An Integral Equation Method Based on Resultant Forces on a Piece-wise Smooth Crack in a Finite Plate

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ABSTRACT

Integral equations for the resultant forces on a piece–wise smooth crack line are formulated and coupled to the standard BEM equations for the outer boundary of a finite plane. The resulting equations are a generalization of the equations for infinite geometries (Cheung and Chen, 1987). The integrals along the crack line, with the dislocation densities as unknowns, contain only a weak logarithmic singularity. An improvement of the numerical formulation at a kink of the crack line is introduced. Two numerical experiments are presented and compared with alternative numerical calculations.

KEYWORDS

Integral equation method; BEM; resultant forces; kinked crack; finite geometries.

INTRODUCTION

Singular integral equations are widely applied for the solution of fracture mechanics problems. For cracks in two dimensional infinite geometries and elastic materials, these equations can for example be derived either by applying the integral transform method, or by using complex potentials (Erdogan, 1983). The resulting integral equations, with dislocation densities as unknowns, are expressions for the tractions along the crack line and contain a Cauchy–type singularity. More general integral equations for finite geometries, also for the tractions along the crack line, can be derived by performing partial integrations of the standard BEM equations (Zang and Gudmundson, 1988a). The numerical formulation of the integral equations is based on a suitable numerical evaluation of the singular integrals. A collocation method is then applied to derive an approximate solution of the equations.

It was however shown by Zang and Gudmundson (1988a) that if a piece–wise smooth crack is considered, the expressions for the tractions on the crack line are not valid at a kink. This fact can give rise to numerical difficulties. Lo (1978) applied a Green's function for a point dislocation which satisfies the traction free boundary conditions on the main crack surfaces. An integral equation for the branched portion of the crack could then be formulated. Problems which can arise from the presence of the kink are avoided by using this method. The method is, however, limited to problems which concern an infinite plane containing a crack formed by two straight lines. For more general problems, a symmetrical constraint equation for a kink was introduced by Zang and Gudmundson (1988a), since in general the deformation close to a corner is dominated by a symmetrical singular mode. However, if a corner is loaded by a pure antisymmetrical load, errors may arise by the application of a symmetrical constraint equation. This problem can be avoided using an antisymmetrical constraint equation at that particular corner, if it could be known beforehand that the corner would be antisymmetrically loaded.

Recently, Cheung and Chen (1987) introduced an alternative integral equation for problems with kinked cracks in infinite plates. The unknows in their equations are still dislocation densities and the integrals represent the expressions for resultant forces along the crack line. Compared to a 1/r singularity for the traction formulation (Erdogan 1983; Zang and Gudmundson, 1988a), their equations only contain a weaker logarithmic singularity. This implies that the equations are valid for every point along the crack line.

Using the same ideas as were presented by Cheung and Chen (1987), integral equations for the resultant forces on an internal piece-wise smooth crack in a finite plate have been derived in this investigation. The unknowns in the equations are still the dislocation densities along the crack line. To handle the singularities of the dislocation densities at a kink, double nodes with identical coordinates were introduced at each kink of the crack line. Numerical results (Zang and Gudmundson, 1988b) have shown that this technique generates much better results compared to the results presented by Cheung and Chen (1987), where an extrapolation equation was used for the numerical formulation at a kink.

Two different numerical examples have been evaluated, a rectangular plate containing a V-shaped crack and a Z-shaped crack respectively. The numerical results have been compared to the solutions using the method presented by Zang and Gudmundson (1988a) and to own finite element calculations.

PROBLEM FORMULATION AND NUMERICAL IMPLIMENTATION

A two-dimensional region Ω, with outer boundary Γ, and an internal piece-wise smooth crack line Γ_c, is loaded by prescribed tractions τ_j on some part of the boundary and prescribed displacements u_j on the other part of the boundary, see Fig 1. Furthermore, the crack surfaces are assumed to be symmetrically loaded, i.e. $\tau_j^+ = -\tau_j^-$, where the superscripts + and − denote the upper and the lower boundary of the crack line respectively. The problem can be formulated (*cf.* Zang and Gudmundson, 1988a) as an integral equation for the displacements on the outer boundary,

$$c_{ij}(p)u_j(p) = \int_\Gamma U_{ij}(p,q)\tau_j(q)d\Gamma - \int_\Gamma T_{ij}(p,q)u_j(q)d\Gamma - \int_{\Gamma_c^-} W_{ij}(p,q)\frac{\partial}{\partial s^-}[\Delta u_j(q)]\,ds^-, \quad p \in \Gamma \tag{1}$$

coupled to the integral equation for the tractions on the crack surfaces,

$$\tau_j(p^-) = n_i(p^-)\left[-\int_\Gamma U_{ijk}(p^-,q)\tau_k(q)d\Gamma + \int_\Gamma T_{ijk}(p^-,q)u_k(q)d\Gamma + \int_{\Gamma_c^-} P_{ijk}(p^-,q)\frac{\partial}{\partial s^-}[\Delta u_k(q)]ds^-\right], \quad p^- \in \Gamma_c^- \tag{2}$$

and the constraint equation for the dislocation densities along the crack line,

$$\int_{\Gamma_c^-} \frac{\partial}{\partial s^-}(\Delta u_j)\,ds^- = 0. \tag{3}$$

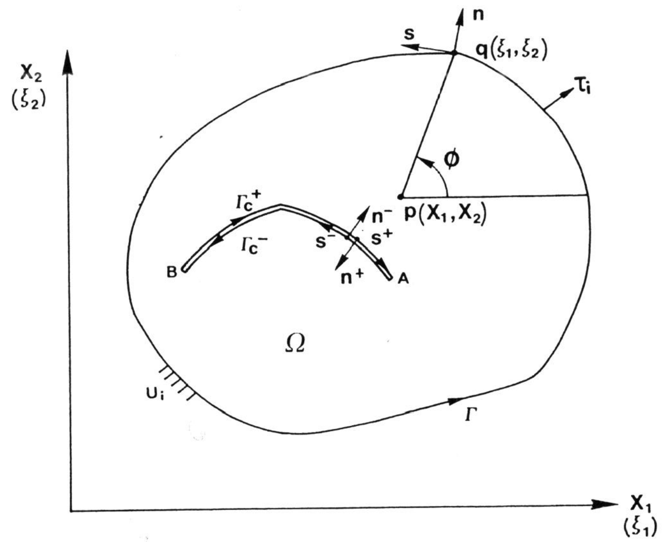

Fig. 1. Geometry and coordinate systems.

In eqs.(1–3), Δu_k denotes the crack opening displacement, $u_k^+ - u_k^-$, and s^- is the local tangential coordinate along Γ_c^-. The integral kernels U_{ij}, T_{ij}, W_{ij}, U_{ijk}, T_{ijk} and P_{ijk} are given by Zang and Gudmundson (1988a).

It was shown (Zang and Gudmundson, 1988a) that eq.(2) is not valid at a kink of the crack line. This can introduce some numerical difficulties when kinks are present on a crack line. To avoid such difficulties, an integral equation for the resultant forces along the crack line is instead introduced in the present investigation.

The integral equations for the resultant forces on the crack line are determined by an integration of eq.(2), along the crack line, from one crack tip A to a ponit $p^- \epsilon \Gamma_c^-$,

$$\underset{p^-\in\,\Gamma_c^-}{F_j(p^-)} = \int_A^{p^-} \tau_j(\lambda)\, d\lambda. \tag{4}$$

After a change in order of integration and an explicit integration with respect to λ, the resultant forces can be written as

$$\underset{p^-\in\,\Gamma_c^-}{F_j(p^-)} = \int_\Gamma H_{jk}(p^-,q)\tau_k(q)d\Gamma + \int_\Gamma G_{jk}(p^-,q)u_k(q)\, d\Gamma + \int_{\Gamma_c^-} K_{jk}(p^-,q)\frac{\partial}{\partial s^-}[\Delta u_k(q)]ds^- + C_j, \tag{5}$$

where for plane deformation

$$H_{jk} = \frac{-1}{4\pi(1-\nu)}\left[2(1-\nu)\,\phi\,\delta_{jk} - (1-2\nu)\ln(r)\,\epsilon_{jk} + (r_{,k}r_{,s})\,\epsilon_{js}\right],$$

$$G_{jk} = \frac{-2\mu}{4\pi(1-\nu)\,r}\left[r_{,s}n_t\,\epsilon_{ts}\,\delta_{jk} + (r_{,j}\epsilon_{ks} + r_{,k}\epsilon_{js})\,r_{,s}r_{,t}n_t\right], \tag{6}$$

$$K_{jk} = \frac{-2\mu}{4\pi(1-\nu)}\left[\ln(r)\,\delta_{jk} - r_{,j}r_{,k}\right],$$

$$tg(\phi) = r_2/r_1, \quad r_j = (\xi_j - x_j), \quad r_{,j} = r_j/r, \tag{7}$$

and where δ_{ij} denotes the Kronecker delta and $\epsilon_{12}=\epsilon_{22}=0$, $\epsilon_{12}=-\epsilon_{21}=-1$.

The integral equation (1) for the displacements on the outer boundary, together with the integral equation (5) for the resultant forces on the crack line, and the constraint equation (3) define the problem to be solved. It is observed that the only singularity in eq.(5) is the logarithmic term in K_{jk}. Since p^- is located on the crack line and only internal cracks are considered, the integrals along the outer boundary contain no singularities.

The integrals in eqs.(1, 3, 5) can be divided into the integrals along the outer boundary Γ, with the displacements u_j or the tractions τ_j as unknowns, and the integrals along the lower crack line Γ_c, with the dislocation densities $\partial/\partial s^-(\Delta u_j)$ as unknowns. In this investigation, the standard BEM technique was directly utilized for the integrals along the outer boundary Γ. For the integrals along the crack line Γ_c, a slightly different boundary element method was applied. The method is described below.

A crack formed by two straight lines is considered. The crack is divided into two segments separated by the kink. Each line is discretized into N_i elements with N_i+1 nodes. The line coordinate s, and the dislocation densities D_i ($D_i = \partial/\partial s^-(\Delta u_i)$), within an element away from the crack tip, are described by standard linear isoparametric shape functions. For instance, the interpolations for the interval $a_k a_{k+1}$ become

$$s = M_1(\eta)s_k + M_2(\eta)s_{k+1},$$
$$D_i = M_1(\eta)D_{i,k} + M_2(\eta)D_{i,k+1}, \tag{8}$$

$$M_1(\eta) = (1-\eta)/2,$$
$$M_2(\eta) = (1+\eta)/2, \tag{9}$$

where s_k denotes the coordinate, $D_{i,k}$ the dislocation densities at node a_k, and $|\eta|\le 1$ the local coordinate for the considered element. For the first element, a_1a_2, which contains the crack tip a_1, the following interpolation is used for the dislocation densities

$$D_i = \sqrt{2/(1+\eta)}\,[M_1(\eta)D_{i,1} + M_2(\eta)D_{1,2}]. \tag{10}$$

For the elements adjacent to a kink, double nodes with identical coordinates are introduced at the kink. In addition, a mesh refinement technique (Strese, 1984) was applied close to points for which singularities in the dislocation densities could be expected.

It follows from above that there are $2(N_i+1)$ unknown nodal dislocation densities for each smooth segment (i) of the crack line. Thus, for a crack composed of two straight lines and with the addition of the two unknown constants C_j in eq.(5), a total of $2(N_1+N_2+3)$ unknowns result. Certain collocation points are then selected in order to generate as many equations as unknowns.

Since only internal crack problems are considered, the following constraint equation has to be fulfilled

$$\int_{\Gamma_c^-} D_i\, ds = 0. \tag{11}$$

Hence for a unique solution to the problem, at least (N_1+N_2+2) collocation points are needed. If all the nodes which are not located on a kink are selected as collocation points, then there will be (N_1+N_2) collocation points and two additional collocation points are needed. These two points can be arranged such that $\eta=\eta^r$ for the point to the right of the kink and $\eta=\eta^l$ for the point to the left of the kink. Zang and Gudmundson (1988b) showed that the numerical solutions are not sensitive to the selection of η^r and η^l. In this investigation $\eta^r = -0.5$, $\eta^l = 0.5$ were used for all the numerical calculations. In this way, as many equations as unknowns can be generated for the problem.

The integrals along Γ_c^- are numerically evaluated by using Gauss–Chebyshev quadrature for nonsingular parts and explicit analytical integration for singular parts.

The stress intensity factors, for example at the crack tip B, see Fig. 1, can be calculated afterwards as

$$K_I = \frac{-2\mu}{(\kappa+1)}\sqrt{2\pi d}\,\left[\overset{*}{D}_1\sin(\alpha) - \overset{*}{D}_2\cos(\alpha)\right],$$
$$K_{II} = \frac{+2\mu}{(\kappa+1)}\sqrt{2\pi d}\,\left[\overset{*}{D}_1\cos(\alpha) + \overset{*}{D}_2\sin(\alpha)\right], \tag{12}$$

where d is the length of the element close to the crack tip B, α is the angle between the X_1 axis and the crack line, and $\overset{*}{D}_1$, $\overset{*}{D}_2$ are the nodal values at the crack tip B.

NUMERICAL RESULTS

Two different crack problems were evaluated by the present method, a rectangular plate containing a V shaped crack and a Z shaped crack respectively. In this investigation linear elements were used for all the integrals. For comparison, calculations were performed by the finite element method and the method by Zang and Gudmundson (1988a).

A V–shaped Crack

A square plate containing a V–shaped crack subjected to uniform shear on the outer boundary is considered, see Fig.2. The outer boundary of the plate was modelled by 72 elements and each straight part of the crack line by 22 elements. For comparison, a finite element calculation using the program ADINA with 8–noded isoparametric elements was performed for the same geometry. At the two crack tips quarter point elements were used to simulate the right singularity. No special consideration was taken to the singularity at the corner B in the FEM calculation.

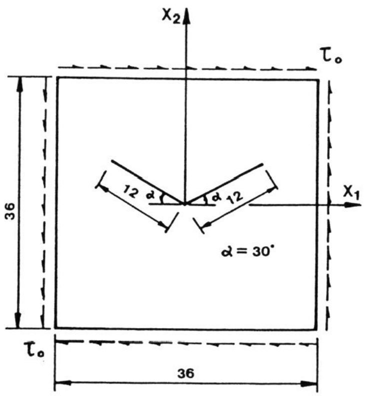

Fig. 2. Geometry and loading for the V–shaped crack.

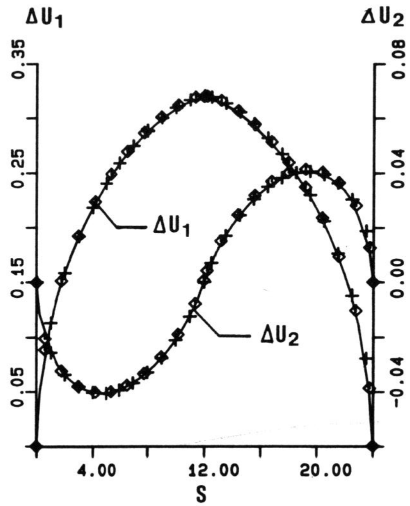

Fig. 3. Crack opening displacements for the V–shaped crack calculated with $E/\tau_0 = 205$, $\nu = 0.3$. The squares denote the present method, the crosses the method by Zang and Gudmundson (1988a) and the solid line the FEM results.

The integral equation for the tractions along the crack line was utilized by Zang and Gudmundson (1988a) to solve the problem. This formulation enforces two constraint equations at each kink. Numerical results presented in that paper showed that symmetrical constraint equations can be justified in almost all situations. In the present problem, however, antisymmetrical constraint equations have to be applied at the kink, since the corner is dominated by a pure antisymmetrical mode. The outer boundary was similarly modelled by 72 elements and for each straight crack line 25 Gauss–point were used.

In Fig.3, the crack opening displacements Δu_1 and Δu_2 are presented by the three methods. It is observed that a good agreement between the solutions of the alternative methods is achieved.

A Z–shaped Crack

A rectangular plate containing a Z–shaped crack is considered, see Fig.4. The plate is subjected to a uniform tension on one end and has a sliding support on the opposite end. The outer boundary of the plate was discretized into 90 linear BEM–elements. The middle straight part of the crack line contained 20 elements and each of the two other straight parts of the crack line contained 5 elements. Totally the plate was modelled by about 250 degrees of freedom. For comparison a finite element calculation using the program ADINA with 8–noded isoparametric elements was performed for the same geometry. Again the method presented by Zang and Gudmundson (1988a) with symmetrical constraint equations at each kink was used to solve the same problem. Using this method, the problem was modelled by 90 elements for the outer boundary, 25 Gauss–points for the middle straight line, and 15 Gauss–points for each of the two other crack lines. In Fig.5 the crack opening displacements computed with the present method are compared to the other two methods. Also in this case the agreement is good.

DISCUSSION

Integral equations for the resultant forces along the crack line was used in this investigation to solve problems of a finite elastic plate containing internal piece–wise smooth cracks. The integrals along the crack line contain a weak logarithmic singularity and they are valid for every point along the crack line. The collocation points can thus be selected in a free manner. Furthermore, no constraint equation, as was the case in the paper by Zang and Gudmundson (1988a), is needed at a kink. The present integral equations can be applied to any internal crack configuration and for all kinds of loading conditions.

In the authors' previous work (Zang and Gudmundson 1988b), cracks with different kink configurations in an infinite plate were examined in detail utilizing the same kind of formulation as in the present analysis. A very good convergency rate was observed for crack problems in both infinite and half infinite geometries. The numerical results in this paper indicate that the present method also can be an effective and reliable tool for investigations of internal crack problems in finite geometries.

From the discussion above, it may be concluded that the method employing integral equations for resultant forces along the crack line can be more efficient than the method using traction type equations.

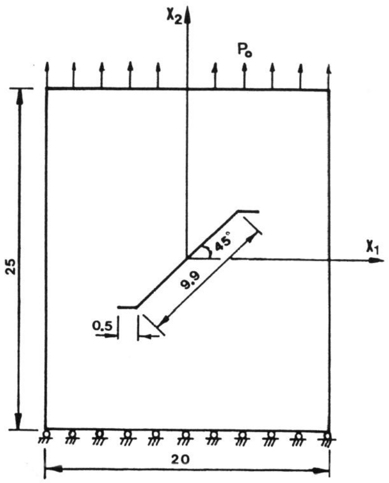

Fig. 4. Geometry and loading for the Z-shaped crack.

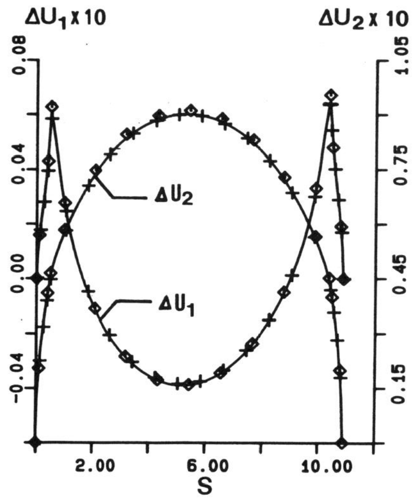

Fig. 5. Crack opening displacements for the Z-shaped crack calculated with $E/P_o = 205$, $\nu = 0.3$. The squares denote the present method, the crosses the method by Zang and Gudmundson (1988a) and the solid line the FEM results.

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