An Economic Finite Element Strategy for LEFM Problems

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ABSTRACT

An economic finite element strategy is developed for the estimation of Stress Intensity Factors (SIF) in Linear Elastic Fracture Mechanics (LEFM) problems involving cracks in structural components. Considering two-dimensional problems, the finite element model consists of 5- noded triangular isoparametric regular/singular elements located at the crack tip and 4-noded quadrilateral elements in the remaining part of the structure. The square root singularity is achieved in the 5-noded elements by moving the mid-side nodes to the quarter point position (as in the case of 8-noded quadrilateral/6-noded triangular elements). Modified Crack Closure Integral (MCCI) method is adopted which generates accurate results for a relatively coarse mesh. The relevant equations for strain-energy-release rates (G) are derived for 5-noded triangular element and the SIF are computed. The model is demonstrated by numerical studies for a centre crack in finite plate under uniaxial tension.

KEY WORDS

Stress intensity factor, Strain-energy-release rate, Singular elements, Modified crack closure integral, 5-noded traingular element.

INTRODUCTION

Finite element technique is the most preferred method in linear elastic fracture mechanics (LEFM) for the determination of stress intensity factors (SIF) at crack tips in structural components. Various approaches (Gallagher 1971, Atluri et al., 1975, Barsoum 1976) using singular and non-singular elements have appeared in literature which is quite extensive. Most of these techniques (except a few approaches with singular elements where SIF are used as degrees of freedom) basically carry out stress analysis of the cracked structure. Using this stress and displacement distributions direct methods such as displacement or force extrapolation (Chan et al., 1970, Raju et al., 1977) and indirect methods where estimation of energy parameters such as strain-energy-release rates (Rybicki et al., 1977, Dattaguru et al., 1982) and J-integral (Rice 1968) are used to determine the SIF. The later methods involving the estimation of energy parameter are generally economical and are capable of predicting accurate results with coarse meshes.

The present paper deals with modified crack closure integral (MCCI) method wherein strain-energy-release rates are estimated leading to the determination of SIF. An economic finite element solution is presented in this paper for cracks in two-dimensional structures using this method. The finite element model consists of 5-noded triangular isoparametric regular/singular elements at the crack tip and 4-noded quadrilateral elements in the remaining portion of the structure. The singularity is achieved at the crack tip by moving the mid-side nodes to quarter points (as presented by Barsoum, 1976 for 8-noded quadrilateral/or 6-noded triangular elements). The MCCI equations for strain-energy-release rate estimation are derived for 5-noded regular/singular elements for both I and II modes of fracture. A procedure, similar to that used earlier (Krishnamurthy et al., 1985 and Ramamurthy et al., 1986) for 8-noded regular/singular isoparametric elements is employed, here, to derive MCCI equations for G_I and G_{II}. Numerical results are presented for the case of central crack in a finite plate under uniaxial tension. The derivation of the equations for G_I, G_{II} and G_{III} for three-dimensional through and part-through (semi and quarter elliptical) cracks using the MCCI are nearly complete (Badari Narayana 1988).

ANALYSIS

Element Shape Functions

5-noded triangular isoparametric serendipity element in real and natural coordinate system is shown in Fig. 1(a) and (b). The shape functions defining the displacement distribution in this element are given by Zienkiewicz (1971)

$$\{u\} = [N_i]\ \{u_i\} \qquad (1)$$

where

$$N_1 = (1 - \xi - \eta)\ (1 - 2\xi - 2\eta); \quad N_2 = -\xi\,(1 - 2\xi - 2\eta)$$

$$N_3 = -\eta\,(1 - 2\xi - 2\eta); \quad N_4 = 4\eta\,(1 - \xi - \eta); \quad N_5 = 4\xi\,(1 - \xi - \eta)$$

and u_i are the nodal displacements.

The edge 1-5-2 (or 1-4-3) in the real and natural coordinate system is as shown in Fig. 1(c). The transformation between the two systems is given by

$$r = a_0 + a_1\,\xi + a_2\,\xi^2 \qquad (2)$$

With the mid-side node is at the mid-points and proceeding in the same way as given by Barsoum, 1976 it can easily be shown that here too, shifting the mid-side node to quarter point, there is a $1/\sqrt{r}$ singularity in $\partial\xi / \partial r$ and so in the strains (or stresses) at the origin 1. So, by the choice of both mid-side nodes 4 and 5 at quarter points on the respective edges, (Fig. 1(d)) isoparametric singular 5-noded element is obtained.

Finite Element Model

A finite element model using isoparametric 5-noded triangular elements near

the crack tip and 4-noded quadrilateral elements in the remaining region is shown in Fig. 2. This is a typical mesh with the dimensions of the finite plate shown (H/w = 6, 2a/w = 0.2 to 0.8 in steps of 0.1) with the model consists 96 elements and 250 degrees of freedom.

Crack Closure Integral

The crack closure integral estimates the strain-energy-release rate (G), based on Irwin's (1958) concept, that when a crack grows by an infinitesimal increment, G is equal to the work required to close the crack to its original length. Taking polar coordinate system with the origin at the crack tip (Fig. 3), the above statement can be written for G_I and G_{II} as

$$G_I = \underset{\Delta a \to 0}{\mathrm{Lt}}\ (1/2\ \Delta a)\int_0^{\Delta a} \sigma_y\ (r{=}x,\ \theta{=}0)\ u_y\ (r{=}\Delta a - x, \theta = \pi)\ dr$$

$$G_{II} = \underset{\Delta a \to 0}{\mathrm{Lt}}\ (1/2\,\Delta a)\int_0^{\Delta a} \sigma_{xy}\ (r{=}x,\theta{=}0)\ u_x\ (r{=}\Delta a - x, \theta = \pi)\ dr \qquad (3)$$

where

$\sigma_y(r,\theta)$, $\sigma_{xy}(r,\theta)$= Stress distribution ahead of the crack tip

$u_x(r,\theta)$, $u_y(r,\theta)$ = Relative sliding and opening displacements between points on the crack faces

Now, SIF for mode I and mode II can be obtained using the standard relations.

Modified Crack Closure Integral

In MCCI the equation (3) to evaluate the crack closure integral is modified in terms of nodal forces and nodal displacements to suit finite element method.

a) In the first place, the equations for strain-energy-release rate will be presented for the non-singular element in which the mid-side nodes are at mid point. Figure 4 shows the nodal forces on the crack extension line and the crack opening displacements at the nodes along the crack. Assuming the distributions for displacements and for the stresses on the crack extension line as

$$u_y(\xi') = a_0 + a_1\,\xi' + a_2\,\xi'^2$$

$$\sigma_y(\xi) = b_0 + b_1\xi + b_2\,\xi^2 \qquad (4)$$

It is possible to derive these distributions in terms of nodel values. In the later case of stresses this is done by finding the equivalent nodal forces for these distributions. Now, the constant a_0, a_1 and a_2 and b_0, b_1 and b_2 could be solved from the nodal values (Fig. 4) as

$a_0 = u_{y,j-2}$, $a_1 = 4\,u_{y,j-1} - 3\,u_{y,j-2}$ and $a_2 = -4\,u_{y,j-1} + 2\,u_{y,j-2}$ and

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = (3/2\ \Delta a) \begin{bmatrix} 6 & -1 & 2 \\ -24 & 10 & -16 \\ 20 & -10 & 20 \end{bmatrix} \begin{bmatrix} F_{y,j} \\ F_{y,j+1} \\ F_{y,j+2} \end{bmatrix} \qquad (5)$$

The final expressions for strain-energy-release rates in mode I and mode II could be derived as

$$G_I = (1/2\,\Delta a)\,[F_{y,j}\,u_{y,j-2} + F_{y,j+1}\,u_{y,j-1}] \text{ and}$$

$$G_{II} = (1/2\ \Delta a)\,[F_{x,j}\,u_{x,j-2} + F_{x,j+1}\,u_{x,j-1}] \qquad (6)$$

The limit as Δa —> 0 is satisfied numerically by using a small element at the crack tip. These expressions are same as those obtained for 8-noded quadrilateral elements with mid-side node at mid-point by Krishnamurthy et al., 1985. Expressions identical to Eq.(6) were used earlier by Buchholz et al., 1984, and Grebner et al., 1985 for linear strain triangular (LST) elements to obtain (G) values for axisymmetric crack problems (in the analysis of debonding of thermally stressed fibre-reinforced composite materials).

b) Now, we will obtain MCCI equations for (G) for 5-noded triangular singular element. The nodal forces and crack opening displacements for 5-noded singular elements are shown in Fig. 5. Here, shifting the mid-side node to quarter point results in strain (or stress) singularity. So, the expressions for displacements and stresses on the crack extension line can be represented as

$$u_y(\xi') = a_0 + a_1\xi' + a_2\,\xi'^2$$

$$\sigma_y(\xi) = (b_0/\xi) + b_1 + b_2\xi \qquad (7)$$

The constants a_0, a_1 and a_2 and b_0, b_1, and b_2 are obtained in terms of the nodal values (Fig. 5) as

$$a_0 = 0,\ a_1 = 4u_{y,j-1} - u_{y,j-2} \text{ and } a_2 = -4u_{y,j-1} + 2u_{y,j-2} \text{ and}$$

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = (3/2\,\Delta a) \begin{bmatrix} 3 & -1/2 & 1 \\ -12 & 5 & -8 \\ 10 & -5 & 10 \end{bmatrix} \begin{bmatrix} F_{y,j} \\ F_{y,j+1} \\ F_{y,j+2} \end{bmatrix} \qquad (8)$$

The transformation for the singular element between and systems can be obtained as

$$(1 + \xi)^2 + (1 + \xi')^2 = 4 \qquad (9)$$

Now using Eqs.(7) and (9) in Eq. (3) and on simplification the expressions for strain-energy-release rates can be derived as

$$G_I = (1/2\,\Delta a)\,[(C_{11}\,F_{y,j} + C_{12}\,F_{y,j+1} + C_{13}\,F_{y,j+2})\,u_{y,j-1}$$
$$+ \ (C_{21}\,F_{y,j} + C_{22}\,F_{y,j+1} + C_{23}\,F_{y,j+2})\,u_{y,j-2}] \text{ and}$$
$$G_{II} = (1/2\,\Delta a)\,[(C\ \ F_{x,j} + C_{12}\,F_{x,j+1} + C_{13}\,F_{x,j+2})\,u_{x,j-1}$$
$$+ \ (C_{21}\,F_{x,j} + C_{22}\,F_{x,j+1} + C_{23}\,F_{x,j+2})\,u_{x,j-2}] \qquad (10)$$

where $C_{11} = 33\pi/2 - 52$, $C_{12} = 17 - 21\pi/4$, $C_{13} = 21\pi/2 - 32$

$C_{21} = 14 - 33\pi/8$, $C_{22} = 21\pi/16 - 7/2$, $C_{23} = 8 - 21\pi/8$

These expressions are same as those obtained for 8-noded singular elements by Ramamurthy et al., 1986. Once again the limit Δa —> 0 is approached numerically by taking the crack tip element size to be as small as possible.

NUMERICAL STUDIES

Numerical studies were carried out on the problem of centre crack in finite plate under uniaxial tension for various crack sizes. The crack tip essentially deforms in mode I. The equations derived above were used in finite element models using both regular and singular 5-noded elements at the crack tip to evaluate SIF. These results were compared with reference solution (Rooke et al., 1976). The maximum deviation with reference solution is within 1.5 percent. It is important to note that degrees of freedom used in the present model are varying between 200-250 where as the degrees of freedom for an equally accurate result with 8-noded quadrilateral elements is more than 300.

CONCLUSION

5-noded regular and singular elements are used at the crack tip for estimation of stress intensity factor (SIF) in 2-dimensional fracture problems. The SIF are estimated through strain-energy-release rates obtained by modified crack closure integral (MCCI). Combining these elements with the MCCI is found to result in an economic solution. The advantage will be more if a similar technique is used for 3-dimensional problems (Badari Narayana 1988).

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Table 1 Comparision of SIF from MCCI method and standard reference (Rooke et al 1976) values: CCT specimens under uniaxial tension

a) SIF from MCCI method using 5-noded triangular regular elements

a/w	Present FEM	$K_I/\sigma\sqrt{\pi a}$ 8-node FEM solution Krishnamurthy et al (1985)	$K_I/\sigma\sqrt{\pi a}$ Reference solution	Column 2 / Column 4	Column 3 / Column 4
1	2	3	4		
0.2	1.0235	1.0272	1.0254	0.0987	1.0017
0.3	1.0737	1.0505	1.0594	1.0134	0.9916
0.4	1.1273	1.1038	1.1118	1.0139	0.9928
0.5	1.1947	1.1887	1.1891	1.0047	0.9996
0.6	1.3234	1.3108	1.3043	1.0146	1.0049
0.7	1.4808	1.4989	1.4842	0.9977	1.0099
0.8	1.8135	1.8244	1.7989	1.0081	1.0141

b) SIF from MCCI method using 5-noded triangular singular elements

a/w	Present FEM	$K_I/\sigma\sqrt{\pi a}$ 8-node FEM Solution Ramamurthy et al (1986)	$K_I/\sigma\sqrt{\pi a}$ Reference Solution	Column 2 / Column 4	Column 3 / Column 4
1	2	3	4		
0.2	1.0124	1.0213	1.0254	0.9873	0.9960
0.3	1.0619	1.0473	1.0594	1.0023	0.9885
0.4	1.1149	1.1026	1.1118	1.0027	0.9917
0.5	1.1787	1.1887	1.1891	0.9913	0.9926
0.6	1.3093	1.3121	1.3043	1.0038	1.0059
0.7	1.4648	1.5007	1.4842	0.9869	0.9884
0.8	1.7925	1.8273	1.7989	0.9964	0.9844

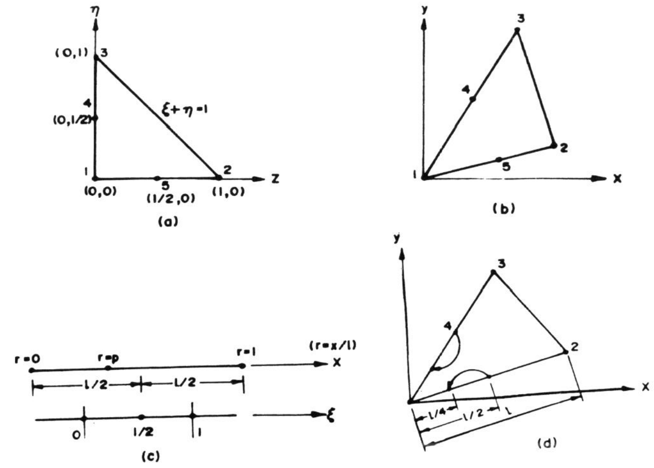

Fig. 1. 5-noded triangular element
a) parent element in ξ – η plane
b) regular element in X-Y plane
c) one dimensional element mapping
d) singular element in X-Y plane

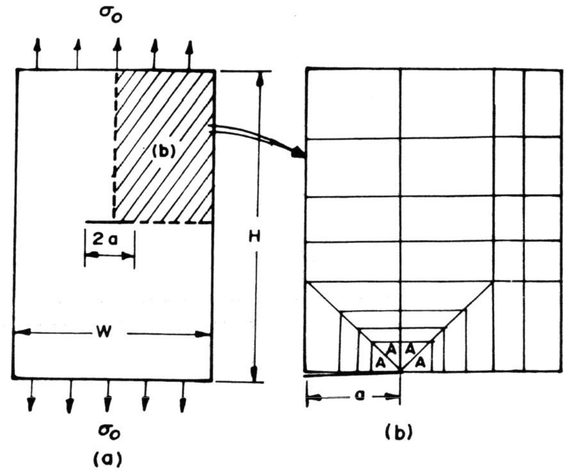

Fig. 2. Center cracked tension specimen
a) Part of the structure analysed
b) Finite element mesh for CCT – specimen

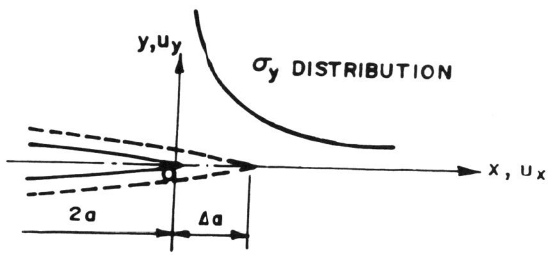

Fig. 3. Original and extended crack configurations

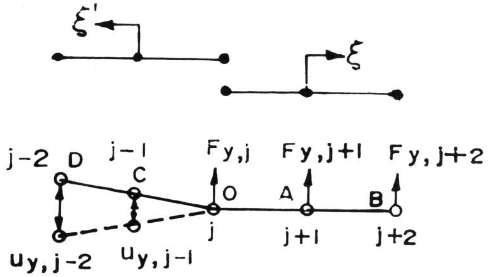

Fig. 4. Crack opening displacements and nodal forces: 5-noded triangular regular elements

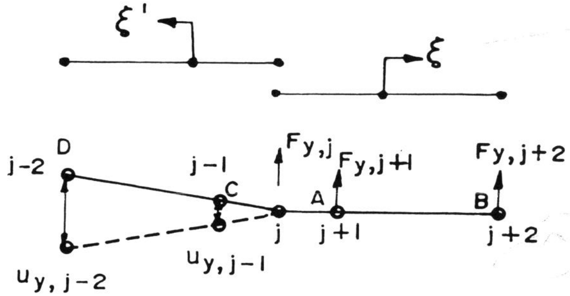

Fig. 5. Crack opening displacement and nodal forces: 5-noded triangular singular elements