About the Penny-shaped Dugdale Crack Under Shear and Triaxial Loading

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ABSTRACT

The three-dimensional problem of a penny-shaped Dugdale crack is considered. Beyond the known mode I case, in particular the case of pure shear loading is modelled and investigated. Analytical closed-form solutions are given that show a marked analogy to the mode I case. By an appropriate superposition the penny-shaped Dugdale crack under triaxial (mixed mode) loading can be treated in a convenient manner. The derived and presented description is of a clear closed-form analytical character which for applicability is useful.

KEYWORDS

Dugdale crack, elastic-plastic, mixed mode, stress intensity factors, three-dimensional crack problem, penny-shaped crack, yield ring

INTRODUCTION

In the field of elastic-plastic fracture mechanics the Dugdale crack concept is a well established tool to describe the localized yielding that accompanies elastic-plastic crack growth. Numerous contributions, generalizations and applications repeatedly have proved the usefulness of the Dugdale crack concept. Above all, the plane problem of a straight Dugdale crack has been considered so far, predominantly for pure mode I loading (Chell, 1976; Dugdale, 1960; Gross, 1984; Hahn and Rosenfield, 1965; Herrmann, 1987; Janson, 1977; Newman, 1968; Seeger, 1973; Tada *et al.*, 1985; Theocaris and Gdoutos, 1974) but also for mode II (Becker and Gross, 1987; Bilby *et al.*, 1963; Tada *et al.*, 1985) and for mixed mode (Becker and Gross, 1988) loading.

Three-dimensional Dugdale crack problems have attained less attention till now (Mattheck and Görner, 1984; Mattheck and Gross, 1985; Tada *et al.*, 1985). In general their analytical treatment involves markedly more intricacy. Nevertheless three-dimensional crack problems are of essential importance — most crack-like defects within materials strictly speaking necessitate a three-dimensional analysis.

MODE I PENNY-SHAPED DUGDALE CRACK

As an idealized defect a penny-shaped crack of radius a is to be considered in an infinitely extended isotropic linear-elastic material. Under remote loading the real crack is enlarged by a "yield ring" to a fictitious penny-shaped crack of radius b in the same plane. This is illustrated in Fig. 1 together with the used cartesian coordinates x, y, z and the cylindrical coordinates r, ϑ, z. On the yield ring a normal stress σ_0 respectively

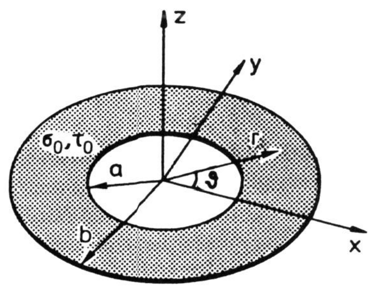

Fig. 1: Penny-shaped-Dugdale crack

a shear stress τ_0 are prescribed. In an idealized way these quantities describe the normal and shear stress proportions that are transmitted by the yield zone when plastification takes place. The yield ring width $b-a$ is to be reconciled with the given stresses in such a manner that along the circular crack front stress singularities are as little as can be. The stress intensity factors $K_I(\vartheta)$, $K_{II}(\vartheta)$, $K_{III}(\vartheta)$ ideally vanish.

This requirement can be met without particular problems in the case of a pure mode I loading, that means for uniaxial tension σ in z-direction. In this case the stress intensity factors K_{II}, K_{III} vanish anyway (as well as τ_0) and the demand $K_I = 0$ is guaranteed by the condition (Tada *et al.*, 1985)

$$\frac{a}{b} = \sqrt{1-\frac{\sigma^2}{\sigma_0^2}} \tag{1}$$

$$\text{respectively} \quad \frac{\sigma}{\sigma_0} = \sqrt{1-\frac{a^2}{b^2}} \quad . \tag{2}$$

Relation (1) is the "penny-shape-analogue" of the following equation which holds in the

case of the corresponding plane mode I Dugdale crack problem:

$$\frac{a}{b} = \cos\frac{\pi\sigma}{2\sigma_0} \quad . \tag{3}$$

Equation (1) gives the following "relative yield zone size":

$$\left(\frac{b-a}{b}\right)_{penny-shaped} = 1-\sqrt{1-\frac{\sigma^2}{\sigma_0^2}} \quad . \tag{4}$$

On the other hand in the case of the corresponding plane problem equation (3) gives

$$\left(\frac{b-a}{b}\right)_{plane} = 1-\cos\frac{\pi\sigma}{2\sigma_0} \quad . \tag{5}$$

For a comparison in Fig. 2 the relative yield zone sizes according to (4) and to (5) are represented as functions of stress. Obviously in the whole range $0 < \sigma < \sigma_0$ the yield ring width of the penny-shaped Dugdale crack is smaller than the yield strip length of the corresponding plane Dugdale crack problem — in the three-dimensional case yielding is somewhat inhibited.

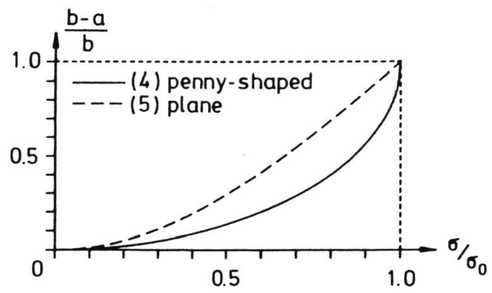

Fig. 2: Yield zone size as function of stress

MODELLING OF THE PENNY-SHAPED DUGDALE CRACK UNDER SHEARMODE LOADING

Basic Setting

The term "shearmode loading" is used here to denote the case that only remote shear stresses τ_{xz} and τ_{yz} are applied with respect to the introduced x-y-z-coordinate system. Without loss of generality it can be assumed that only $\tau_{xz} = \tau$ is different from zero. Differently from the pure mode I case the adherence to the penny-shaped form of the fictitiously extended crack (radius b) is an additional model assumption in the shearmode case. This assumption is justified in the same measure as the Dugdale crack condition of vanishing stress intensity factors can be fulfilled. For the Dugdale crack modelling the decisive advantage of the penny-shape form is that it makes the corresponding three-dimensional crack problem mostly accessible to a closed-form analytical treatment. A survey of appropriate methods and formulas for example has been given by Kassir and Sih (Kassir and Sih, 1975).

From the given homogeneous remote shear stress $\tau_{xz} = \tau$ the following stress intensity factors K_{II} and K_{III} (Tada *et al.*, 1985) result for the penny-shaped crack lying in the x-y-plane (Fig. 3):

$$\begin{aligned} K_{II}^G(\vartheta) &= \frac{4}{\sqrt{\pi}(2-\nu)}\tau\sqrt{b}\cos\vartheta \quad , \\ K_{III}^G(\vartheta) &= -\frac{4(1-\nu)}{\sqrt{\pi}(2-\nu)}\tau\sqrt{b}\sin\vartheta \quad . \end{aligned} \tag{6}$$

The quantities K_{II}^G and K_{III}^G vary along the circular crack front — they are functions of the angle ϑ. From an exclusive loading of the Dugdale yield rings on the other hand

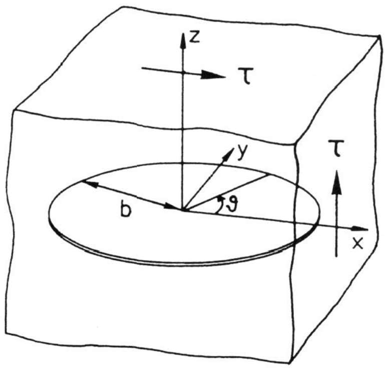

Fig. 3: Penny-shaped Griffith crack under pure shear

stress intensity factors $K_{II}^F(\vartheta)$ and $K_{III}^F(\vartheta)$ result that still have to be derived. Ideally the yield ring width $b - a$ and the given stresses harmonize in such a way that the Dugdale crack conditions

$$\begin{aligned} K_{II}^G(\vartheta) + K_{II}^F(\vartheta) &= 0 \quad , \\ K_{III}^G(\vartheta) + K_{III}^F(\vartheta) &= 0 \end{aligned} \tag{7}$$

hold identically along the circular crack front. Unlike in the corresponding plane case the relations (7) comprise two conditions for functions of ϑ. For an underlying constant shear stress value $\tau_{xz} = \tau_0$ on the yield ring it cannot be expected that the conditions (7) can be fulfilled both identically by the mere choice of the yield ring width $b - a$.

But it turns out that the requirement (7) can at least be met in the sense of a good approximation.

For the derivation of the stress intensity factors K_{II}^F, K_{III}^F use is made of already available, rather general formulas (Kassir and Sih, 1975). For a penny-shaped crack (radius b) by a pure shear load of the crack faces the following boundary conditions are supposed to be given:

$$\begin{aligned} \sigma_z(r,\vartheta,0) &= 0\,, & r \geq 0\,, 0 \leq \vartheta < 2\pi\,, \\ \tau_{rz}(r,\vartheta,0) &= q_c(r,\vartheta)\,, & 0 \leq r < b\,, 0 \leq \vartheta < 2\pi\,, \\ \tau_{\vartheta z}(r,\vartheta,0) &= q_s(r,\vartheta)\,, & 0 \leq r < b\,, 0 \leq \vartheta < 2\pi\,, \\ u_r(r,\vartheta,0) = u_\vartheta(r,\vartheta,0) &= 0\,, & r \geq b\,, 0 \leq \vartheta < 2\pi\,. \end{aligned} \tag{8}$$

The prescribed shear stresses are specified by the functions $q_c(r,\vartheta)$ and $q_s(r,\vartheta)$ whereby the following fourier series expansions hold:

$$\begin{aligned} q_c(r,\vartheta) &= \sum_{n=0}^{\infty} a_n(r)\cos n\vartheta \quad , \\ q_s(r,\vartheta) &= \sum_{n=1}^{\infty} b_n(r)\sin n\vartheta \quad . \end{aligned} \tag{9}$$

For each n ($n = 1, 2, \ldots$) two auxiliary functions are introduced by a_n and b_n:

$$\begin{aligned} \Phi_1(t) &= -\frac{t^{-n+\frac{3}{2}}}{(2-\nu)G\sqrt{2\pi}}\int_0^t r^n[a_n(r) - b_n(r)]\frac{dr}{\sqrt{t^2-r^2}} \quad , \\ \Phi_2(t) &= \frac{\nu}{2}\Phi_1(t) + \frac{t^{-n-\frac{1}{2}}}{2G\sqrt{2\pi}}\left\{\int_0^t r^{n+2}[a_n(r) + b_n(r)]\frac{dr}{\sqrt{t^2-r^2}}\right. \\ &\quad \left. + \frac{(1+2n)\nu}{2-\nu}\int_0^t r^n[a_n(r) - b_n(r)]\sqrt{t^2-r^2}\,dr\right\} \quad . \end{aligned} \tag{10}$$

Then the stress intensity factors $K_{II}(\vartheta)$, $K_{III}(\vartheta)$ along the circular crack edge can be represented in the following way (Kassir and Sih, 1975):

$$\begin{aligned} K_{II}(\vartheta) &= -\frac{2}{\sqrt{\pi}b^{\frac{3}{2}}}\int_0^b r^2 a_0(r)\frac{dr}{\sqrt{b^2-r^2}} + \frac{2\sqrt{2}G}{b}\sum_{n=1}^{\infty}[\Phi_1(b) - \Phi_2(b)]\cos n\vartheta \quad , \\ K_{III}(\vartheta) &= -\frac{2\sqrt{2}G}{b}\sum_{n=1}^{\infty}[(1-\nu)\Phi_1(b) + \Phi_2(b)]\sin n\vartheta \quad . \end{aligned} \tag{11}$$

Constant Shear Stress on Yield Ring

Let a yield ring shear stress τ_0 be prescribed that is of a constant amount and a constant direction (Fig. 4). Then from (9) and (11) the following quantities q_c, q_s and stress intensity factors K_{II}^F, K_{III}^F result:

$$\begin{aligned} q_c(r,\vartheta) &= \tau_0 \cos\vartheta H(r-a) \quad , \\ q_s(r,\vartheta) &= -\tau_0 \sin\vartheta H(r-a) \quad , \quad r<b \quad , \qquad (12) \\ &\text{with} \quad H(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \quad \text{(Heaviside-function)} \quad , \end{aligned}$$

$$\begin{aligned} K_{II}^F(\vartheta) &= -\frac{4}{\sqrt{\pi}(2-\nu)}\tau_0\sqrt{b}\cos\vartheta\sqrt{1-\frac{a^2}{b^2}}\left(1-\frac{\nu}{2}\frac{a^2}{b^2}\right) \quad , \\ K_{III}^F(\vartheta) &= \frac{4(1-\nu)}{\sqrt{\pi}(2-\nu)}\tau_0\sqrt{b}\sin\vartheta\sqrt{1-\frac{a^2}{b^2}}\left(1+\frac{\nu}{2(1-\nu)}\frac{a^2}{b^2}\right) \quad . \qquad (13) \end{aligned}$$

With the given stress intensity factors (6) and (13) the Dugdale crack conditions (7)

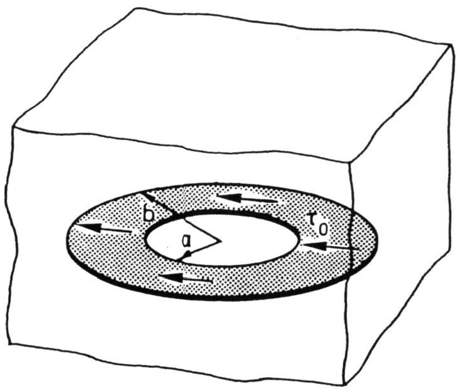

Fig. 4: Penny-shaped crack under yield ring shear stress

can be satisfied in a good approximation if

$$\frac{\nu}{2}\frac{a^2}{b^2} \ll 1 \quad \text{and} \quad \frac{\nu}{2(1-\nu)}\frac{a^2}{b^2} \ll 1 \quad . \qquad (14)$$

The conditions (7) then give the following relationship between the quantities τ, τ_0, a and b:

$$\frac{a}{b} = \sqrt{1-\frac{\tau^2}{\tau_0^2}} \qquad (15)$$

$$\text{respectively} \quad \frac{\tau}{\tau_0} = \sqrt{1-\frac{a^2}{b^2}} \quad . \qquad (16)$$

The analogy to the pure mode I relations (1), (2) is obvious. In the most inconvenient case, namely when the yield ring is very small ($a \approx b$), the conditions (14) are satisfied in the same measure as the factor $\nu/2$ is smaller than 1. Things are the more favourable the larger the yield ring width is, as this gives a decreasing factor a^2/b^2.

Slightly Variable Shear Stress on Yield Ring

By a slightly modified yield ring shear stress the Dugdale crack conditions (7) both can be satisfied even exactly. To this end the quantities q_c and q_s are supposed to be of the following kind:

$$\begin{aligned} q_c(r,\vartheta) &= \tau_c \cos\vartheta H(r-a) \quad , \\ q_s(r,\vartheta) &= -\tau_s \sin\vartheta H(r-a) \quad . \end{aligned} \qquad (17)$$

This corresponds to a shear stress loading on the yield ring whose "radial component"

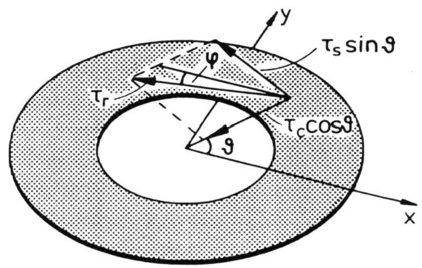

Fig. 5: Yield ring shear stress

τ_{rz} and "circumferential component" $\tau_{\vartheta z}$ are of different angular amplitudes:

$$\begin{aligned} \tau_{rz}(r,\vartheta,0) &= \tau_c \cos\vartheta H(r-a) \quad , \\ \tau_{\vartheta z}(r,\vartheta,0) &= -\tau_s \sin\vartheta H(r-a) \quad . \end{aligned} \qquad (18)$$

Unlike the presuppositions before, the now considered stress vector with the components (18) in general has a non-constant amount as well as a non-constant direction. The resultant amount τ_r is given by

$$\tau_r = \sqrt{\tau_c^2\cos^2\vartheta + \tau_s^2\sin^2\vartheta} \quad . \qquad (19)$$

The angular deflection φ between shear stress vector and negative x-axis-direction (Fig. 5) is given by

$$\varphi = \vartheta - \arctan\frac{\tau_s \sin\vartheta}{\tau_c\cos\vartheta} \quad . \qquad (20)$$

By (10), (11) the following stress intensity factors K^F_{II}, K^F_{III} result from (17):

$$\begin{aligned} K^F_{II}(\vartheta) &= -\frac{2}{3\sqrt{\pi}(2-\nu)}\sqrt{b}\cos\vartheta \cdot \\ &\sqrt{1-\frac{a^2}{b^2}}\left\{\tau_c\left(5-\nu+\frac{a^2}{b^2}-2\nu\frac{a^2}{b^2}\right)+\tau_s\left(1+\nu-\frac{a^2}{b^2}-\nu\frac{a^2}{b^2}\right)\right\} \quad , \\ K^F_{III}(\vartheta) &= \frac{2}{3\sqrt{\pi}(2-\nu)}\sqrt{b}\sin\vartheta \cdot \\ &\sqrt{1-\frac{a^2}{b^2}}\left\{\tau_c\left(1-2\nu-\frac{a^2}{b^2}+2\nu\frac{a^2}{b^2}\right)+\tau_s\left(5-4\nu+\frac{a^2}{b^2}+\nu\frac{a^2}{b^2}\right)\right\} \quad . \end{aligned} \tag{21}$$

The Dugdale crack conditions (7) with K^G_{II}, K^G_{III} according to (6) and K^F_{II}, K^F_{III} according to (21) can be satisfied exactly and the validity of (15) respectively (16) can be preserved if the following choice of the quantities τ_c and τ_s is made:

$$\begin{aligned} \tau_c &= \tau_0\frac{8-8\nu+2\nu^2+(4+2\nu-2\nu^2)a^2/b^2}{8-8\nu+2\nu^2+(4-4\nu+\nu^2)a^2/b^2} \quad , \\ \tau_s &= \tau_0\frac{8-8\nu+2\nu^2+(4-10\nu+4\nu^2)a^2/b^2}{8-8\nu+2\nu^2+(4-4\nu+\nu^2)a^2/b^2} \quad . \end{aligned} \tag{22}$$

According to these two relations the shear stress quantities τ_c and τ_s depend on the ratio a/b. For $a/b = 0$ they are equal to τ_0, otherwise they are of different values. In Fig. 6 this is depicted for $\nu = 0.3$. In the least convenient case $a/b = 1$ the reciprocal difference

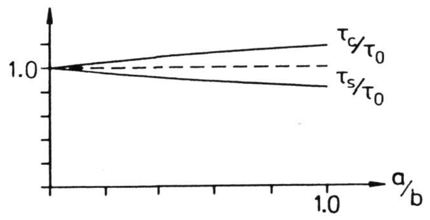

Fig. 6: Shear stress quantities τ_c und τ_s

takes its maximum, but even in this case τ_c and τ_s lie within a tolerance range of less than $\pm 20\%$ with respect to τ_0. In consequence of (19) this finds a corresponding counterpart in the resulting $\tau_r(\vartheta)$-behavior. In Fig. 7 $\tau_r(\vartheta)$ as well as $\varphi(\vartheta)$ are represented for three different ratios a/b. The deviations of the angle φ from zero are not immoderate. Even in the least convenient case $a/b = 1$ the angle φ does not exceed a range of $\pm 11°$.

In all, the yield ring modelling given by the relations (17) – (22) does not give rise to considerable deviations from a yield shear stress τ_0 that with respect to amount and direction is constant. This sustains the adequacy of the presented simpler Dugdale crack modelling, given by an actually constant prescribed yield stress τ_0, and supports its use as a good approximation.

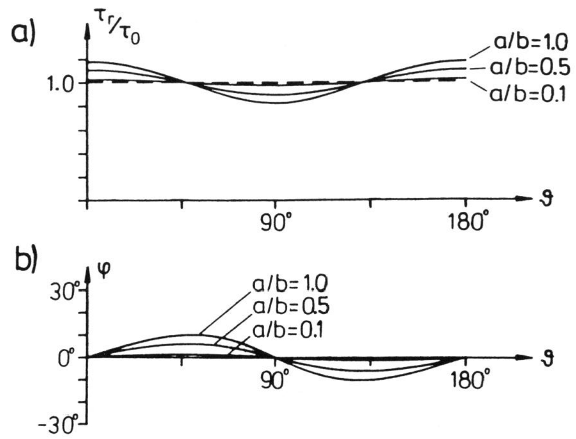

Fig. 7: a) τ_r-behavior
b) φ-behavior

PENNY-SHAPED DUGDALE CRACK UNDER TRIAXIAL LOADING

A more general, triaxial loading is given for the penny-shaped Dugdale crack if the homogeneous remote stresses σ_z ($\sigma_z > 0$), σ_x, σ_y, τ_{xz}, τ_{yz} and τ_{xy} are applied simultaneously.

The stress component σ_z gives rise to a mode I loading ($\sigma = \sigma_z$). The components τ_{xz} and τ_{yz} produce a shearmode loading with the effective resultant shear stress

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} \tag{23}$$

in the direction

$$\vartheta = \arctan\frac{\tau_{yz}}{\tau_{xz}} \quad . \tag{24}$$

In order to meet the Dugdale crack requirements (7) as well as this of vanishing K_I-stress intensity factor both a normal stress σ_0 and a shear stress τ_0 (in the direction given by (24)) are prescribed on the hypothetical Dugdale yield ring. Doing so corresponds with a superposition of the already considered mode I and shearmode loading cases, that with respect to the yield ring size is essentially nonlinear.

Including the special cases that either $\sigma_0 = 0$ or $\tau_0 = 0$, the validity of (1) and/or (15) can be described by the following single complex relation:

$$\frac{a}{b} = \sqrt{1-\left(\frac{\sigma_z + i\sqrt{\tau_{xz}^2+\tau_{yz}^2}}{\sigma_0 + i\tau_0}\right)^2} \quad . \tag{25}$$

This is equivalent to the following two real equations:

$$\frac{a}{b} = \sqrt{1 - \frac{\sigma_z^2 + \tau_{xz}^2 + \tau_{yz}^2}{\sigma_0^2 + \tau_0^2}} \quad , \tag{26}$$

$$\sigma_z \tau_0 = \sigma_0 \sqrt{\tau_{xz}^2 + \tau_{yz}^2} \quad . \tag{27}$$

Compared with the pure mode I respectively shearmode problems, equation (27) is an additional mixed mode condition. For given σ_z, τ_{xz}, τ_{yz} it prescribes the reciprocal ratio of σ_0 and τ_0. This allows to interrelate the two stresses σ_0, τ_0 to a single yield stress quantity σ_{yield}. It is proposed to do so by use of the von Mises yield criterion $\frac{3}{2}\sigma'_{ij}\sigma'_{ij} = \sigma_{yield}^2$. Herein σ'_{ij} denotes the deviatoric stress components. On the Dugdale yield ring this gives

$$\sigma_0^2 + \sigma_x^2 + \sigma_y^2 - \sigma_0(\sigma_x + \sigma_y) - \sigma_x\sigma_y + 3\tau_0^2 + 3\tau_{xy}^2 = \sigma_{yield}^2 \quad . \tag{28}$$

Hereby the influence of the "crack-parallel" remote stresses σ_x, σ_y and τ_{xy} "within" the yield ring is taken into account by correspondingly equal quantities σ_x, σ_y, τ_{xy}. The relations (27) and (28) together allow to determine the stresses σ_0 and τ_0 explicitly from σ_{yield} and the given load σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}:

$$\sigma_0 = \sigma_z \left\{ -\frac{\sigma_z(\sigma_x + \sigma_y)}{2(\sigma_z^2 + 3\tau_{xz}^2 + 3\tau_{yz}^2)} + \frac{\sqrt{\sigma_z^2(\sigma_x + \sigma_y)^2 + 4(\sigma_z^2 + 3\tau_{xz}^2 + 3\tau_{yz}^2)(\sigma_{yield}^2 - \sigma_x^2 - \sigma_y^2 + \sigma_x\sigma_y - 3\tau_{xy}^2)}}{2(\sigma_z^2 + 3\tau_{xz}^2 + 3\tau_{yz}^2)} \right\} ,$$

$$\tau_0 = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} \left\{ -\frac{\sigma_z(\sigma_x + \sigma_y)}{2(\sigma_z^2 + 3\tau_{xz}^2 + 3\tau_{yz}^2)} + \frac{\sqrt{\sigma_z^2(\sigma_x + \sigma_y)^2 + 4(\sigma_z^2 + 3\tau_{xz}^2 + 3\tau_{yz}^2)(\sigma_{yield}^2 - \sigma_x^2 - \sigma_y^2 + \sigma_x\sigma_y - 3\tau_{xy}^2)}}{2(\sigma_z^2 + 3\tau_{xz}^2 + 3\tau_{yz}^2)} \right\} . \tag{29}$$

The presented modelling of a shearmode and a triaxially loaded penny-shaped Dugdale crack is of a pleasing closed-form analytical character. The given formulas are easy to survey, which for applicability is welcome.

REFERENCES

Becker, W. and D. Gross (1987). About the mode II Dugdale crack solution. *International Journal of Fracture*, 34, 65-70.

Becker, W. and D. Gross (1988). About the Dugdale crack under mixed mode loading. *International Journal of Fracture*, 37, 163-170.

Bilby, B. A., A. H. Cottrell, F. R. S. and K. H. Swinden (1963). The spread of plastic yield from a notch. *Proc. Roy. Soc.*, A272, 304-314.

Chell, G. G. (1976). Bilby, Cottrell and Swinden model solutions for centre and edge cracked plates subject to arbitrary mode I loading. *International Journal of Fracture*, 12, 135-147.

Dugdale, D. S. (1960). Yielding of steel sheets containing slits. *J. Mech. Phys. Solids*, 8, 100-104.

Gross, D. (1984). Dugdale-Modell-Näherung für die Scheibe mit Randriß unter Biegung. *Zeitschrift für Flugwissenschaften und Weltraumforschung*, 8, 387-391.

Hahn, G. T. and A. R. Rosenfield (1965). Local yielding and extension of a crack under plane stress. *Acta Metallurgica*, 13, 293-306.

Herrmann, K. (1987). On the applicability of the Dugdale crack model to fibre-reinforced composites. *Zeitschrift für Angewandte Mathematik und Mechanik*, 67, T196-T197.

Janson, J. (1977). Dugdale-crack in a material with continuous damage formation. *Engineering Fracture Mechanics*, 9, 891-899.

Kassir, M. K. and G. C. Sih (1975). *Mechanics of fracture, Vol. 2: Three-dimensional crack problems.* Noordhoff International Publishing, Leyden.

Mattheck, C. and F. Görner (1984). Leak prediction by use of a generalized Dugdale model for semi-elliptical surface flaws in plates under tension loading. In: L. Faria: *Life Assessment of Dynamically Loaded Materials and Structures, Vol. 2, The Proceedings of the 5th European Conference on Fracture, Lisbon, 17th-21st Sept. 1984*, pp. 631-642.

Mattheck, C. and D. Gross (1985). A Dugdale model including strain hardening for through wall and surface cracks. In: *Transactions of the 8th International Conference on Structural Mechanics in Reactor Technology, Vol. G, Brussels, 19th-23rd Aug. 1985*, pp. 187-194.

Newman, J. C. (1968). Fracture of cracked plates under plane stress. *Engineering Fracture Mechanics*, 1, 137-154.

Seeger, T. (1973). Ein Beitrag zur Berechnung von statisch und zyklisch belasteten Rißscheiben nach dem Dugdale-Barenblatt-Modell. PhD thesis, Technische Hochschule Darmstadt.

Tada, H., P. C. Paris and G. R. Irwin (1985). *The stress analysis of cracks handbook*, 2nd ed.. Paris Productions Inc., St. Louis.

Theocaris, P. S. and E. E. Gdoutos (1974). The modified Dugdale-Barenblatt model adapted to various fracture configurations in metals. *International Journal of Fracture*, 10, 549-564.