Finite Element Method of Thin Shell J Integral Evaluation

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ABSTRACT

The finite element method has been applied on thin shell J integral evaluation. Thin shell J integral has been recently derived using Gurtin´s approach, which has been shown to be path dependent, unless the crack is placed along a generatrix of a cylindrical shell. In any other case two line and one surface integral should be added to the Rice´s J integral in order to regain its path independency. Anyhow, even in the case of path dependency of Rice´s J integral, evaluation of the complete integral expression is relatively an easy task in the computational mechanics, having in mind finite element method. Therefore, in this paper the finite element method analysis of the thin shell J integral has been presented, together with an example regarding axial crack in the cylindrical shell.

KEYWORDS

Thin shell, J integral, path dependency, computational mechanics, finite element method.

INTRODUCTION

Thin shell analysis requires generally numerical methods in order to include the middle surface shapes and boundary conditions which can not be treated efficiently by the analytical methods. Furthermore, the cracked thin shell is a problem itself, since the most popular fracture mechanics parameter, namely J integral, is path dependent, unless the crack is positioned along the generatrix of a cylindrical shell. Anyhow, path dependency does not eliminate a possibility of using the conservation law of J integral type for the same purpose. As it was shown recently, (Sedmak et al., 1988), an energy release rate due to unit crack growth can be identified with an integral expression comparising the line integral, analogous to Rice´s J integral, and three additional integrals (one surface and two line integrals) "recovering" the path independency of thin shell J integral. Evaluation of the complete integral expression is relatively easy task in computational mechanics, at least not more complicated than if J integral were path independent. This is doubtlessly thru if finite element method is employed, since there is no essential difference in its application for the evaluation of all integral terms involved. Therefore, the aim of this paper is the finite element analysis of cracked thin shell, using thin shell J integral as the basic fracture mechanics parameter. In this stage of analysis, we have chosen the standard example - cylindrical shell with an axial crack, enabling us to use thin shell J integral as a path independent integral. The procedure for complete expression evaluation is presented, but an example is still lacking.

Thin shell J integral

Based on the analysis given by Sedmak et al. (1988), we define here the following integral expression:

$$J=\int_{\Gamma}[W\delta_1^{\alpha}-({}_R\mathbf{N}^{\alpha}\mathbf{u}_1+{}_R\mathbf{M}^{\alpha}\boldsymbol{\kappa}_1)]e^1 n_{\alpha}dL-\int_{D}[W\delta_1^{\alpha}-({}_R\mathbf{N}^{\alpha}u_1+{}_R\mathbf{M}^{\alpha}\kappa_1)]B_{\alpha}^1\mathbf{e}\cdot\mathbf{N}dA-\int_{S^-}W^-\mathbf{e}\cdot\mathbf{m}dL+\int_{S^+}W^+\mathbf{e}\cdot\mathbf{m}dL \quad (1)$$

where W denotes the strain energy, $\mathbf{u}_1$ and $\boldsymbol{\kappa}_1$ derivatives of displacement and director displacement vector over the coordinate θ^1 along the crack, B_{α}^1 mixed components of the second fundamental form, **e** unit vector of crack propagation direction, **m** unit outward normal to the crack face, **N** unit outward normal to the middle surface (Fig. 1), n_{α} are the components of an outward unit normal to the contour of integration Γ and ${}_R\mathbf{N}^{\alpha}$ and ${}_R\mathbf{M}^{\alpha}$ denote the stress vector (membrane and bending, respectively) measured in the reference configuration.

As it was shown (Sedmak et al., 1988), the integral expression (1), called here the thin shell J integral, can be identified as an energy release rate due to unit crack growth along the coordinate θ^1. For the cylindrical shell with an axial crack, only the first integral, which resembles strongly to Rice´s J integral, in expression (1) remains, since $\mathbf{e}\cdot\mathbf{m}=0$ and $\mathbf{e}\cdot\mathbf{N}=0$. In any other case, either $\mathbf{e}\cdot\mathbf{m}\neq 0$ or $\mathbf{e}\cdot\mathbf{N}\neq 0$, causing path dependence of the first integral. In order to illustrate this, we mention here that for the cylindrical shell with a circumferential crack $\mathbf{e}\cdot\mathbf{m}=0$, but $\mathbf{e}\cdot\mathbf{N}\neq 0$, producing the surface integral in the expression (1). The same situation can be observed for the spherical shells, if the crack is along any of the main circles.

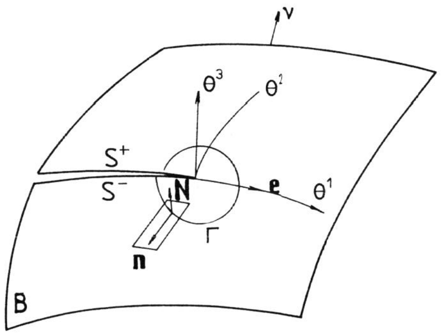

Figure 1. Material surface with an edge crack

Finite element method for thin shell J integral evaluation

We apply here simple and effective quadrilateral isoparametric element in the scope of the Galerkin method for boundary value problems, what is described elsewhere (Berković, 1982a, b). It is to be mentioned here, that no singular element is used, since an extrapolation technique has been used in a manner described fully by Sedmak and Berković, (1986). The accuracy of such a procedure is of the same order as with the similarly graded mesh with singular elements. It should also be mentioned that the procedure applied here takes only linear elasticity into account, although the approach is nonlinear and can be easily adopted for both material and geometrical nonlinearity.

Taking the displacement and stress field as already known (we have used software DSTATA, developed in the Aeronautical Institute in Žarkovo, Yugoslavia for this purpose), their post-processing toward J integral evaluation is described in what follows. Having in mind expression (1) we define now all quantities needed:

$$W=\frac{H}{4}[\frac{1}{2}S_o^{\alpha\beta}\overset{o}{\gamma}_{\alpha\beta}+\frac{3}{2}S_1^{\alpha\beta}\overset{1}{\gamma}_{\alpha\beta}-\frac{HB}{4A}(S_1^{\alpha\beta}\overset{o}{\gamma}_{\alpha\beta}+S_o^{\alpha\beta}\overset{1}{\gamma}_{\alpha\beta})+S_o^{\alpha 3}\overset{o}{\gamma}_{\alpha 3}+3S_1^{\alpha 3}\overset{1}{\gamma}_{\alpha 3}-\frac{HB}{2A}(S_1^{\alpha 3}\overset{o}{\gamma}_{\alpha 3}+\frac{3}{5}S_o^{\alpha 3}\overset{1}{\gamma}_{\alpha 3})] \quad (2)$$

where H is shell thickness and $S_o^{\alpha\beta}$, $S_1^{\alpha\beta}$, $S_o^{\alpha 3}$, $S_1^{\alpha 3}$, $\overset{o}{\gamma}_{\alpha\beta}$, $\overset{1}{\gamma}_{\alpha\beta}$, $\overset{o}{\gamma}_{\alpha 3}$, $\overset{1}{\gamma}_{\alpha 3}$ are coefficients of the Legendre´s polynomials for stress and strain tensors. The are defined by the following relations:

$$\overset{o}{\gamma}_{\alpha\beta}=2\mathbf{x}_{\alpha}\mathbf{u}_{\beta} \qquad \overset{1}{\gamma}_{\alpha\beta}=\frac{1}{3}(\mathbf{x}_{\alpha}\boldsymbol{\kappa}+\mathbf{H}_{\beta}\mathbf{u}_{\alpha}) \quad (3)$$

$$\overset{o}{\gamma}_{\alpha 3}=\mathbf{x}_{\alpha}\boldsymbol{\kappa}+\mathbf{u}_{\alpha}\mathbf{H} \qquad \overset{1}{\gamma}_{\alpha 3}=\frac{1}{6}(\mathbf{H}_{\alpha}\boldsymbol{\kappa}+\boldsymbol{\kappa}_{\alpha}\mathbf{H}) \quad (4)$$

$$S_o^{\alpha\beta}=A^{\alpha\beta\chi\psi}\overset{o}{\gamma}_{\chi\psi} \qquad S_1^{\alpha\beta}=A^{\alpha\beta\chi\psi}\overset{1}{\gamma}_{\chi\psi}+B^{\alpha\beta\chi\psi}\overset{o}{\gamma}_{\chi\psi} \quad (5)$$

$$S_o^{\alpha 3}=\frac{2\mu H^2}{3}A^{\alpha\beta}\overset{o}{\gamma}_{\beta 3} \qquad S_1^{\alpha 3}=\frac{6\mu H^2}{S}A^{\alpha\beta}\overset{1}{\gamma}_{\beta 3}+\frac{2\mu H^2}{S}B^{\alpha\beta}\overset{o}{\gamma}_{\beta 3} \quad (6)$$

with $A^{\alpha\beta}=\mathbf{x}_{\alpha}\mathbf{x}_{\beta}$ denoting the first base metric tensor, where symbol $(\ )_{\alpha}$ stands for the partial derivative with regard to convective coordinates θ^{α}. $A^{\alpha\beta\chi\psi}$ and $B^{\alpha\beta\chi\psi}$ are tensor of elastic constants, given by:

$$A^{\alpha\beta\chi\psi}=\frac{E}{1+\nu}(\frac{\nu}{1-\nu}A^{\alpha\beta}A^{\chi\psi}+A^{\alpha\chi}A^{\beta\psi}) \quad (7)$$

$$B^{\alpha\beta\chi\psi}=\frac{EH}{3(1-\nu)}[\frac{\nu}{1-\nu}(A^{\alpha\beta}B^{\chi\psi}+A^{\chi\psi}B^{\alpha\beta})+A^{\alpha\chi}B^{\beta\psi}+A^{\beta\psi}B^{\alpha\chi}] \quad (8)$$

where E denotes Young´s elasticity modulus, ν Poison´s ratio and finally, μ is the shear modulus, $\mu=\frac{E}{2(1+\nu)}$.

Membrane and bending stress vectors can be expressed through the coefficients $S_o^{\alpha\beta}$, $S_1^{\alpha\beta}$, $S_o^{\alpha 3}$ and $S_1^{\alpha 3}$, as follows:

$$ {}_R\mathbf{N}^\alpha = \mathbf{X}_\beta S_o^{\alpha\beta} + \frac{1}{2}\left(\mathbf{H}_\beta - \frac{HB}{A}\mathbf{X}_\beta\right)S_1^{\alpha\beta} + \frac{1}{2}\mathbf{H}S_o^{\alpha 3} \tag{9} $$

$$ {}_R\mathbf{M}^\alpha = \mathbf{X}_\beta S_1^{\alpha\beta} + \frac{1}{6}\left(\mathbf{H}_\beta - \frac{HB}{A}\mathbf{X}_\beta\right)S_o^{\alpha\beta} + \frac{1}{2}\mathbf{H}\left(S_1^{\alpha 3} - \frac{HB}{10A}S_o^{\alpha 3}\right) \tag{10} $$

Before transforming the aferementioned relations from the convective physical coordinates θ^α into the isoparametric nondimensional coordinates of a finite element, ξ^α, following relations should be writen down:

$$ n_1 dL = \sqrt{A}d\theta^2 \qquad n_2 dL = -\sqrt{A}d\theta^1 \tag{11} $$

so that the first integral in the expression (1) can be writen as follows:

$$ J=\int_\Gamma({}_R\mathbf{N}^2\mathbf{u}_1+{}_R\mathbf{M}^2\boldsymbol{\kappa}_1)\sqrt{A}d\theta^1+[W-({}_R\mathbf{N}^1\mathbf{u}_1+{}_R\mathbf{M}^1\boldsymbol{\kappa}_1)]\sqrt{A}d\theta^2 \tag{12} $$

We can write now the general transformation between the coordinates as follows;

$$ d\theta^\alpha = \frac{\partial\theta^\alpha}{\partial\xi^\beta}d\xi^\beta \tag{13} $$

Anyhow, we shall not use here such a general transformation because it is more cumbersome than instructive, but ruther the specialized one given for a quadrilateral finite element and cylindrical shell (Fig. 2):

$$ \frac{\partial\theta^\alpha}{\partial\xi^\beta} = \begin{cases} a & \alpha = \beta = 1 \\ b & \alpha = \beta = 2 \\ 0 & \alpha \neq \beta \end{cases} \tag{14} $$

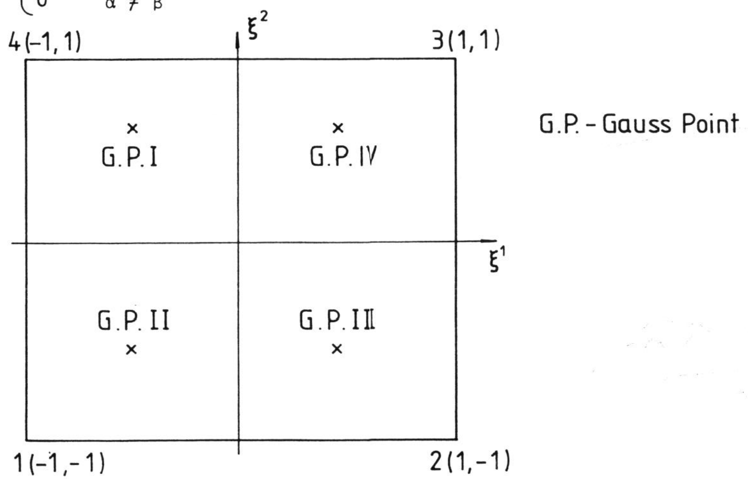

Figure 2. Quadrilateral isoparametric element

where a and b denote the lengths of element sides. Now, all quantities appearing in expression (1), can be transformed as follows;

$$ d\theta^1 = a\cdot d\xi^1 \qquad d\theta^2 = b\cdot d\xi^2 \tag{15} $$

$$ \mathbf{u}_1 = \frac{\partial u}{\partial\theta^1} = \frac{\partial u}{\partial\xi^\alpha}\frac{\partial\xi^\alpha}{\partial\theta^1} = \bar{\mathbf{u}}_1\frac{1}{a} \qquad \boldsymbol{\kappa}_1 = \dots = \bar{\boldsymbol{\kappa}}_1\frac{1}{a} \tag{16} $$

$$ {}_R\mathbf{N}^1 = {}_R\bar{\mathbf{N}}^1\frac{\partial\theta^1}{\partial\xi^1} = {}_R\bar{\mathbf{N}}^1 a \qquad {}_R\mathbf{M}^1 = \dots = {}_R\bar{\mathbf{M}}^1 a \tag{17} $$

$$ {}_R\mathbf{N}^2 = {}_R\bar{\mathbf{N}}^2\frac{\partial\theta^2}{\partial\xi^2} = {}_R\bar{\mathbf{N}}^2 b \qquad {}_R\mathbf{M}^2 = \dots = {}_R\bar{\mathbf{M}}^2 b \tag{18} $$

$$ A = \det A_{\alpha\beta} = A_{11}A_{22}-A_{12}^2 = \frac{\bar{A}_{11}\bar{A}_{22}-\bar{A}_{12}^2}{a^2b^2} = \frac{\bar{A}}{a^2b^2} \tag{19} $$

where "-" indicates isoparametric coordinates. Therefore, expression (10) can be writen as:

$$ J=b\int_\Gamma({}_R\mathbf{N}^2\mathbf{u}_1+{}_R\mathbf{M}^2\boldsymbol{\kappa}_1)\sqrt{A}d\xi^1+[W-({}_R\mathbf{N}^1\mathbf{u}_1+{}_R\mathbf{M}^1\boldsymbol{\kappa}_1)]\sqrt{A}\xi^2 \tag{20} $$

where "-" has been omited due to clearity of the expression.

Now the interpolation function for a quadrilateral isoparametric element should be introduced:

$$ p^k = \frac{1}{4}[(1-\xi^1)(1-\xi^2) \quad (1+\xi^1)(1-\xi^2) \quad (1-\xi^1)(1+\xi^2) \quad (1+\xi^1)(1+\xi^2)] \tag{21} $$

so that the position vector **X** and director **H** can be presented inside an element as follows:

$$ \mathbf{X} = \mathbf{X}_k p^k \qquad \mathbf{H} = \mathbf{H}_k p^k \tag{22} $$

where index k denotes a finite element node (k=1,2,3,4). For the isoparametric element the same representation is valed for the displacement **u** and director displacement $\boldsymbol{\kappa}$:

$$ \mathbf{u} = \mathbf{u}_k p^k \qquad \boldsymbol{\kappa} = \boldsymbol{\kappa}_k p^k \tag{23} $$

since all of the "nodes" values ($\mathbf{X}_k$, $\mathbf{H}_k$, $\mathbf{u}_k$, $\boldsymbol{\kappa}_k$) are independent if ξ^α, it can be writen:

$$ \mathbf{X}_\alpha = \mathbf{X}_k p^k_\alpha; \quad \mathbf{H}_\alpha = \mathbf{H}_k p^k_\alpha; \quad \mathbf{u}_\alpha = \mathbf{u}_k p^k_\alpha; \quad \boldsymbol{\kappa}_\alpha = \boldsymbol{\kappa}_k p^k_\alpha \tag{24} $$

where p^k_α denotes $\partial p^k/\partial\xi^\alpha$. Therefore, any other quantity appearing in the expression (20) can be represented using (22) - (24), as it is shown, e.g., for the first metric tensor:

$$A_{\alpha\beta} = \mathbf{X}_\alpha \mathbf{X}_\beta = X_k p^k_\alpha X_l p^l_\beta \tag{25}$$

Procedure for the surface integral term evaluation

Although basically the same, the evaluation of surface integral term in (20) involves somewhat different procedure. First of all, an area differential (dA) can be writen as follows:

$$dA = \sqrt{A}d\theta^1 d\theta^2 = \frac{\sqrt{A}}{ab} d\xi^1 d\xi^2 \tag{26}$$

and second fundamental metric tensor as:

$$B^1_1 = \bar{B}^1_1; \qquad B^1_2 = \bar{B}^1_2 \frac{a}{b} \tag{27}$$

The product $\mathbf{e}\cdot\mathbf{N}$ can be writen as:

$$\mathbf{e}\cdot\mathbf{N} = e^\alpha N_\alpha = e^1 N_1 = N_1 \tag{28}$$

giving the expression

$$\frac{1}{a^2 b}\int_D [W\delta^\alpha_1 - ({}_R\mathbf{N}^\alpha \mathbf{u}_1 + {}_R\mathbf{M}^\alpha \boldsymbol{\kappa}_1)]B \frac{1}{a}\sqrt{A}N\, d\xi^1 d\xi^2 \tag{29}$$

Finally, we mention that the line integrals (along S±) can be evaluated in the same way as expression (20), except for the product $\mathbf{e}\cdot\mathbf{m}$, which should be transformed as follows:

$$\mathbf{e}\cdot\mathbf{m} = \mathbf{e}^\alpha m_\alpha = e^1 m_1 = m_1$$

RESULTS AND DISCUSSION

Cylindrical shell under internal pressure with an axial crack is used to illustrate the procedure described in this paper. In order to compare the results obtained here with a literature data, we have chosen the example given by Barsoum et al. (1979), which is solved using displacement extrapolation for stress intensity factors evaluation. Since stress intensity factors are separated on the membrane and bending components (Barsoum et al., 1979), J integral, defined here, should be separated in the same way, using relations:

$$J_M = \int_\Gamma (W_M\delta^\alpha_1 - {}_R\mathbf{N}^\alpha \mathbf{u}_1)e^1 n_\alpha dL \tag{30}$$

$$J_B = \int_\Gamma (W_B\delta^\alpha_1 - {}_R\mathbf{M}^\alpha \boldsymbol{\kappa}_1)e^1 n_\alpha dL \tag{31}$$

where W_M and W_S denote membrane and bending strain energy, respectively. Finally we need relations

$$J_M = \frac{K_M^2}{E}H \qquad J_B = \frac{K_B^2}{E}\frac{H}{3} \qquad A_M = \frac{K_M H}{pR\sqrt{\Pi a}} \qquad A_B = \frac{K_B H}{pR\sqrt{\Pi a}} \tag{32}$$

Results for A_M and A_B are given in Fig. 3 for different values of cracked shell parameter $\lambda = \frac{a}{\sqrt{RH}}\sqrt[4]{12(1-\nu^2)}$. For the example chosen R=40, H=1, ν=0.3 other data as follows: E=210 GPa, p = 1 MPa.

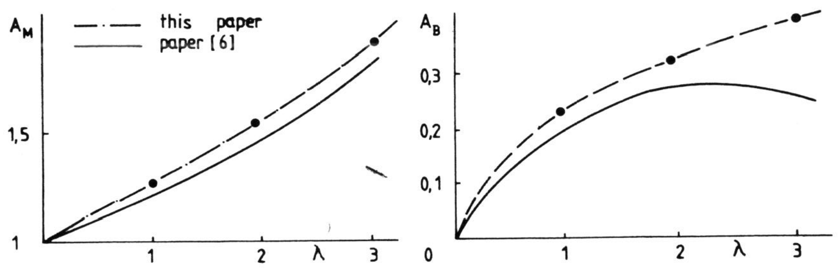

Figure 3. Results for the membrane and bending stress intensity factors

As one can see from the Fig. 3. the agreement between the results is up to 95%, except for the higher values of λ for bending stress intensity factor. The reason for this is probably due to the coupling of membrane and bending terms in (2). Anyhow, having in mind that the membrane stress intensity factor is dominant in this case, the combined value of these two factors is still in good agreement.

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