi}{2} a_{11} \,\mathrm{Im}(\mu_1 + \mu_2) \\ C &= \frac{\pi}{2}\left\{ -a_{22} \,\mathrm{Im}\left(\frac{1}{\mu_1 \mu_2}\right) + a_{11} \,\mathrm{Im}(\mu_1 \mu_2)\right\} \end{aligned} \tag{161}$$

where a_{ij} are material constants in the relation $\varepsilon_i = a_{ij}\sigma_j$ $(i,j = 6)$ and μ_j are the complex roots of the characteristic equation, $a_{11}\mu_j^4 - 2a_{16}\mu_j^3 + (2a_{12} + a_{66})\mu_j^2 - 2a_{26}\mu_j + a_{22} = 0$. [See Sih and Liebowitz (1968) for further details.] In the case of isotropy, (162) reduces to

$$\begin{aligned} &A = \frac{1}{H}; \quad B = \frac{1}{H}; \quad C = 0 \\ &H = E/(1-\nu^2) \ \text{plane strain}; \quad H = E \ \text{plane stress} \end{aligned} \tag{162}$$

We now consider the *simultaneous action* of *2 load-systems* on the cracked body. The load system is:

$$(\lambda^1 \hat{t}_i^1 + \lambda^2 \hat{t}_i^2) \ \text{at}\ S_t; \quad \text{and} \quad (\lambda^1 \hat{f}_i^1 + \lambda^2 \hat{f}_i^2) \ \text{in}\ V \tag{163}$$

When (163) is used in (157) one obtains:

$$\mathcal{G}\,\mathrm{d}a + \mathrm{d}\lambda^R C_{Rn}\lambda^n = -\mathrm{d}\pi; \qquad R,n = 1,2 \tag{164}$$

or

$$\frac{\partial \mathcal{G}}{\partial \lambda^R} = \frac{\mathrm{d}C_{Rn}}{\mathrm{d}a}\lambda^n \tag{165}$$

and

$$\frac{\mathrm{d}C_{Rn}}{\mathrm{d}a} = \int_{S_t} \hat{t}_i^n \frac{\mathrm{d}\hat{u}_i^R}{\mathrm{d}a}\,\mathrm{d}S + \int_V \hat{f}_i^n \frac{\mathrm{d}\hat{u}_i^R}{\mathrm{d}a}\,\mathrm{d}V \qquad R,n = 1,2 \ \text{load cases} \quad i = 1,2,3 \tag{166}$$

The K-factors under the combined mode loading are

$$K_I = \hat{K}_I^m \lambda^m; \qquad K_{II} = \hat{K}_{II}^m \lambda^m \tag{167}$$

[sum on m = 1,2] and

$$\mathcal{G} = A\hat{K}_I^m \hat{K}_I^n \lambda^m \lambda^n + B\hat{K}_{II}^m \hat{K}_{II}^n \lambda^m \lambda^n + C\hat{K}_I^m \hat{K}_{II}^n \lambda^m \lambda^n \tag{168}$$

[sum $m,n = 1,2$]. Using (168) in (165), and observing that the resulting equation is valid for arbitrary λ^1 and λ^2 one obtains:

$$(2A + C)\hat{K}_I^R \hat{K}_I^n + (2B + C)\hat{K}_{II}^R \hat{K}_{II}^n = \frac{\mathrm{d}C_{nR}}{\mathrm{d}a} \tag{169}$$

$$= \int_{S_t} \hat{t}_i^n \frac{\mathrm{d}\hat{u}_i^R}{\mathrm{d}a}\,\mathrm{d}S + \int_V \hat{f}_i^n \frac{\mathrm{d}\hat{u}_i^R}{\mathrm{d}a}\,\mathrm{d}V \qquad R,n = \text{load cases} \tag{170}$$

Let R be a known reference load-state, for which the solution, i.e., $\hat{K}_I^R$, $\hat{K}_{II}^R$, and $(\mathrm{d}\hat{u}_i^R/\mathrm{d}a)$ are known, and n is an arbitrary load-state for which the mixed mode factors K_I^n and K_{II}^n are to be computed. Eq. (170) is thus a single equation governing the two unknowns K_I^n and K_{II}^n. By writing Eq. (170) for *two known and linearly independent reference states*, two equations for two unknowns K_I^n and K_{II}^n can be obtained. The solution of these equations can be seen to be:

$$K_I = \frac{\hat{K}_{II}^{R2}\lambda^m}{K^R(2A+C)}\frac{\mathrm{d}C_{mR1}}{\mathrm{d}a} - \frac{\hat{K}_{II}^{R1}\lambda^m}{K^R(2A+C)}\frac{\mathrm{d}C_{mR2}}{\mathrm{d}a} \quad (\text{no sum on } m) \tag{171}$$

$$K_{II} = \frac{\hat{K}_I^{R1}\lambda^m}{K^R(2B+C)}\frac{\mathrm{d}C_{mR2}}{\mathrm{d}a} - \frac{\hat{K}_I^{R2}\lambda^m}{K^R(2B+C)}\frac{\mathrm{d}C_{mR1}}{\mathrm{d}a} \quad (\text{no sum on } m) \tag{172}$$

and

$$K^R = \hat{K}_I^{R1}\hat{K}_{II}^{R2} - \hat{K}_I^{R2}\hat{K}_{II}^{R1} \tag{173}$$

where $(\mathrm{d}C_{mR1}/\mathrm{d}a)$ etc. are defined from (166) by replacing R by $R1$ etc. It is seen that for K^R to be *nonzero*, the reference states should not be both of either Mode I or Mode II. Furthermore, in a general anisotropic body with an arbitrarily oriented crack, the reference states $R1$ and $R2$ can be taken to either loads on external surfaces, or tractions on the crack-faces themselves.

Thus, to evaluate the mixed-mode load factors for any reference state m, one only needs the appropriately normalized weight-functions, $(\mathrm{d}u_i^{R1}/\mathrm{d}a)$ and $(\mathrm{d}u_i^{R2}/\mathrm{d}a)$. In the following we discuss some recent work on computational methods for these weight functions, for anisotropic or isotropic materials.

5.1 Weight Functions, Using Finite Element / Boundary Element Models of only Uncracked Structures:

Recently Chen and Atluri (1988a,b) and Rajiyah and Atluri (1988a,b) developed simple methods for computing weight functions using finite element or boundary element models of only the *uncracked* structure.

It is worth noting that, due to the complications of the fundamental solutions (for a point load) for a general anisotropic medium, the boundary element method is not convenient for application to the anisotropic soilds. However, it is well known that the Galerkin finite element method does not have this restriction. Chen and Atluri (1988) use the finite-element alternating method (for general anisotropic solids) and Rajiyah and Atluri (1988) use the boundary-element alternating method (for isotropic solids) in computing the weight functions.

While the load-systems are, in general, considered to be at S_t and in V, it is convenient to consider only the complementary problem of tractions on the crack-face alone, and consider the case, for simplicity, when the body forces are zero. Thus, hence forth we treat S_t to be the crack-face alone. It is seen that the weight-functions (du_i^R/da) on the crack-face will be singular (of the $r^{-1/2}$ type, where r is the distance from the crack-tip). Thus, special quadrature rules are needed to integrate the quantity $t_i^m\ [du_i^R/da]$ on the crack-face [Chen and Atluri (1988)].

The following solution procedure is adopted to compute the weight-functions for an embedded or edge crack in a general anisotropic, finite-dimensional structure, when the crack is oriented arbitrarily with respect to the material axes of anisotropy.

(A) Consider two different reference states: one a normal pressure (say constant) on the crack-face and the second a shear traction (say constant) the crack-face. These two load-states are labelled $R1$ and $R2$ respectively. Henceforth it is understood that the following steps are carried out for states $R1$ and $R2$ respectively.

(B) The start with, treat the problem as one of an *infinite domain*. As discussed in Section 3.3 of this paper, expand the applied tractions on the crack face in the form [Gladwell and England (1977)]:

$$\tau_{yy} - i\tau_{xy} = -[p(t) + is(t)] = \sum_{n=1}^{N} b_n U_{n-1}(t) \qquad |t| \le a \tag{174}$$

where $U_{n-1}(t)$ is the Chebyshev polynomial of the second kind, defined as:

$$U_n = \sin[(n+1)t]/\sin\theta; \qquad t = a\cos\theta \tag{175}$$

and b_n are the parameters determined by curve-fitting. For this applied loading on the crack-face in an infinite anisotropic body, the solution for the K-factors, far-field stresses, and crack-face displacements, can be derived [Gladwell and England (1977)], as:

$$K_I - iK_{II} = \mp\sqrt{\pi a}\sum_{n=1}^{N} b_n \qquad x = \pm a \tag{176}$$

$$\begin{aligned} u_x &= 2\mathrm{Re}[p_1\phi(z_1) + p_2\psi(z_2)] \\ u_y &= 2\mathrm{Re}[q_1\phi(z_1) + q_2\psi(z_2)] \end{aligned} \tag{177}$$

$$\begin{aligned} \tau_{xx} &= 2\,\mathrm{Re}[s_1^2\phi'(z_1) + s_2^2\psi'(z_2)] \\ \tau_{yy} &= 2\,\mathrm{Re}[\phi'(z_1) + \psi'(z_2)] \\ \tau_{xy} &= -2\,\mathrm{Re}[s_1\phi'(z_1) + s_2\psi'(z_2)] \end{aligned} \tag{178}$$

$$\begin{aligned} 2\phi(z_1) &= \left(\frac{s_2}{s_2 - s_1}\right)\sum_{n=1}^{N}\mathrm{Re}(b_n)\frac{R_n(z_1)}{n} + \frac{1}{(s_1 - s_2)}\sum_{n=1}^{N} i\,\mathrm{Im}(b_n)\frac{R_n(z_1)}{n} \\ 2\psi(z_1) &= \left(\frac{s_1}{s_1 - s_2}\right)\sum_{n=1}^{N}\mathrm{Re}(b_n)\frac{R_n(z_2)}{n} + \frac{1}{(s_2 - s_1)}\sum_{n=1}^{N} i\,\mathrm{Im}(b_n)\frac{R_n(z_2)}{n} \end{aligned} \tag{179}$$

For further details of the definitions of various parameters in (178), see Gladwell and England (1977) (note the coordinate system: x along the crack, $x = \pm a$ for the crack-tips, and y normal to the crack).

(C) Compute the K-factors for step (B) using (176).

(D) Compute the crack-face displacements for step (B) using (177). Note that, once the coefficients b_n of (174) are known, the crack-face displacements u_x and u_y from (179) and (177) are known in an *analytical* form, with their dependence on the crack length being explicitly known [see Gladwell and England (1977) for details].

(E) Compute the *tractions at the boundaries of the given finite-dimensional structure*, using (178). Call these residual boundary-traction system as T (recall that steps (B) onwards are repeated for reference systems $R1$ and $R2$ of step (A)).

(F) From the analytical expressions for u_i at the crack-face as determined in step (D), determine the *analytical* expression for (du_i/da) by differentiating u_i w.r.t. a. Note that (du_i/da) will be infinite at the crack-tip $x = \pm a$.

It is important to remember that the present step (F) is still *based on an analytical solution.* No *finite element or boundary-element* models, and no *virtual-crack extensions*, and *no finite-difference* methods are used in computing (du_i^R/da).

(G) Now consider the *finite element model of the uncracked structure* of the *given geometry,*and anisotropic material [In the case of *isotropic* material, a boundary-element model (with only the boundary being descritized) of the uncracked structure is far more efficient [Rajiyah and Atluri (1988)]. Apply the reverse of the traction system T as determined in step (E) above, on the boundaries of the *uncracked structural model.* From the finite-element (or boundary element) solution, find the *tractions at the location of the crack* in the uncracked structure, and label this traction system as R_c.

(H) Reverse the system R_c on the crack faces. Go back to step (B), and repeat steps (B), (C), (D), (E), and (F), for this system R_c on the crack face in an *infinite domain. Repeat steps (B) to (H)* until *convergence is obtained,* i.e., the traction system T in step (E) is *negligible.*

(J) The weight-functions for a finite-dimensional structure of the given geometry, and given crack-orientated, are obtained by summing up all the values of (du_i^{R1}/da) [and du_i^{R2}/da] obtained in step (F) *for all iterations* until convergence is established.

A number of problems has been solved in Chen and Atluri (1988), and Rajiyah and Atluri (1988) to demonstrate the ease and accuracy of the above procedures.

It is important to note, *the crack is not numerically modelled* at all. The dependence of crack plane displacements on crack length as evaluated in steps (D) and (F), for infinite domains, is explicitly known; and *thus the weight-functions are evaluated in the above alternating method, in an analytical sense without using numerical differentiations.* Furthermore, the above mentioned procedures have been documented in the work of Chen and Atluri (1988) to work very well for anisotropic materials with arbitrarily oriented cracks.

For a two-dimensional anisotropic problem, it is possible to develop a boundary element method for mixed-mode crack analysis, *wherein a straight crack is explicitly included in the formulation and not modeled by boundary elements*, by using the fundamental solutions for an infinite cracked anisotropic plate. This was developed by Snyder and Cruse (1975). This boundary integral equation is:

$$C_{ij}U_j = \int_{\partial\Omega} [u^{\star}_{ji}t_j - t^{\star}_{ji}u_j]\, \mathrm{d}A \tag{180}$$

It should be noted that the crack surface, S_c, is not a part of $\partial\Omega$ in (180), as the crack is explicitly accounted for in the fundamental solution. By differentiating (180) w.r.t. a, one obtains the integral equations for $(\mathrm{d}u_j/\mathrm{d}a)$:

$$C_{ij}(\mathrm{d}u_j/\mathrm{d}a) = \int_{\partial\Omega} [u^{\star}_{ji}\frac{\mathrm{d}t_j}{\mathrm{d}a} - t^{\star}_{ji}\frac{\mathrm{d}u_j}{\mathrm{d}a} + \frac{\mathrm{d}u^{\star}_{ji}}{\mathrm{d}a}t_j - \frac{\mathrm{d}t^{\star}_{ji}}{\mathrm{d}a}u_j]\, \mathrm{d}A \tag{181}$$

By taking the limit on the left-hand-side to $\partial\Omega$, one can solve the *boundary* integral equation for the unknown values of $(\mathrm{d}u_j/\mathrm{d}a)$ and $(\mathrm{d}t_j/\mathrm{d}a)$ at $\partial\Omega$. Once these data at $\partial\Omega$ is known, Eq. (181) simply becomes an integral relation for the interior values of $(\mathrm{d}u_j/\mathrm{d}a)$. Since the crack surface is interior to $\partial\Omega$ in the formulation, the crack-surface weight functions can be determined from (181). Such procedures have been reported, along with some examples, by Cruse and Raveendra (1988). An advantage of this procedure is its ability to decouple the vector components of crack tip behavior easily.

5.2 Weight-Functions by Using Direct Finite Element / Boundary Element Modeling of the Cracked Structure

The earlier class of modeling used only F.E.M. or B.E.M. models of the *uncracked* structure, while the crack was accounted for in some *analytical* fashion. If the material is nonhomogeneous, or if the crack exists in a complicated structural construction such as a bi-material plate or a stiffened plate, etc., the direct *numerical* modeling of the crack itself is unavoidable, to determine the weight functions. We discuss here some advances made recently in this direction. The discussion is limited to the case of *isotropy*.

Consider the analytic relation

$$\mathcal{G} = \int_{S_t} t_i \frac{\mathrm{d}u_i}{\mathrm{d}a}\,\mathrm{d}S + \int_V f_i \frac{\mathrm{d}u_i}{\mathrm{d}a}\,\mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}a}\int_V W\,\mathrm{d}V \tag{182}$$

(wherein the existence of S_u with nonzero values of prescribed u_i is ignored, for simplicity). In the context of a finite element method, wherein the *cracked structure is modeled* directly by finite elements, Eq. (182) may be written as:

$$\mathcal{G} = \boldsymbol{Q}\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}a} - \boldsymbol{q}\mathbf{K}\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}a} - \frac{1}{2}\boldsymbol{q}\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}a}\boldsymbol{q} = -\frac{1}{2}\boldsymbol{q}\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}a}\boldsymbol{q} \tag{183}$$

where $\boldsymbol{Q}$ is the generalized nodal force vector (due to applied loading at arbitrary S_t and in V); $\boldsymbol{q}$ is the nodal displacement vector, and $\mathbf{K}$ the stiffness matrix. Eq. (183) follows from (182), since, at equilibrium, $\mathbf{K}\boldsymbol{q} \equiv \boldsymbol{Q}$.

When a finite element mesh is used near the crack tip, a small change in crack length, by $\mathrm{d}a$, affects only the elements in a core immediately surrounding the crack tip. This is the basic idea behind the stiffness derivative method [Parks (1974) and Hellen (1975)]. Let *the small domain near the crack tip* be V_ϵ. Thus;

$$\mathcal{G} = -\frac{1}{2}\boldsymbol{q}_\epsilon \frac{\mathrm{d}\mathbf{K}_\epsilon}{\mathrm{d}a}\boldsymbol{q}_\epsilon \tag{184}$$

where $(\ \ )_\epsilon$ indicates the quantity $(\ \ )$ in the region V_ϵ. In (184), $(\mathrm{d}\mathbf{K}_\epsilon/\mathrm{d}a)$ is determined by the finite difference relation:

$$\frac{\mathbf{K}_\epsilon(a+\Delta a) - \boldsymbol{K}_\epsilon(a)}{\Delta a}$$

Eq. (184) can be applied to the *reference state*, to determine the K-factors for the reference state. However, if the reference state is of mixed mode loading, for isotropic materials, $\mathcal{G} = (K_I^2 + K_{II}^2)/H$, and a mode separation is necessary. *Thus within the core-region* V_ϵ [*which is certainly much smaller than the cracked structure*], one may decompose $\boldsymbol{q}^\epsilon$ into mode I and mode II parts; by using the relations:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} u_1^I \\ u_2^I \end{Bmatrix} + \begin{Bmatrix} u_1^{II} \\ u_2^{II} \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} u_1 + u_1' \\ u_2 - u_2' \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} u_1 - u_1' \\ u_2 + u_2' \end{Bmatrix} \tag{185}$$

where 1 and 2 are directions along and normal to the crack axis, respectively, and where $(\ \ )$ denotes a quantity at a point p in the upper portion of the cracked plate, and $(\ \ )'$ is the respective quantity at a point p' which is the mirror image in the crack axis of p. Thus $\boldsymbol{q}_\epsilon = \boldsymbol{q}_\epsilon^I + \boldsymbol{q}_\epsilon^{II}$.

Thus

$$\frac{(K_I^R)^2}{H} = -\frac{1}{2}\boldsymbol{q}_\epsilon^I \frac{\mathrm{d}\mathbf{K}_\epsilon}{\mathrm{d}a}\boldsymbol{q}_\epsilon^I \tag{186}$$

and

$$\frac{(K_{II}^R)^2}{H} = -\frac{1}{2}\boldsymbol{q}_\epsilon^{II} \frac{\mathrm{d}\mathbf{K}_\epsilon}{\mathrm{d}a}\boldsymbol{q}_\epsilon^{II} \tag{187}$$

where $\boldsymbol{q}_\epsilon^I$ is the vector of appropriate nodal displacements u_i^I, etc. Now we consider the problem of determining the weight functions for the reference state of mixed mode loading. To this end consider the finite element equilibrium equation for the entire *cracked structure* loaded under mixed mode reference load:

$$\mathbf{K}\boldsymbol{q} = \boldsymbol{f} \tag{188}$$

For a fixed-loading, the weight-functions *everywhere in the structure* can be derived, from (188), as:

$$\mathbf{K}\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}a} = -\frac{\mathrm{d}\boldsymbol{K}_\epsilon}{\mathrm{d}a}\boldsymbol{q} \equiv \frac{\mathrm{d}\boldsymbol{K}_\epsilon}{\mathrm{d}a}q_\epsilon \tag{189}$$

From a solution of (189), $(\frac{\mathrm{d}u_i}{\mathrm{d}a})$ is determined for the *reference state* (*which can, in general, be a mixed-mode loading*), at all nodes at S_t, and in V (including the crack-face). The above method, for a pure mode I problem was presented by Parks and Kamenetsky (1979).

Now, we present a simple extension of the "stiffness-derivative" weight-function evaluation method for *mixed-mode* problems. In as much as the stiffness matrix ($\mathbf{K}$) and its derivative ($\mathrm{d}\mathbf{K}/\mathrm{d}a$) are evaluated once and for all for the structure, Eq. (188) can be solved for *two reference states*, at least one of them mixed-mode, $R1$ and $R2$; Eqs. (186) and (187) can be solved for mixed-mode K-factors for the two reference states, i.e., K_I^{R1}, K_{II}^{R1}, K_I^{R2}, and K_{II}^{R2}. Likewise, Eq. (189) can be solved for the two different reference states, to determine ($\mathrm{d}u_i^{R1}/\mathrm{d}a$) and ($\mathrm{d}u_i^{R2}/\mathrm{d}a$) *everywhere in the structure* (i.e. at S_t, in V, and on the crack-face) as desired. *For any other arbitrary state of mixed-mode loading*, Eqs. (171) and (172) may be used for determining the mixed-mode K-factors. Note that this simple procedure leads to *weight-functions everywhere in the structure* (external surfaces, crack faces, and within the body) as may be desired.

On the other hand, Sha and Yang (1985), instead of using the procedure as discussed in the above paragraph for mixed-mode problems, proceed to consider only pure-mode I and pure-mode II weight functions, ($\mathrm{d}u_i^I/\mathrm{d}a$) and ($\mathrm{d}u_i^{II}/\mathrm{d}a$), by decomposing the displacement *everywhere in the cracked structure* (*not* only *in* V_ϵ) using Eq. (185). If the weight-functions are sought at the external boundary, pairing the points on the external boundary, and their *mirror images* is geometrically impossible, for *arbitrary-shaped* structures, with arbitrarily oriented cracks. Sha and Yang (1985) consider the following equations:

$$\mathbf{K}\frac{\mathrm{d}q^I}{\mathrm{d}a} = -\frac{\mathrm{d}\mathbf{K}_\epsilon}{\mathrm{d}a}q^I \tag{190}$$

and

$$\mathbf{K}\frac{\mathrm{d}q^{II}}{\mathrm{d}a} = -\frac{\mathrm{d}\mathbf{K}_\epsilon}{\mathrm{d}a}q^{II} \tag{191}$$

Note that q^I (q^{II}) *is simply a vector of appropriate displacements* u_i^I (u_i^{II}) *at each node in the structure*, which may not always be geometrically feasible.

Another method for weight-functions which obviates the need for a finite difference evaluation of ($\mathrm{d}\mathbf{K}_\epsilon/\mathrm{d}a$) is based on the equivalent domain-integral method for evaluating the energy-release rate. This method has recently been developed by Nikishkov (1988) and Chen and Atluri (1988). As discussed by Nikishkov and Atluri (1987), the energy-release rate in a 2-D elastic problem can be written (as the equivalent domain integral representation of the J-integral), as:

$$\mathcal{G} = -\frac{1}{F}\int_{V_s}\left[W\frac{\mathrm{d}s}{\mathrm{d}x_1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1}\frac{\partial s}{\partial x_j}\right]\mathrm{d}V \tag{192}$$

where V_s is any arbitrary region near the crack-tip [V_s is much smaller than the total region V]; s is any arbitrary but continuous function which is equal to 1 at the crack-tip, and goes to zero at the boundary of V_s; $F = 1$ in two dimensional crack problems; W is the stress-work density, and u_i are displacements.

Suppose that the region V_s in (192) is taken to be the same as the region V_ϵ considered in Eq. (184) [Even otherwise, if V_ϵ is smaller than V_s; since ($\mathrm{d}\boldsymbol{K}/\mathrm{d}a$) may be taken to be zero in the region $V_s - V_\epsilon$; one may rewrite (184) as $\mathcal{G} = -\frac{1}{2}q_s(\mathrm{d}\mathbf{K}_s/\mathrm{d}a)q_s$, without loss of generality]. Suppose that one introduces a finite element interpolation:

$$\begin{aligned} u_i &= N^k u_i^k \qquad K = 1,\cdots N \text{ nodes} \\ &= \boldsymbol{N}\boldsymbol{q} \qquad i = 1,2 \end{aligned} \tag{193}$$

Then,

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}N^k_{,j}u^k_i \tag{194}$$

Also, we introduce the finite element interpolations,

$$s = N^k s^k \tag{195}$$

Using (193–195) in (192), one has:

$$\begin{aligned} \mathcal{G} &= -\frac{1}{F}\int_{V_s}\left[\frac{1}{2}N^k_{,j}N^L_{,1} - N^k_{,1}N^L_{,j}\right]\sigma_{ij}u_i^k s^L]\,\mathrm{d}V \\ &\equiv -\frac{1}{2}T_i^k u_i^k \qquad k = 1,\cdots,N \text{ nodes} \qquad i = 1,2 && (196) \\ &\equiv -\frac{1}{2}Q_s^{\star t}\cdot q_s && (197) \end{aligned}$$

where the definition of $Q_s^\star$ is apparent. Thus, when $V_s \equiv V_\epsilon$; comparing (197) with (184), one has:

$$\frac{\mathrm{d}\mathbf{K}_\epsilon}{\mathrm{d}a}q_\epsilon \equiv Q_\epsilon^\star \tag{198}$$

Note that $Q_\epsilon^\star$ is computed from a simple integral over V_ϵ as in (196a) and (197). Eq. (198) shows *that a finite-difference evaluation* of ($\mathrm{d}\mathbf{K}_\epsilon/\mathrm{d}a$) as in Eq. (184) *can be avoided* if the identity in (198) is used, and the energy-release-rate can be computed using (197).

For a fixed reference loading (*which can in general be of the mixed-mode type*), the weight functions everywhere in the structure (including at the external boundary, S_t, the crack-face, or in V), can now be obtained, using (189):

$$\mathbf{K}\frac{\mathrm{d}q}{\mathrm{d}a} \equiv -\frac{\mathrm{d}\boldsymbol{K}_\epsilon}{\mathrm{d}a}q_\epsilon \equiv Q_\epsilon^\star \tag{199}$$

where $Q_\epsilon^\star$ is computed from the domain-integral over V_ϵ as apparent from (197).

Equation (199) is solved for *two arbitrary reference states* [which are both not either of Mode I or of Mode II type, with loading being either on the external boundary, or on the crack-face] to find ($\mathrm{d}u_i^{R1}/\mathrm{d}a$) and ($\mathrm{d}u_i^{R2}/\mathrm{d}a$) *that are required* in Eqs. (171) and (172) in order to compute, the mixed-mode K-factors for any other given arbitrary load-state. Note that in (199), $\mathbf{K}$ is computed only once; and $Q_\epsilon^\star$ is computed separately for each *reference state*. However, examining (196a) and (197) it is seen that the only quantity that is different in integral for $Q_\epsilon^\star$ in the two *reference states* is σ_{ij} in V_ϵ.

In order to use (171) and (172) to compute the mixed-mode K-factors for any given arbitrary load-state is the only additional informations needed are the mixed-mode K-factors (K_I^{R1}, K_I^{R2}), and (K_{II}^{R1}, K_{II}^{R2}) for the *two reference states*. For isotropic materials, the mode-decomposition of the energy-release rate of Eq. (196a) can be accomplished by decomposing the displacement, strain, and stress fields in the *core region* V_ϵ *near the crack tip* [V_s is much smaller than V, the total domain]. The displacement decomposition is already given in (185), while the stress decomposition can be written as:

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} = \begin{Bmatrix}\sigma_{11}^I\\ \sigma_{22}^I\\ \sigma_{12}^I\end{Bmatrix} + \begin{Bmatrix}\sigma_{11}^{II}\\ \sigma_{22}^{II}\\ \sigma_{12}^{II}\end{Bmatrix} \equiv \frac{1}{2}\begin{Bmatrix}\sigma_{11}+\sigma'_{11}\\ \sigma_{22}+\sigma'_{22}\\ \sigma_{12}-\sigma'_{12}\end{Bmatrix} + \frac{1}{2}\begin{Bmatrix}\sigma_{11}-\sigma'_{11}\\ \sigma_{22}-\sigma'_{22}\\ \sigma_{12}+\sigma'_{12}\end{Bmatrix} \tag{200}$$

where, ( ) and ( )' are quantitites at a point p in the "upper side" of the crack, and at a point p' which is a mirror image of p in the crack-plane. Thus, for a reference state, the individual nodal intensities are computed from:

$$\frac{(K_I^R)^2}{H} = -\frac{1}{F}\int_V \left[W^I \frac{\partial s}{\partial x_1} - \sigma_{ij}^I \frac{\partial u_i^I}{\partial x_1}\frac{\partial s}{\partial x_j}\right] dV \qquad (201)$$

and

$$\frac{(K_{II}^R)^2}{H} = -\frac{1}{F}\int_V \left[W^{II} \frac{\partial s}{\partial x_1} - \sigma_{ij}^{II} \frac{\partial u_i^{II}}{\partial x_1}\frac{\partial s}{\partial x_j}\right] dV \qquad (202)$$

where

$$W^I = \frac{1}{2}\sigma_{ij}^I u_{ij}^I; \qquad W^{II} = \frac{1}{2}\sigma_{ij}^{II} u_{ij}^{II} \qquad (203)$$

This completes the algorithm for determining the mixed mode K-factors for an arbitrary given loading, using the weight functions for reference states, derived from *the equivalent domain integral method*, which avoids the need for a finite-difference evaluation of the stiffness derivative ($d\mathbf{K}/da$) as in the approaches for Mode I problem given by Parks and Kamenetzky (1979).

Special variational technique for determining directly the weight functions that are singular in the vicinity of the crack-tip (crack front) (and hence have unbounded strain-energy) has been presented by Sham (1987). This variational technique handles both traction and mixed boundary conditions. A finite element implementation of the variational principle has also been given by Sham (1987) and this leads to a unified approach in the direct finite element computation of weight functions for all three fracture modes. Sham and Zhou (1988a) have presented weight functions for a semi-infinite crack in a full space of arbitrary anisotropy; in particular, the results for monoclinic solids are presented in closed form. Using such a solution, Sham and Zhou (1988a) applied the variational technique of Sham (1987) to determining weight functions for homogeneous and piecewise homogeneous anisotropic (where crack tips do not terminate at the material interafce) bodies. Employing the weight functions obtained, they also evaluated the stress intensity factors of a matrix crack in an idealized model of a fiber-reinforced composite laminate under curing conditions. Sham and Bueckner (1988) have recently introduced antiplane strain weight functions for an interface notch in an isotropic bi-crystal and Sham (1988a) has used the same variational technique to determine these notch-interface weight functions. Generalizaing Beuckner's (1971) weight function concepts, Sham (1988b) has developed higher order weight functions for calculating power expansion coefficients of an elastic field in a two-dimensional body in the absence of body forces, Integration formulas for the expansion coefficient, in analogy to those for stress intensity factors, are given for interior points and crack tips. Some of these expansion coefficients at an interior point can be related to the image force of a discrete dislocation and those for the crack tip correspond to important fracture parameters as discussed by Sham (1988b).

5.3 Weight Functions and Influence Functions for 3-D Crack Problems

As aptly noted by Swedlow (1988), "since fracture and fatigue are demonstrably three-dimensional, the technology base needed to describe these processes must be of the

same dimensionality if only to describe these events, let alone predict them". Thus, simple numerical methods of engineering interest are in need of development for 3-dimensional problems. Along with the domain integral methods, the weight function approaches to 3-D problems present interesting possibilities in this regard.

Here, we consider, for simplicity, only mode I crack problems in 3-D isotropic elastic solids. For isotropic solids, containing cracks of arbitrary shape, under mode I loading, Rice (1972) has derived the counterpart of Eqs. (171) and (172) [for mode I], as follows:

$$\int_{\Gamma_c} \frac{2}{H}[K_I K_I^R \delta\ell] d\Gamma = \int_{S_t} t_i \delta_\ell u_i^R \, dS + \int_V f_i \delta_\ell u_i^R \, dV \qquad (204)$$

where Γ_c is the crack-front; K_I is the stress-intensity factor (which varies along Γ_c) for any arbitrary loading t_i at S_t and f_i in V; K_I^R is stress-intensity the factor (which varies along Γ_c) for the reference loading t_i^R at S_t and f_i^R in V; S_t is the loaded-surface (which may be taken to be the crack-face, without loss of generality); $\delta\ell$ is a smooth function along $d\Gamma$ denoting the infinitesmal advance of the crack in a direction locally normal to Γ; $\delta_\ell u_i^R$ will denote the *first-order variation* in u_i^R to a change in the crack-front i.e., $\delta_\ell u_i^R$ is a function of the location in S_t and V, and H is a material constant.

Rice (1985) has presented results for the first order variation of an elastic displacement field associated with the arbitrary incremental planar advance of the location of the front of a *half-plane crack* in a loaded elastic full space, and also discussed the relation of such results to a 3-D weight function theory, and derived an expression for the distribution of the mode-I K-factor for a slightly curved crack-front. Later Gao and Rice (1986) extended this work to the mixed mode case.

Recently Bueckner (1987) has presented analytical results for 3-D weight functions, for a penny-shaped and a half-plane crack in an elastic full-space, under mixed-mode conditions. Employing these results, and Sham's (1987) variational technique, Sham and Zhou (1987b) have determined the Mode I weight functions for both penny-shaped and elliptical cracks in finite bodies.

Here, our objective is to discuss crack-surface weight functions for embedded or surface cracks of the elliptical geometry, i.e., *we treat S_t to be the crack-face* in (204) and ignore the body forces. For surface or corner flaws of semi- or quarter-elliptical geometry respectively, engineeering theories of fatigue crack-growth are often based on the consideration of the K-factors at the major and minor axis locations ($x = a$, and $y = b$), respectively. Thus, one often thinks of a "two-parameter" characterization of K-factor variation along the crack-front, for the given arbitrary loading. There are two alternative approaches for the above "two-parameter" characterization. One is directly in terms of the "local" values (or values at major and minor axis points on the ellipse) K_I^* and K_I^{**}; and the other is in terms of "local weighted average" values along specified portions of the crack front, $\bar{K}_I^*$ and $\bar{K}_I^{**}$, defined as:

$$(\bar{K}_I^*)^2 = \frac{1}{\delta A_1}\int_{\Gamma_c} K_I^2 \delta\ell \, d\Gamma$$
$$(\bar{K}^{**})^2 = \frac{1}{\delta A_2}\int_{\Gamma_c} K_I^2 \delta\ell\Gamma \qquad (205)$$

where K_I and $\delta\ell$ are as in Eq. (204), and δA_1 and δA_2 are changes in crack area due to a virtual change in the length of the major and minor axes, respectively. The weight functions for these weighted average values, $\bar{K}_I^*$ and $\bar{K}_I^{**}$ have been defined by Busuner (1976), and used for residual life estimations of complex structures by Besuner (1976) and Cruse and Besuner (1977). In the following we sketch a procedure for determing the weight functions directly for K_I^* and K_I^{**}.

Suppose for the *reference load state*, the K-factor variation, $K_I^R(\Gamma)$ is determined from the finite element alternating technique as discussed in Section 3 of this paper. Thus, an analytical expression for K_I^R as a function of the crack-front coordinate ϕ, is known, wherein the appropriate coefficients are determined from the finite element alternating method. We consider the case of the given loading, and introduce a *trial solution* for K_I for this loading, in terms of its values, K_I at $x = a$ (denoted as K_I^*) and K_I at $y = b$, denoted as K_I^{**}.

In as much as $\delta\ell$ in Eq. (204) is arbitrary, we introduce two such trial variations: (i) one wherein the major axis is extended by (da) such that the equation of the ellipse is $(x/(a+da))^2 + (y/b)^2 = 1$; and (ii) the other wherein the ellipse is $(x/a)^2 + (y/(b+db))^2 = 1$. Thus, $\delta\ell$ is given in terms of (da) or (db) and the elliptical angle. Also, as discussed in Section 3, the crack surface displacements for the reference load state are known in the form of analytical expressions wherein the coefficients are determined from the finite element alternating method. Thus, for $\delta\ell$ as in case (i) and (ii), the first-order variations in u_i^R at S_t can be determined from analytical expressions where coefficients are numerically determined in the alternating method. Thus, when the known expressions for $K_I^R(\Gamma)$, $\delta\ell(\Gamma)$; and $\delta_\ell u_i^{(R)}$ are used, and the two-parameter trial function for K_I (for the given state) are used in Eq. (204), one obtains 2 algebraic equations, (one for each of the two cases of $\delta\ell$ listed above) governing the two unknowns K_I^* and K_{II}^{**}. Thus, one obtains a *weight-function representation* for the *stress-intensity factors at the major and minor axes* of the elliptical (or part-elliptical) flaw under the given state of mode I loading.

Using the VNA analytical solution for an embedded elliptical crack in an infinite body as detailed in Section 2 of this paper, Nishioka and Atluri (1988) derived an analytical expression for the first order variation of the crack-face displacement field due to a geometrical perturbation in the major and minor axes of an elliptical crack. Using this, Nishioka and Atluri (1988) have developed the above described weight-function representation for the K-factors at the major and minor axes of the (part)-elliptical crack, in the forementioned 2 parameter characterization.

If K_I variation under the action of an arbitrary crack-face traction is needed all along the crack-front, the *influence function* concept is more useful. For a given surface flaw in given structural geometry, one can generate a stress-intensity factor variation all along the crack-front, for a *given polynomial loading* on the crack-face, using the alternating method described in Section 3. If the given arbitrary crack-face pressure is then decomposed into individual polynomial variations multiplied by an appropriate constant, then the K-factor variation for the given load can easily be determined by a (weighted) linear superposition of the various influence functions. This has been done for several problems [Nishioka and Atluri (1982); O'Donoghue, Nishioka, and Atluri (1984b)].

In general, the advantage of the weight function concept over that of the influence functions is that the former can be used for localized crack-face forces, including point forces, while the latter is more suited for more distributed crack-face tractions of the polynomial type.

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