A Method for Mechanical State Characterization of Inelastic Composite Laminates with Damage

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ABSTRACT

A method using a work potential is described for the characterization of mechanical behavior of inelastic composites with damage, but without significant time-dependent behavior. It is based on the theoretically and experimentally motivated assumption of path-independence of mechanical work over limited ranges of stress or strain states. This method and, for comparison, an approach employing plasticity theory are illustrated with the special case of a unidirectional-fiber laminate or ply. Use of the work-potential method for a multidirectional-fiber laminate is discussed in the concluding remarks.

KEYWORDS

Composites, Laminates, Damage, Inelasticity, Plasticity

1. INTRODUCTION

Considerable progress has been made in recent years on the development of high strength-to-weight, tough structural composites. This behavior is achieved in-part by laminating individual plies of unidirectional, continuous fiber-reinforced plastic or metal. The laminates are resistant to crack growth through the thickness if two or more fiber orientations are used. Delamination and cracking within plies is reduced by using ductile matrices. For organic polymer matrices, the ductility is obtained by adding toughening agents, such as rubber particles, to normally brittle crosslinked resins, or by using resins with little or no crosslinking (Johnston, 1987). These improvements in material performance place increased demands on the structural designer and those concerned with the micromechanics of composites if inelasticity is due to both plastic deformation and damage or if it has to be considered under a wider range of conditions than for the brittle matrix systems.

Traditionally, matrix ductility has been treated using incremental plasticity theory (Christensen, 1979) while micro- and macrocracking of composites have been analyzed using linear elasticity theory (Wang and

Haritos, 1987). In this paper we discuss an approach to characterizing inelastic composite material behavior which is based on total rather than incremental strains. Also, the approach uses the same mathematical formalism for inelasticity due to plastic deformation as due to cracking on various scales and other damage mechanisms; the term inelasticity, as used here, refers to any stable behavior in which stress or load is not always a single-valued function of strain or displacement. It is believed that this unified approach simplifies the problem of understanding and predicting mechanical behavior of composites with damage. Fatigue and time-dependent behavior and thermal effects are not treated here, although approaches have been proposed in the papers which motivated the present study (Schapery, 1987a, 1988).

Schapery (1987a) has shown theoretically that the stresses and mechanical work of deformation are often independent of many details of the deformation history when the inelasticity is due to micro- and macrocracking. However, cracking is not the only mechanism that produces this behavior. Indeed, it has been observed for a rubber-toughened, graphite/epoxy composite in which there are probably significant effects of shear banding in the matrix (possibly initiated or enhanced by cavitation of rubber particles) (Yee, 1987). This limited path-independence was used by Schapery (1988) to develop a constitutive theory that treats different inelastic mechanisms within the same mathematical framework. Also, as shown by Schapery (1987a), fracture analysis is simplified when this theory is valid because of the applicability of certain equations for relating changes in local and global energies.

Figure 1 illustrates one type of path-independence we have found for the rubber-toughened composite. Rectangular composite bars with an angle-ply layup (alternating fiber angle, $\theta = \pm 35^o$, with respect to the axial direction) were subjected to various axial and torsional deformations

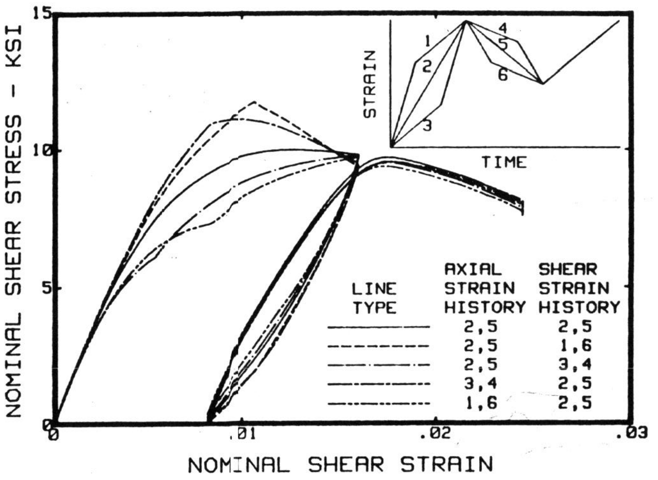

Fig. 1. Shear stress-strain curves for proportional and nonproportional straining of an angle-ply laminate; Hexcel T2C 145/F155 graphite/epoxy $[\pm 35^\circ]_{6s}$; 0.15" thick X 0.5" wide X 8.75" long. From Lamborn and Schapery (1988).

through controlled movement of the end-grips. The different deformation paths are identified in Fig. 1 by number; for example, the bottom line type is used for axial history 1 during the first loading period and axial history 6 during unloading, while the corresponding torsional histories are 2 and 5. The "nominal" shear stress and shear and axial strains are quantities which are proportional to the torque, twist, and axial displacement, respectively; the proportionality coefficients depend only on the specimen dimensions, and are introduced to minimize the effect of specimen-to-specimen size differences.

At the end of the first loading period, the five different strain paths result in practically the same stress (Fig. 1) and total work. The same behavior holds for the unloading and reloading. In contrast, unreinforced aluminum bars exhibit significant path-dependence (Lamborn and Schapery, 1988); we do not know if fiber-reinforced aluminum would exhibit less path-dependence.

Unloading and reloading behavior of the graphite/epoxy material under pure axial or torsional straining is similar to that shown in Fig. 1; there is significant hysteresis and the average slope of the loop decreases with increasing strain at the unloading point. The stress during loading does not usually exhibit a maximum point prior to fracture, in contrast to that in Fig. 1. We are now investigating the damage state as a function of deformation history using similar specimens; significant edge delaminations have been found at the highest stresses for deformation histories like those in Fig. 1.

The primary effects of deformation history on the composite appear to be associated with the sign of (nominal) strain rate and the strain magnitude when the sign last changed. Although a more precise definition of limited path-independence was given by Schapery (1988) here we shall just refer to differences between loading, unloading, and reloading curves, and suppose that for each case there is no effect of path (which is approximately true for the data in Fig. 1).

The local stresses and strains (as opposed to the "nominal" quantities in Fig. 1) are distributed very nonuniformly throughout the specimens used in these axial-torsional tests, and thus the results cannot be used directly in a basic material characterization of the composite. However, it is unlikely that the specimens' overall behavior would exhibit limited path-independence if the ply-level constitutive equations did not reflect this type of behavior.

The discussion in Sections 2-4 is concerned primarily with the characterization of the behavior of a undirectional-fiber laminate consisting of one or more plies under the assumption of this limited path-independence. Special versions of the theory (Schapery, 1988) are used here to illustrate it for composites. Specifically, Section 2 considers nonlinear loading and unloading behavior, and expresses the inelasticity in terms of one parameter S which represents the effect of microstructural changes on the overall stress-strain behavior; such S-parameters provide the inelasticity and, in the context of some thermodynamic formulations, are called internal state variables. Section 3 contrasts the theory with a plasticity model based on the normality rule, and uses the characterization in Section 2 as an example. In Section 4 another illustration is given by using a linear approximation for unloading behavior. Concluding remarks in Section 5 discuss in-part the use of unidirectional ply characterization in laminates with ply-level and larger scales of damage.

2. A CONSTITUTIVE EQUATION WITH NONLINEAR UNLOADING BEHAVIOR

Figure 2 shows a unidirectional laminate or ply and the coordinate notation, in which the x_1 axis is parallel to the fibers; the x_3 axis is normal to the ply plane. The stresses σ_i and strains ε_i ($i = 1,2,...6$) are mechanical variables referred to the principal material coordinates x_i. In most of the discussion it will be convenient to use this single index notation. As is customary, $i = 4,5,6$ are used for the shearing variables; the relationship between single and double indexed variables for plane stress is

$$\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{12} = \sigma_6 \qquad (1)$$
$$\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad 2\varepsilon_{12} = \varepsilon_6$$

A constitutive equation will be proposed which accounts for nonlinear loading and unloading behavior and which is consistent with the path-independence of work discussed in the Introduction as well as the nonlinear behavior reported by Lou and Schapery (1971) and Sun and Chen (1987); the reader is referred to these two papers for the experimental data, as space does not permit its reproduction here. Specifically, a strain energy density $w = w(\varepsilon_i, S)$ is assumed to exist, where the microstructure state is defined by S; only one structure parameter S will be used here, although more could be introduced, if necessary. By definition of w,

$$\sigma_i = \partial w/\partial \varepsilon_i \qquad (2)$$

In both aforementioned references strains are expressed in terms of stresses, and thus it is helpful to eliminate w in favor of a so-called dual strain energy density $w_\sigma = w_\sigma(\sigma_i, S)$,

$$w_\sigma \equiv w - \sigma_i \varepsilon_i \qquad (3)$$

(Throughout this paper the summation convention is employed, in which a repeated index implies summation over its range.) By using (2) and introducing differential changes in (3), it follows in the usual way that

$$\varepsilon_i = -\partial w_\sigma/\partial \sigma_i \qquad (4)$$

A form of w_σ discussed by Schapery (1988, Eq. (A24)) is proposed now for characterizing ply behavior,

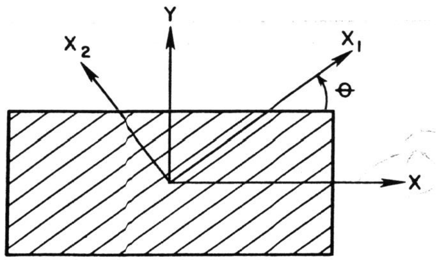

Fig. 2. Unidirectional composite and coordinates.

$$w_\sigma = w_{\sigma o} + P(\sigma_o, S) \qquad (5)$$

where $w_{\sigma o} = w_{\sigma o}(\sigma_i)$, $\sigma_o = \sigma_o(\sigma_i)$, and P are presently arbitrary functions. The mechanical work during processes in which S changes can be shown to be independent of path if and only if $S = S(\sigma_o)$. ; proof of this statement may be made by the same method as used in a study of w (Schapery, 1988, Appendix A). The function $S(\sigma_o)$ can be absorbed in the functional dependence of P on S, and thus we may use $S = \sigma_o$ whenever S changes without any actual limitation in the model. Whether S varies or is constant, the strains are obtained from (4) and (5),

$$\varepsilon_i = \varepsilon_i^e - \frac{\partial P}{\partial \sigma_o} \frac{\partial \sigma_o}{\partial \sigma_i} \qquad (6)$$

where, by definition,

$$\varepsilon_i^e \equiv -\partial w_{\sigma o}/\partial \sigma_i \qquad (7)$$

The ε_i^e are defined through derivatives of a fully path-independent potential, $w_{\sigma o}$, and thus it is appropriate to call them "elastic strains". All stress-history effects are in the second term in (6), which gives the "inelastic strains".

In order to obtain a constitutive equation that agrees with Sun and Chen's experimental data we select for σ_o the quadratic form,

$$\sigma_o = (a_{ij}\, \sigma_i\, \sigma_j)^{\frac{1}{2}} \qquad (8)$$

where the a_{ij} are constants; as the antisymmetric components of a_{ij} have no effect on σ_o, we may suppose $a_{ij} = a_{ji}$. Now,

$$\partial \sigma_o/\partial \sigma_i = a_{ij}\, \sigma_j / \sigma_o \qquad (9)$$

and thus from (6),

$$\varepsilon_i = \varepsilon_i^e - \frac{\partial P}{\partial \sigma_o} a_{ij}\, \sigma_j / \sigma_o \qquad (10)$$

During structure-change processes $S = \sigma_o$, as noted previously, and therefore the coefficient $\partial P/\partial \sigma_o$ depends on only σ_o. For such processes we may thus write

$$\varepsilon_i = \varepsilon_i^e + \varepsilon_o\, a_{ij}\, \sigma_j / \sigma_o \qquad (11)$$

where

$$\varepsilon_o = \varepsilon_o(\sigma_o) \equiv -(\partial P/\partial \sigma_o) \text{ evaluated at } S = \sigma_o \qquad (12)$$

In the terminology of plasticity theory, (11) is for "loading" processes.

Without fiber fracture, the strain in the fiber direction is essentially independent of stress history in most structural composites; thus, as assumed by Sun and Chen, $a_{1j} = a_{j1} = 0$. We suppose further that the composite is orthotropic, regardless of stress-history, where the axes x_i are the principal material axes; this condition implies the only a_{ij} which do not vanish are a_{22}, a_{23}, a_{33}, a_{44}, a_{55}, a_{66}, as well as $a_{32} (= a_{23})$. There are really only five independent constants because σ_o may be normalized with respect to a constant without limiting the generality of (5); this normalization will be done by simply letting $a_{22} = 1$. If all stresses vanish except for σ_2, (8) reduces to $\sigma_o = |\sigma_2|$; thus σ_o becomes the applied stress for the case of uniaxial tensile loading normal to the fibers.

For plane stress, $\sigma_3 = \sigma_4 = \sigma_5 = 0$, so that (8) reduces to

$$\sigma_o = (\sigma_2^2 + a_{66}\,\sigma_6^2)^{1/2} \tag{13}$$

From (11),

$$\varepsilon_1 = \varepsilon_1^e \tag{14}$$

$$\varepsilon_2 = \varepsilon_2^e + \varepsilon_o \sigma_2/\sigma_o \tag{15}$$

$$\varepsilon_6 = \varepsilon_6^e + a_{66}\,\varepsilon_o \sigma_6/\sigma_o \tag{16}$$

For uniaxial tension normal to the fibers, $\sigma_o = \sigma_2$, as noted previously. Equation (15) then shows that ε_o reduces to the inelastic component of ε_2. For general stress states ε_o is at most a function of σ_o, according to (12).

By introducing some additional specializations, including the assumption that the ε_i^e are linear in the σ_i, we will finally arrive at Sun and Chen's findings for uniaxial loading of unidirectional, rectangular specimens. Namely, for loading in the x direction (cf. Fig. 2),

$$\sigma_1 = \cos^2\theta\ \sigma_x, \quad \sigma_2 = \sin^2\theta\ \sigma_x, \quad \sigma_6 = -\sin\theta\ \cos\theta\sigma_x \tag{17}$$

where σ_x is the applied force/area. The axial strain ε_x may be expressed in terms of the strains in (14)-(16) using the second-order tensor transformation rule,

$$\varepsilon_x = \cos^2\theta\ \varepsilon_1 + \sin^2\theta\ \varepsilon_2 - \sin\theta\ \cos\theta\ \varepsilon_6 \tag{18}$$

Substitution of (14)-(17) into (18) yields

$$\varepsilon_x = \varepsilon_x^e + h^2 \varepsilon_o\ \sigma_x/\sigma_o \tag{19}$$

where ε_x^e is the elastic axial strain, and

$$h \equiv (\sin^4\theta + a_{66}\sin^2\theta\ \cos^2\theta)^{1/2} \tag{20}$$

Observe also from (13) and (17) that

$$\sigma_o = h\sigma_x \tag{21}$$

We can obtain the function $h(\theta)$ used by Sun and Chen by multiplying (20) by $\sqrt{3/2}$. Equation (19) is the same as derived by them from a plasticity model for loading behavior; this model will be discussed in Section 3.

Experimental information on $\varepsilon_x - \sigma_x$ behavior for two fiber angles θ may be used with (19) to evaluate a_{66} and the function $\varepsilon_o = \varepsilon_o(\sigma_o)$. (Alternatively, one may use data from several fiber angles to determine the a_{66} which minimizes the data spread in the $\varepsilon_o(\sigma_o)$ plot.) Results from tests at other fiber angles then serve to check (19). A simple power law

$$\varepsilon_o = A\ \sigma_o^n \tag{22}$$

where A and n are positive constants, was reported by Sun and Chen to fit

the data out to specimen failure ($\varepsilon_x \approx 1\%$); for a boron/aluminum composite $n = 5.8$ and $a_{66} = 4$, whereas for graphite/epoxy $n = 3.7$ and $a_{66} = 2.5$. (The constant a_{66} used by Sun and Chen is one-half of the a_{66} used here.) Values of $a_{66} = 4$ and $n = 2.4$ have been obtained recently by Mignery and Schapery (1988) from studies of unidirectional and angle-ply laminates of the same rubber-toughened graphite/epoxy material used to develop the curves in Fig. 1. Although the latter exponent ($n = 2.4$) is smaller than that reported by Sun and Chen for an untoughened unidirectional graphite/epoxy material ($n = 3.7$), the angle-ply stress-strain curves (Mignery and Schapery, 1988) exhibit a larger degree of nonlinearity because the total axial strain range is approximately 5%, as compared to 1% in the former study.

In the much earlier work of Lou and Schapery (1971) it was found that the parameter σ_o in (13) accounted for the effect of stress state on the functions used to characterize nonlinear viscoelastic behavior of a glass/epoxy composite. The motivation for the use of this parameter came in part from the observation that the octahedral shear stress τ_{oct} can normally be used to correlate multiaxial yielding of plastics (just as for metals). As a simplification, the matrix was viewed as a uniformly stressed layer of material sandwiched between layers of rigid fiber material; i.e., the lines in Fig. 2 at the angle θ were imagined to define layers rather than fibers. Using the principal material axes, Fig. 2, this shear stress is

$$\tau_{oct} = \tfrac{1}{3}[(\bar\sigma_1 - \bar\sigma_2)^2 + (\bar\sigma_2 - \bar\sigma_3)^2 + (\bar\sigma_3 - \bar\sigma_1)^2 + 6(\bar\sigma_4^2 + \bar\sigma_5^2 + \bar\sigma_6^2)]^{1/2} \tag{23}$$

where the $\bar\sigma_i$ in this equation are the stresses in a matrix layer.

For a matrix in plane stress $\bar\sigma_2$ and $\bar\sigma_6$ are the same as the stresses σ_2 and σ_6 acting on a composite consisting of parallel layers of matrix and reinforcement material. A factor ν_e was also introduced, as defined by the relationship $\bar\sigma_1 = \nu_e\sigma_2$. For a linear elastic, isotropic matrix ν_e is the Poisson's ratio, and for an incompressible elastic or rigid-plastic matrix $\nu_e = 0.5$. Use of these idealizations in (23) yields

$$\tau_{oct} = (2/3c)^{1/2}\,(\sigma_2^2 + c\sigma_6^2)^{1/2} \tag{24a}$$

where

$$c \equiv 3/(1 - \nu_e + \nu_e^2) \tag{24b}$$

As reported by Lou and Schapery (1971) a finite element analysis of a linear elastic composite with a square array of fibers was made to predict the average octahedral shear stress in the matrix. Apart from a numerical factor, (24) was found to be a fairly good approximation to this average. Considering c to be the arbitrary constant a_{66}, it is seen that (24a) and (13) are equivalent parameters for characterizing nonlinear behavior. It is also of interest to find from (24b) that $c = 4$ when $\nu_e = \frac{1}{2}$ and $c = 3.88$ when $\nu_e = 0.35$; the former value is the same as found experimentally by Sun and Chen (1987) for the boron/aluminum composite and by Mignery and Schapery (1988) for the rubber-toughened graphite/epoxy composite; the latter value of c was reported by Lou and Schapery (1971) for glass/epoxy material.

Most of the experimental work reported above is for proportional loading, (17). However, that of Mignery and Schapery (1988) involves nonproportional loading of the plies in an angle-ply layup. These studies provide limited experimental support for (11). We are currently making additional studies of angle-ply and unidirectional laminates under loading, unloading and reloading to address the applicability of (10) and (42) for toughened and untoughened graphite/epoxy composites.

It should be observed that the difference between loading and unloading curves in the model (6) is characterized by one scalar factor $\partial P/\partial\sigma_o$, where $P = P(\sigma_o, S)$. The loading curves, $d\sigma_o/dt>0$, are predicted by using $S = \sigma_o$. For unloading, $d\sigma_o/dt<0$, the thermodynamic requirement of positive entropy production and the path-independence of the unloading work are violated unless S is constant Schapery (1988). Consequently, for arbitrary stress histories, S is always the largest value of σ_o up to the current time.

This representation does not account for the difference between unloading and reloading curves. Tonda and Schapery (1987) were able to account for this difference for an untoughened graphite/epoxy composites using linear viscoelasticity theory; the approach to combining the effects of viscoelasticity and structure changes was developed earlier (Schapery, 1981). Whether or not this approach is able to account for all of the hysteresis is not presently known. It may be necessary to introduce another S-parameter which is activated at the start of reloading.

3. THE NORMALITY RULE FOR INELASTIC STRAINS

Let us now compare the normality rule employed in plasticity theory to predict plastic strain increments with the type of normality contained in (4). Following Sun and Chen (1987), we take $\sigma_o^2 = k$ as the yield condition, where k is a scalar that varies with the amount of plastic straining. Plastic strains are introduced in the same way as is commonly done for metals,

$$d\epsilon_i^p \equiv d\epsilon_i - d\epsilon_i^e \quad (25)$$

where $d\epsilon_i^p$, $d\epsilon_i$, and $d\epsilon_i^e$ are infinitesmal changes in plastic, total, and elastic strains, respectively. The elastic strains are assumed to be linear in the stresses,

$$\epsilon_i^e = S_{ij}\sigma_j \quad (26)$$

where S_{ij} are the constant compliances. The associated flow rule for plastic strain increments is

$$d\epsilon_i^p = \frac{\partial\sigma_o^2}{\partial\sigma_i}\, d\lambda \quad (27)$$

where $d\lambda$ is a scalar. This equation shows that $d\epsilon_i^p$ is a vector which is normal to the surface σ_o = constant. From (8) and (27),

$$d\epsilon_i^p = 2a_{ij}\sigma_j d\lambda \quad (28)$$

For proportional stressing $\sigma_i = k_i\sigma_o$ (where the k_i are constants) (28) may be integrated to obtain the total plastic strains,

$$\epsilon_i^p = \frac{\partial\sigma_o^2}{\partial\sigma_i}(\int\sigma_o d\lambda)/\sigma_o \quad (29)$$

which is also a vector normal to the surface σ_o = constant. The total strain is

$$\epsilon_i = \epsilon_i^e + \epsilon_i^p \quad (30)$$

which may be compared to the strain (6) derived from a dual strain energy density. The "inelastic strain" vector in (6),

$$\epsilon_i^I \equiv -\frac{\partial P}{\partial\sigma_o}\frac{\partial\sigma_o}{\partial\sigma_i} \quad (31)$$

is normal to the surface σ_o = constant, just as ϵ_i^p in (29). However, in contrast to ϵ_i^p, the normality of ϵ_i^I exists for proportional and non-proportional stressing. Observe also that this normality is preserved during unloading and reloading; recall that the coefficient $\partial P/\partial\sigma_o$ depends on both σ_o and S, and that $S = \sigma_o$ only when σ_o is equal to its largest value (considering all values up to the current time).

Consider next for further comparison a type of normality discussed by Rice (1971) for incremental inelastic strains. He developed (4) from thermodynamics with internal variables and used it in a study of inelastic behavior of metals; S is one of possibly many internal variables. A change in strain due to infinitesimal changes in both σ_i and S is, from (4),

$$d\epsilon_i = -\frac{\partial^2 W_\sigma}{\partial\sigma_i\partial\sigma_j}d\sigma_j + \frac{\partial G}{\partial\sigma_i}dS \quad (32)$$

where

$$G \equiv -\partial W_\sigma/\partial S \quad (33)$$

Rice observed that when elastic and inelastic strains are defined through increments, as expressed by the first and second terms in (32), respectively, the incremental inelastic strain,

$$d\epsilon_i^I \equiv \frac{\partial G}{\partial\sigma_i}dS \quad (34)$$

is normal to the "yield" surface G = constant. In fracture mechanics G (33) is called the "energy release rate". When there are two or more structure parameters S_m (m = 1, 2, ...),

$$d\epsilon_i^I = \frac{\partial G_m}{\partial\sigma_i}dS_m \quad (35)$$

where

$$G_m \equiv -\partial W_\sigma/\partial S_m \quad (36)$$

Thus, the m^{th} component of $d\epsilon_i^I$ is normal to the respective surface, Sm = constant, as noted by Rice.

When we use the special form for W_σ in (5), Rice's incremental elastic and inelastic strains become

$$d\epsilon_i^e \equiv -\frac{\partial^2 W_\sigma}{\partial\sigma_i\partial\sigma_j}d\sigma_j = -\frac{\partial^2 W_{\sigma o}}{\partial\sigma_i\partial\sigma_j}d\sigma_j - \frac{\partial^2 P}{\partial\sigma_i\partial\sigma_j}d\sigma_j \quad (37)$$

$$d\epsilon_i^I \equiv \frac{\partial G}{\partial\sigma_i}dS = -\frac{\partial^2 P}{\partial S\partial\sigma_i}dS = -\frac{\partial^2 P}{\partial S\partial\sigma_o}dS\frac{\partial\sigma_o}{\partial\sigma_i} \quad (38)$$

Notice that $d\epsilon_i^I$ is normal to the surface σ_o = constant and that an increment in the elastic strain defined in (7) is equal to only the first term in (37). Observe also that the tangent elastic compliance matrix $-\partial^2 W_\sigma/\partial\sigma_i\partial\sigma_j$ used in defining the incremental elastic strains in (32) is a function of the structure parameter S as well as stresses, while that based on the elastic strain in (7), $-\partial^2 W_{\sigma o}/\partial\sigma_i\partial\sigma_j$, depends only on the stresses.

4. A CONSTITUTIVE EQUATION WITH LINEAR UNLOADING BEHAVIOR

In characterizing the effect of damage on composite material behavior, it is commonly assumed that the material is linearly elastic when damage is constant. This linearity assumption is equivalent to using a dual strain energy density in which stress dependence is limited to first and second order terms,

$$w_\sigma = -b_o - b_i\sigma_i - \frac{1}{2} b_{ij}\sigma_i\sigma_j \tag{39}$$

where b_o, b_i and b_{ij} may be functions of one or more structure parameters S_m. In this case the strains (4) are

$$\epsilon_i = b_i + b_{ij}\sigma_j \tag{40}$$

The residual strains b_i and compliances b_{ij} may vary with stress history through changes in S_m; only one S will be used here. The strain energy density is related to w_σ through (3), and may be written as

$$w = c_o + c_i\,\epsilon_i + \frac{1}{2} c_{ij}\,\epsilon_i\,\epsilon_j \tag{41}$$

which provides the stresses

$$\sigma_i = c_i + c_{ij}\,\epsilon_j \tag{42}$$

The relationship between the b's and c's may of course be obtained by comparing (40) and (42). These second-order energies may be sufficiently general to predict ply stress-strain behavior if the unloading and reloading curves can be approximated by the same straight line whose position (c_i) and slope (c_{ij}) vary with S (as shown in Fig. 3).

The work ($\int\sigma_i d\epsilon_i$) and dual work ($-\int\epsilon_i d\sigma_i$) during structure-change processes are independent of path or history if and only if (Schapery, 1988),

$$-\frac{\partial w_\sigma}{\partial S} = g \quad \text{or} \quad -\frac{\partial w}{\partial S} = g \tag{43}$$

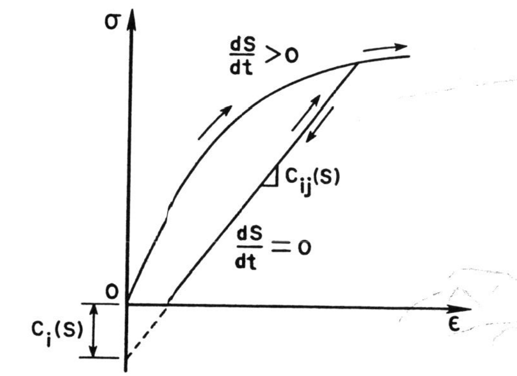

Fig. 3. Stress-strain behavior according to (42), showing loading, unloading and reloading.

where g is at most a function of S; the quantity g is the specific fracture energy if S is the fracture surface area of a crack. (Equation (43) is not limited to the second-order energies (39) and (41).) As shown by Schapery (1988) S can always be chosen so that (43) reduces to

$$-\frac{\partial w_\sigma}{\partial S} = 1 \quad \text{or} \quad -\frac{\partial w}{\partial S} = 1 \tag{44}$$

Observe that the term c_o ($= -b_o$) can be omitted as it can be absorbed in g in (43).

It should be added that the derivatives $\partial w_\sigma/\partial S$ and $\partial w/\partial S$ are always equal which may be easily shown by taking the differential of (3). Equation (44) provides the relationship for predicting S as a function of stress or strain. Thermodynamic theory requires $dS/dt \geq 0$ (Schapery, 1988); thus, if (44) predicts $dS/dt < 0$, S is actually constant and (44) is to be disregarded.

For the second order energy (39) with $b_o = 0$, the equation for S is

$$\frac{db_i}{dS}\sigma_i + \frac{1}{2}\frac{db_{ij}}{dS}\sigma_i\sigma_j = 1 \tag{45}$$

Although (39) is only of second order in the stresses, it is still sufficiently general to mathematically represent Sun and Chen's data discussed in Section 2. Indeed, this may be done by assuming the b_i are constants and then using

$$b_{ij} = S_{ij} + B\,S^r a_{ij} \tag{46a}$$

where

$$r = \frac{n-1}{n+1}, \qquad B = A^{1-r}(2/r)^r \tag{46b}$$

Also, S_{ij} are the constant elastic compliances, and a_{ij}, A, and n are the constants appearing in (8) and (22); observe that $0 < r < 1$. Equations (45) and (46) yield

$$S = (Br/2)^{(n+1)/2}\sigma_o^{(n+1)} \tag{47}$$

During loading, $d\sigma_o/dt > 0$, (47) is used in (46a) to predict instantaneous values of b_{ij}. For unloading, $d\sigma_o/dt < 0$, the coefficients b_{ij} are constant because S has a constant value equal to that at the start of unloading. Upon reloading, S again changes in accordance with (47) when σ_o reaches its largest past value. Unloading and reloading data are not reported by Sun and Chen (1987), and thus the range of applicability of this particular model cannot be assessed at this time. It is important to notice that this phenomenological characterization is not necessarily limited to brittle or to ductile composites, as Sun and Chen's results are for both types.

Finally, we should mention that the theory based on path-independence of work has been successfully employed in limited studies of particle-reinforced rubber (Schapery, 1987b), and a thermoplastic composite (Dan Jumbo et al., 1987). In the former case nearly all nonlinear behavior was expressed in terms of S-dependence of b_{ij}; in the latter case the residual strains b_i, instead of b_{ij}, were used to account for most nonlinearities. The small amount of nonlinearity that was not adequately represented by the second-order energy functions was apparently due to the large strains ($\approx$ 60%) in the filled rubber specimens and fiber or microfibril alignment (causing an increase in modulus for loading in the fiber direction) in the thermoplastic composite.

5. CONCLUDING REMARKS

A possible approach to predicting multidirectional-fiber laminate behavior would consist of using a unidirectional ply energy density, such as given by (5) or (41), with the usual displacement assumptions of lamination theory (Christensen, 1979). Delaminations and their growth could be accounted for essentially in the same way as done for linear and nonlinear elastic laminates, but with additional bookkeeping when there is any appreciable difference between loading and unloading stress-strain behavior. The work of deformation (which is equal to $w_T \equiv w + S$ if the second equation in (44) is used to predict S) is treated just like strain energy in nonlinear elastic fracture mechanics (Schapery, 1987a); in particular, w_T is used in strain energy release rate and J integral calculations.

With brittle-matrix composites, a significant number of transverse ply-level cracks may develop prior to structural failure (Johnston, 1987). These cracks are somewhat planar with the plane parallel to the fibers and perpendicular to the lamination plane. Typically, after rapid growth, they are arrested at the ply boundaries. If more than one fiber orientation is used, a laminate usually is capable of supporting loads well above that at crack initiation. Whether or not one S-parameter is sufficient to account for a general type of inelasticity which includes transverse cracks requires further study. It should be observed that even with only one parameter, an appreciable effect of these cracks on the laminate behavior may be taken into account through the way w or w_T depends on S; for example, b_{ij} may have the form in (46a) at small S, and then a considerably different form at large S when transverse cracks develop. Physically, S may reflect micro-damage (e.g. rubber particle cavitation) and plastic deformation until transverse cracks develop, and then at larger S-values account for these mechanisms as well as transverse crack density. If the effects of crack density and its growth are not sensitive to properties of adjacent plies with different fiber angles, an experimental program could use the simple angle-ply layup. Similar observations can be made for distributed interior delaminations (Harris et al., 1987); however, at least two plies would comprise the basic element of a laminate.

We are presently using these ideas to characterize and predict the mechanical response of untoughened and toughened graphite/epoxy laminates, recognizing that the proposed method has to be considered as tentative until a significant amount of additional experimental and analytical studies are made. Such studies should help to establish the range of validity of the work-potential method as well as define the experimental program needed for a complete characterization. Micromechanical models of damage in linear elastic composites (Wang and Haritos, 1987) should be helpful in analytically modeling the effect of distributions of cracks on moduli or compliances, and thus reduce the experimental effort. Schapery (1987b) used this approach in an elementary model to relate the orthotropic elastic properties of a particulate composite to a statistical distribution function which characterized the damage, and employed an evolution equation like (44) to predict the change in properties through an S-parameter which is an overall measure of the damage. A similar procedure should be applicable to laminates.

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