A Singular Element for a Variable Crack Length

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ABSTRACT

By means of conformal mapping on a fixed region we built up a singular crack tip element available for a variable crack length which gives the proper singularity for displacement fields, and allows us to compute any order of the total energy derivative whith respect to the length of the crack.

INTRODUCTION

The prediction of the behaviour of cracked solids in plane elasticity requires in particular the knowledge of the stress intensity factors linked to a global quantity: The energy release rate G or the first derivative of the total potential at equilibrium E whith respect to the crack length l.

Nevertheless, for a complete prediction of the evolution it can also be necessary to study higher order derivatives of E. The stability depends on the second derivative and further, higher order derivatives are indispensable for postcritical behaviour of interactive tension crack problems (Nemat Nasser-1980 [1], Q.S.Nguyen-1985 [2]).

With the traditionnal Finite Element Method, in the presence of a crack we have to refine the mesh around the crack tip in order to take into account high stress gradients in this area. Many have been trying to solve these difficulties by using several kinds of crack tip elements whith shape functions that give the proper singularity (Tracey-1971[3], Barsoum-1976[4]). Tsamasphyros (1986[5]) uses conformal mapping which opens the crack tip and thus generates an optimum mesh on the regular transformed region.

However, a complete tool for numerical prediction of crack behaviour is still hard to implement. The increase of crack length requires either locally remeshing or breaking elements. This makes still very difficult the numerical evaluation of higher order derivatives of the energy E. Obstacles occur when one studies the velocities problem because $\dot{u}$ and $\dot{\sigma}$ are of higher order of singularity than u and σ and don't belong to the usual functionnal spaces. Stolz and Q.S.Nguyen(1985[6]) circumvent this difficulty by solving the elastic problem in a referential moving with the crack tip. Destuynder(1982[7]) proposes using a fixed reference region which can be the initial state of the body analogous to the Lagrangien method in the case of displacements.

In this paper, we are dealing with the problem of straight cracks in two-dimensionnal bodies. We will also show how to generalize our process in the case of curved cracks in plane elasticity. Our purpose is to create a singular element located at the crack tip, of the shape of the unit circle with a radial crack of length $h \leqslant 1$. The element is transformed into the inside of the upper half unit circle by means of a conformal mapping ω. The stiffness $\mathcal{K}(\omega)$ of the element will be calculated by solving the elastic problem in the reference domain with arbitrary boundary conditions. We undertake this task by a boundary element method based on Muskelishvili potentials(1933[8]).

Besides providing the proper singularity, this method allows us to build continuously with respect to h the total energy function E(h). In particular, one can for instance calculate the accurate length of crack h_0 if it does exist which equilibrates the external loading and corresponds to a stable state ie:
h_0 as $G=2\gamma$ and $E''_{h^2}>0$
This corresponds to a minimization with respect to h of the functional total energy plus dissipated energy $E+2\gamma h$ on the set of lengths h which correspond to an advance of the crack(Fedelich and Berest 1987[9]).

FORMULATION OF THE BOUNDARY INTEGRAL EQUATION

S being an elastic cracked two-dimensionnal solid and D the singular element at the crack tip ; let's denote (u_x, u_y) the displacement field components and (T_x, T_y) the traction components on an element ds of outward normal n.

$$\sigma_{xx}\cos(n,x) + \sigma_{xy}\cos(n,y) = T_x$$
$$\sigma_{xy}\cos(n,x) + \sigma_{yy}\cos(n,y) = T_y$$

we note $T = T_x + iT_y$, $U = u_x + iu_y$, $z = x+iy$ we specify with the exponent+ the values inside D and with - the values outside D and:

$$F(z) = i\int_{s_0}^{s(z)} T\,ds \qquad \text{(see fig.1)}$$

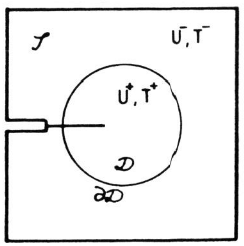

fig.1

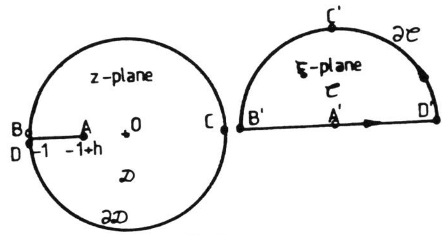

fig. 2

Let's handle the elastic problem inside the element D with given external forces T^+ or displacements U^+. The boundary conditions on ∂D can be expressed in term of two holomorphic functions in D, $\Phi(z)$ and $\Psi(z)$:

$$2\gamma U^+(z) = \chi\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)} \qquad \text{where } \chi = \frac{\lambda+3\mu}{\lambda+\mu} \tag{1}$$

$$F(z) = \Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi(z)} \tag{2}$$

The conformal mapping:

$$z = \omega(\xi) = \frac{-\xi^2+h-1}{-(h-1)\xi^2+1} \; ; \qquad D = \omega(C) \tag{3}$$

transforms the interior of the upper half unit circle into the inside of the unit circle with a radial crack of length h (see fig 2). In particular the point 0 in the ξ-plane becomes the crack tip in the z-plane thus $\omega'(0) = 0$

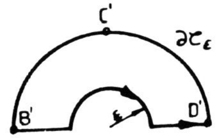

fig. 3

Let's rewrite the boundary conditions in term of $\varphi(\xi)=\Phi(\omega(\xi))$ $f(\xi)=F(\omega(\xi))$, $u^+(\xi)=U^+(\omega(\xi))$, $t^+(\xi)=T^+(\omega(\xi))$ and $\psi(\xi)=\Psi(\omega(\xi))$ on the half circle ∂C_ε once removed the half circle of center 0 and radius ε (see fig 3)

$$2\mu u^+(\sigma) = \chi\varphi(\sigma) - \omega(\sigma)\frac{\overline{\varphi'(\sigma)}}{\omega'(\sigma)} - \overline{\psi(\sigma)} \quad \text{where } \sigma \in \partial C_\varepsilon \tag{4}$$

$$f(\sigma) = \varphi(\sigma) + \omega(\sigma)\frac{\overline{\varphi'(\sigma)}}{\omega'(\sigma)} + \overline{\psi(\sigma)} \tag{5}$$

For all ξ_L outside ∂C_ϵ the integral $\frac{1}{2i\pi}\int_{\partial C_\epsilon}\frac{\overline{\psi(\sigma)}}{\sigma-\xi_L}d\sigma$ vanishes because $\overline{\psi(\sigma)}$ is the boundary value of some function $\psi(\xi)$ holomorphic inside ∂C_ϵ. We deduce from relation (5) :

$$\frac{1}{2i\pi}\int_{\partial C_\epsilon}\frac{\overline{f(\sigma)}}{\sigma-\xi_L}d\sigma = \frac{1}{2i\pi}\int_{\partial C_\epsilon}\frac{\overline{\varphi(\sigma)}}{\sigma-\xi_L}d\sigma + \frac{1}{2i\pi}\int_{\partial C_\epsilon}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)(\sigma-\xi_L)}\varphi'(\sigma)d\sigma \tag{6}$$

In the same way $\frac{\varphi'(\sigma)}{\omega'(\sigma)}$ is holomorphic inside ∂C_ϵ hence:

$$\frac{1}{2i\pi}\int_{\partial C_\epsilon}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)(\sigma-\xi_L)}\varphi'(\sigma)d\sigma = \frac{1}{2i\pi}\int_{\partial C_\epsilon}\frac{\overline{\omega(\sigma)} - \overline{\omega(0)}}{\omega'(\sigma)(\sigma-\xi_L)}d\sigma \tag{7}$$

Being ensured that the kernel $\mathcal{G}(\sigma) = \dfrac{\overline{\omega(\sigma)} - \overline{\omega(0)}}{\omega'(\sigma)}$ is bounded as σ tends to 0 we can let ε go to the value 0 and ∂C_ϵ to ∂C. Therefore :

$$A(\xi_L) = \frac{1}{2i\pi}\int_{\partial C}\frac{\overline{\varphi(\sigma)}}{(\sigma-\xi_L)}d\sigma + \frac{1}{2i\pi}\int_{\partial C}\frac{\mathcal{G}(\sigma)}{(\sigma-\xi_L)}\varphi'(\sigma)d\sigma \tag{8}$$

where: $A(\xi_L) = \frac{1}{2i\pi}\int_{\partial C}\frac{\overline{f(\sigma)}}{(\sigma-\xi_L)}d\sigma$

Let ξ_L tend to some point σ_L of ∂C (remaining outside ∂C). We can evaluate successively the limit of each integral in (8).

$$\lim_{\xi_L\to\sigma_L} A(\xi_L) = A(\sigma_L) = \frac{1}{2i\pi}\int_{\partial C}\frac{\overline{f(\sigma)} - \overline{f(\sigma_L)}}{(\sigma-\sigma_L)}d\sigma$$

Using the fact that $\varphi'(\xi)$ is holomorphic inside ∂C we obtain:

$$\lim_{\xi_L\to\sigma_L}\frac{1}{2i\pi}\int_{\partial C}\frac{\mathcal{G}(\sigma)}{(\sigma-\xi_L)}\varphi'(\sigma)d\sigma = \frac{1}{2i\pi}\int_{\partial C}\mathcal{H}(\sigma,\sigma_L)\varphi'(\sigma)d\sigma \tag{9}$$

where $\mathcal{H}(\sigma,\sigma_L) = \mathcal{G}(\sigma)-\mathcal{G}(\sigma_L)$

At last, as a consequence of the Plemelj formulae:

$$\lim_{\xi_L\to\sigma_L}\frac{1}{2i\pi}\int_{\partial C}\frac{\overline{\varphi(\sigma)}}{(\sigma-\xi_L)}d\sigma = -c\overline{\varphi(\sigma)} + \frac{1}{2i\pi}\int^{*}_{\partial C}\frac{\overline{\varphi(\sigma)}}{(\sigma-\xi_L)}d\sigma \tag{10}$$

Taking the principal value of the integral on the right-hand side and $c=1/2$ when $\sigma_L \neq \pm 1$ and $c=1/4$ when $\sigma_L = \pm 1$ (corners). Using the fact that $\varphi(\sigma)$ must be the boundary value of a function holomorphic in C we have:

$$-c\varphi(\sigma_L) - \frac{1}{2i\pi}\int^{*}_{\partial C}\frac{\varphi(\sigma)}{(\sigma-\xi_L)}d\sigma = 0 \tag{11}$$

Taking the conjugate of (11) and adding to (10) one obtains :

$$\lim_{\xi_L\to\sigma_L}\frac{1}{2i\pi}\int_{\partial C}\frac{\overline{\varphi(\sigma)}}{(\sigma-\xi_L)}d\sigma = -2c\,\overline{\varphi(\sigma)} + \frac{1}{2i\pi}\int_{\partial C}\overline{\varphi(\sigma)}\,d\mathrm{Log}\left(\frac{\sigma-\sigma_L}{\overline{\sigma-\sigma_L}}\right)$$

which is a regularization of the kernel of the integral of the right hand side of (6). Finally integrating by parts in (9) we obtain a boundary integral equation on φ with regular kernel :

$$-2c\overline{\varphi(\sigma_L)} + \frac{1}{2i\pi}\int_{\partial C}\overline{\varphi(\sigma)}d\mathrm{Log}\left(\frac{\sigma-\sigma_L}{\overline{\sigma-\sigma_L}}\right) - \frac{1}{2i\pi}(\varphi(1)[\![\mathcal{H}(1,\sigma_L)]\!]+\varphi(-1)[\![\mathcal{H}(-1,\sigma_L)]\!])$$

$$-\frac{1}{2i\pi}\int_{\partial C}\frac{\partial\mathcal{H}(\sigma,\sigma_L)}{\partial\sigma}\varphi(\sigma)d\sigma = A(\sigma_L) \tag{12}$$

Where $[\![\mathcal{H}(x,\sigma_L)]\!]$ denotes the discontinuity of $\mathcal{H}(\sigma,\sigma_L)$ through the value x. (these discontinuities vanish except for $\sigma_L = \pm 1$)

NUMERICAL RESOLUTION OF EQUATION (12)

Practically we solve equation (12) by discretizing the contour ∂C and using appropriate shape functions for φ, t^+, f, and u^+ on each element.

On the real axis all the functions are interpolated by polynomials of σ. But on the upper half circle t^+, u^+, and f are interpolated by polynomials of σ and $\bar{\sigma}$:

$$f(\sigma) = \sum_{m=1}^{n} f_m P_m(\sigma,\bar{\sigma}) \quad ; \quad p_m(\sigma,\bar{\sigma}) = a^m_{-k}\overline{\sigma^k}+\ldots+a^m_0+\ldots+a^m_k\sigma^k \quad ; \quad 2k+1=n$$

Using the fact that the edges of the crack are traction free it can be shown that $\varphi(\xi)$ can be analytically continued in the entire circle by $\tilde{\varphi}(\xi) = -\dfrac{\omega(\xi)}{\overline{\omega'(\bar{\xi})}}\bar{\varphi}'(\xi) - \bar{\psi}(\xi)$ (with the classical notation $\bar{h}(\xi) = \overline{h(\bar{\xi})}$). Thus it can be developped in a series of positive powers of ξ inside C and up to ∂C. We can therefore retain (which is confirmed by numerical results) only positive powers of σ for shape functions of φ:

$$\varphi(\sigma) = \sum_{m=1}^{p+1} \varphi_m Q_m(\sigma) \quad ; \quad Q_m(\sigma) = b_0^m + \ldots + b_p^m \sigma^p$$

By choosing appropriatly the a_i^m and the b_i^m the coefficients f_m and φ_m take the values of f and φ at the nodes σ_L which allows us to assemble easily the stiffness matrix. One is driven to compute the solution of a linear system:

$$[K(\omega)]\{\varphi\} = [C]\{f\} \tag{13}$$

([] denoting matrix and {} column vectors).

The system is unsolvable without any complementary condition on φ due to the fact that every function $i\alpha\omega(\xi)+\gamma+i\gamma'$ (α,γ,γ' reals) is solution of the homogeneous system (12). We can set for instance $\varphi(0)=0$ and $\mathrm{Im}\varphi(i)=0$. The relation between f and the tractions t^+:

$$f(\sigma) = i\int_{s_0}^{s(\sigma)} t^+(\zeta)\,|\omega'(\zeta)|\,ds \tag{14}$$

can be put under matrix form:

$$\{f\} = [B(\omega)]\{t^+\} \tag{15}$$

by adding (4) and (5) we obtain a relation between φ, u^+, and f,:

$$2\mu u^+(\sigma) + f(\sigma) = (\chi+1)\varphi(\sigma) \tag{16}$$

Finally, with equations (13), (15), and (16) we find a linear relationship between the tractions and the displacements at the nodes σ_L on the boudary ∂C:

$$2\mu[K(\omega)]\{u^+\} = ((\chi+1)[C]-[K(\omega)])[B(\omega)]\{t^+\} \tag{17}$$

This system in order to be attached to a Finite Element program for the whole structure has to be completed by the continuity conditions through ∂D :

$$U^-(z) = u^+(\omega^{-1}(z)) \text{ and } T^-(z)+t^+(\omega^{-1}(z))=0 \tag{18}$$

CALCULATION OF DERIVATIVES OF φ

Under a constant external loading we impose a variation $\delta\omega$ of the geometry ω. As long as we work on a fixed domain, we can obtain by a simple derivation the integral equation on $\delta\varphi$. Rewriting equation (12) under a compact form :

$$L\varphi = A \tag{19}$$

For a given variation $\delta\omega$, one has to solve a boundary integral equation with the same kernel as equation (12) and a right hand side dependent on φ and $\delta\omega$:

$$L\delta\varphi = \delta A - \delta L\varphi \tag{20}$$

where:

$$\delta L\varphi = -\frac{1}{2i\pi}(\varphi(1)[\![\delta\mathcal{H}(1,\sigma_L)]\!] + \varphi(-1)[\![\delta\mathcal{H}(-1,\sigma_L)]\!]) - \frac{1}{2i\pi}\int_{\partial C}\frac{\partial\delta\mathcal{H}(\sigma,\sigma_L)}{\partial D}d\sigma$$

$$\delta A = \frac{1}{2i\pi}\int_{\partial C}\frac{\overline{\delta f(\sigma)-\delta f(\sigma_L)}}{(\sigma-\sigma_L)}d\sigma$$

$$\delta f(\sigma) = i\int_{s_0}^{s(\sigma)} t^+(\zeta)\frac{\mathrm{Re}\left(\delta\omega(\zeta)\,\overline{\omega'(\zeta)}\right)}{|\omega'(\zeta)|} + |\omega'(\zeta)|\frac{\partial t^+(\zeta)}{\partial\sigma}\frac{\delta\omega(\zeta)}{\omega'(\zeta)}\,ds$$

We obtain the relation between δu^+, δf, and $\delta\varphi$ by derivating the relation (16) at a fixed point σ:

$$2\mu\,\delta u^+(\sigma) + \delta f(\sigma) = (\chi+1)\delta\varphi(\sigma) \tag{21}$$

By derivating relations (18) at a fixed point z we find the boundary conditions necessary to attach this element to the whole structure:

$$\delta U^-(z) = \delta u^+(\omega^{-1}(z)) - \frac{\partial u^+(\omega^{-1}(z))}{\partial\sigma}\frac{\delta\omega(\omega^{-1}(z))}{\omega'(\omega^{-1}(z))} \quad \text{and} \tag{22}$$

$$\delta T^-(z) + \delta t^+(\omega^{-1}(z)) - \frac{\partial t^+(\omega^{-1}(z))}{\partial\sigma}\frac{\delta\omega(\omega^{-1}(z))}{\omega'(\omega^{-1}(z))} = 0 \;;$$

This problem corresponds to a classical elastic problem on the initial geometry with given displacement and traction discontinuities $\left(-\frac{\partial u^+}{\partial\sigma}\frac{\delta\omega}{\omega'}\right.$ and $\left.-\frac{\partial t^+}{\partial\sigma}\frac{\delta\omega}{\omega'}\right)$. Once having solved this problem we can calculate the derivative $\frac{\partial E}{\partial h}$ of the total energy. One can easily iterate this process in order to calculate higher orders derivatives of φ: $\delta\varphi,\ldots,\delta^n\varphi$ as functions of $\varphi,\ldots,\delta^{n-1}\varphi$ and $\omega,\ldots,\delta^n\omega$. We obtain by this way the successive derivatives of the total energy $\frac{\partial^n E}{\partial h^n}$.

A POSSIBLE GENERALIZATION: NON STRAIGHT CRACKS

By composing the conformal mapping ω with another conformal mapping of the unit circle ω_2 such as $\omega_2(\bar{\xi}) \neq \overline{\omega_2(\xi)}$ we find another conformal mapping $\omega_2 \circ \omega$ of the inside of the upper half unit circle C into the inside of a region $\mathfrak{D}$ with a curved crack which is image of the segment[-1,-1+h] by ω_2. this process requires the following adaptations: For a given shape of crack in a body, one has to built the function ω_2 (for example by minimizing the area between the real crack and the curve image of [-1,-1+h]; therefore, the shape of the region $\mathfrak{D}$ is free). Moreover, for an initial crack ω_0, we have to build a set of authorized evolutions $\delta\omega$, which correspond to an increase of length but preserve the initial shape of crack ω_0.

CONCLUSION

Once having solved the elastic problem in the fixed reference geometry with arbitrary given forces, we have created a special crack tip element which can be inserted in a Finite Element program. We have shown that this element prevents us from having to remesh at every crack progression. It can therefore be used for calculating any order of derivative of the total energy E.

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