A General Damage Criterion for Solids

C. C. HSIAO and Y. S. CHENG
Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis MN 55455, USA

ABSTRACT

A generalized damage criterion is introduced which can be reduced to any of a number of known criteria by introducing appropriate values for the material coefficients. In the present paper a simple method for determining an approximate expression for the criterion of craze initiation is also given.

KEYWORDS

Damage Criterion, Yielding, Crazing, Fracturing, Static and Dynamic Loading, Fatigue.

INTRODUCTION

Recently a criterion for time-dependent craze initiation has been proposed (Chern and Hsiao, 1985; Hsiao, 1987). Now this criterion is found to be applicable not only for yielding and crazing but also for fracturing under both static and dynamic loads. Thus this new criterion is a new generalized theory of strength of solids. In the present paper a simple method is also given for determining approximate expressions for craze initiation. Experimental results illustrating the method of obtaining an approximate expression for the craze initiation in polystyrene are given.

GENERAL DAMAGE CRITERION

The general damage criterion may be stated as follows: damage would commence in a solid medium under load if and when the magnitude S of the deviatoric stress tensor overcomes the intrinsic flow resistance S_o in the medium:

$$S \geq S_o \,. \tag{1}$$

In terms of the stress tensor components σ_{ij} or the principal stresses σ_i, S may be written in the following form:

$$S = \frac{1}{\sqrt{6}}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)\right]^{1/2}$$
$$= \frac{1}{\sqrt{6}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2} \tag{2}$$

For different kinds of damage the intrinsic flow resistance of the material may be different. Since S_o is a function of material parameters such as density, micro structure and interatomic or intermolecular forces, the interacting forces are in general associated with the average distance between molecules or atoms. Among many other quantities the specific free volume Δv may provide an important dominating clue to the possible structural changes, and may serve as the important parameter to characterize the intrinsic flow resistance S_o. Using series expansion S_o may be expressed as follows (Chern and Hsiao, 1985):

$$S_o = \sum_{n=-\infty}^{\infty} a_n(\Delta v)^n\ , \tag{3}$$

where the coefficients a_n's are functions of materials' parameters only

The specific free volume Δv, in general, is a function of time t, the first invariant σ_{ii} of the stress tensor σ_{ij}, temperature T and possibly some other quantities such as chemical concentration and the environments.

For linear viscoelastic or elastic media, including most glassy polymers below their glass transition temperature $\Delta v(x,t)$ is the volume strain or the change in volume per unit volume at a given point which can be obtained by summing the principal strains $\varepsilon_i(x,t)$ or $\varepsilon_{ii}(x,t)$ to give the first invariant of the strain tensor as follows:

$$\Delta v(x,t) = = \varepsilon_1+\varepsilon_2+\varepsilon_3$$
$$= \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\ , \tag{4}$$

with zero strains referring to the original unaltered volume at reference temperature T_o. Based upon the linear viscoelasticity theory, $\varepsilon_{ii}(x,t)$ can be expressed as the sum of a thermal expansion term and a convolution integral term containing creep compliance $J_2(t)$ and stress rate in dilatation $\dot{\sigma}_{ii}(x,t)$ as shown below

$$\Delta v(x,t) = 3\alpha_o\theta(T) + \int_{-\infty}^{t} J_2(\xi-\eta)\dot{\sigma}_{ii}(x,\tau)\, d\tau\ , \tag{5}$$

where α_o is the linear thermal coefficient of expansion, $\theta(t)$ is the temperature function, and $\xi=t\phi(T)$ and $\eta=\tau\phi(T)$ as defined by temperature-time shift principle for thermorheologically simple viscoelastic materials (Schwarzl and Staverman, 1952).

For linear elastic media at reference temperature T_o, the following reduction can be found from thermoviscoelastic properties of rheological simple materials

$$\left.\begin{aligned} \phi(T_o) &= 1 \\ \theta(T_o) &= 0 \\ J_2(t) &= \frac{1}{3K}u(t) \end{aligned}\right\} \tag{6}$$

with K as the elastic bulk modulus and u(t) is the unit step function. Then Eq. (5) can be reduced to:

$$\Delta v(x,t) = \frac{\sigma_{ii}}{3K}\ , \tag{7}$$

which is to be expected as it can be easily obtained from linear elasticity theory directly.

Therefore, for linear viscoelastic media a general criterion for damage initiation may be expressed as (Chern and Hsiao, 1985; Hsiao, 1987):

$$S \geq \sum_{n=-\infty}^{\infty} a_n\left[3\alpha_o\theta(T) + \int_{-\infty}^{t} J_2(\xi-\eta)\frac{\partial}{\partial\tau}\sigma_{ii}(x,\tau)\, d\tau\right]^n\ , \tag{8}$$

For linear elastic media it may be expressed as:

$$S \geq \sum_{n=-\infty}^{\infty} a_n\left(\frac{1}{3K}\sigma_{ii}\right)^n\ , \tag{8}$$

INITIAL DAMAGE — YIELDING AND CRAZING

(a) Let $a_o = \sigma_y/\sqrt{3}$, $a_n = 0$ $(n \neq 0)$, where σ_y is the yield strength, then the general damage criterion is reducible to

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2\sigma_y^2\ . \tag{10}$$

Eq. (10) is the distortional strain energy criterion put forward by Huber and von Mises long ago.

(b) Let $a_o = \sigma_y/2$, $a_n = 0$ $(n \neq 0)$, under the condition of plane strain with Poisson's ratio $\nu = 0.5$, the general damage criterion is reducible to the maximum shear stress criterion proposed by Tresca:

$$|\sigma_1 - \sigma_2| = \sigma_y \,. \tag{11}$$

(c) Let $a_o = \tau_y$, $a_1 = -fK$, $a_n = 0$ (if $n \neq 0,1$), where τ_y is the yield strength in shear, f is the frictional coefficient, then under the condition of plane strain and $\nu = 0.5$, the general damage criterion is reducible to Mohr-Coulomb formulation as follows:

$$\tau_{max} = \tau_y - f\sigma_n \,, \tag{12}$$

where σ_n represents the normal stress.

(d) Let $a_{-1} = B/4K$, $a_o = A/2$, $a_n = 0$ (if $n \neq -1,0$), where A,B are material constants, under the condition of plane strain and $\nu = 0.5$, the general damage criterion is reducible to the form reported by Sternstein and Ongchin (1969):

$$|\sigma_1 - \sigma_2| = A + \frac{B}{\sigma_1+\sigma_2+\sigma_3} \,. \tag{13}$$

(e) Let $a_{-1} = 2X/9K$, $a_o = 2Y/3$, $a_n = 0$ (if $n \neq -1,0$), where X,Y are material constants, under the condition of plane strain and $\nu = 0.5$, the general damage criterion is reducible to the form suggested by Oxborough and Bowden (1973):

$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \frac{X}{\sigma_1+\sigma_2+\sigma_3} + Y \,. \tag{14}$$

(f) Let $a_{-1} = 0.015\pi\mu^2 d^3 \sigma_y/kTK$, $a_n = 0$ (if $n \neq -1$), where μ is the modulus in shear, d is the molecular diameter displacement, k is Boltzmann's constant, the general damage criterion is reducible to the form similar to that of Argon's (Argon, 1973; Argon and Hannoosh, 1977) as shown below:

$$\frac{\sigma_1+\sigma_2+\sigma_3}{2\sigma_y} = -\ln(\dot{\beta}_o t) + \frac{0.00225\pi\mu^2 d^3}{kTS} \,, \tag{15}$$

with $\dot{\beta}_o$ a constant, as the rate increment of porosity.

AN EXPERIMENTAL METHOD OF DETERMINING AN APPROXIMATE EXPRESSION FOR CRAZE INITIATION

Using a sheet specimen of polystyrene containing a central circular hole of radius R=0.03969 cm (1/64 in) under a tensile stress σ_o = 48.4 MPa.(7020 psi), one can apply the following approximate expression for craze initiation

$$S \geq \frac{a_{-1}}{3\alpha_o\theta(T) + \int_{-\infty}^{t} J_2(\xi-\eta)\dot{\sigma}_{ii}(t,\tau)\,d\tau} + a_o + a_1\left[3\alpha_o\theta(T) + \int_{-\infty}^{t} J_2(\xi-\eta)\dot{\sigma}_{ii}(x,\tau)\,d\tau\right] , \tag{16}$$

to the region around the central circular hole. The boundary equation of the damage area is expressible as follows (Timoshenko and Goodier, 1951):

$$\begin{aligned}&\Big[9\frac{R^8}{r^8} - 6(2+\cos 2\theta)\frac{R^6}{r^6} + \left(-1+4\cos 2\theta + \frac{40}{3}\cos^2 2\theta\right)\frac{R^4}{r^4} \\ &\quad + \left(4 - \frac{10}{3}\cos 2\theta - 8\cos^2 2\theta\right)\frac{R^2}{r^2} + \frac{4}{3}\Big]^{1/2} \\ &= \frac{2a_{-1}}{\sigma_0^2} \cdot \frac{3K}{\left(1-2\frac{R^2}{r^2}\cos 2\theta\right)} + \frac{2a_o}{\sigma_o} + \frac{2a_1}{3K}\left(1-2\frac{R^2}{r^2}\cos 2\theta\right),\end{aligned} \tag{17}$$

where r and θ are the polar coordinates with the origin in the center of the hole.

Using a Helium-Neon Laser system the picture of the craze area around the hole as shown in Fig. 1 has been obtained.

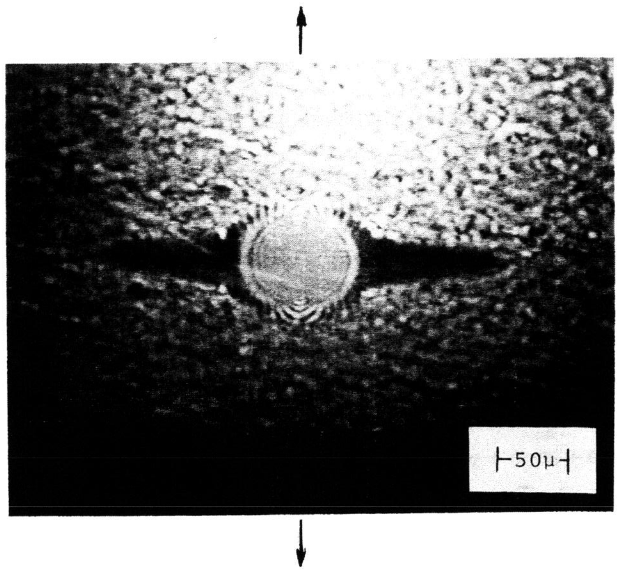

Fig. 1. Damaged Region of Polystyrene Around a Circular Hole under a Vertical Tension as shown.

By choosing several points on the boundary in one of the quadrants of the damaged area, and using data processing methods, the approximate expression for the damage initiation was found to be as below

$$S \geq \frac{0.8041\ (\mathrm{MPa})}{3\alpha_o\theta(T) + \int_{-\infty}^{t} J_2(\xi-\eta)\dot{\sigma}_{ii}(x,\tau)\,d\tau} - 264.6\ (\mathrm{MPa})$$

$$+\ 26618\ (\mathrm{MPa})\left[3\alpha_o\theta(T) + \int_{-\infty}^{t} J_2(\xi-\eta)\dot{\sigma}_{ii}(x,\tau)\,d\tau\right] \tag{18}$$

for polystyrene.

In short-time test, the material may be considered as linearly elastic, then Eq. (18) can be reduced to the following form for practical application.

$$S \geq \frac{8484\ (\mathrm{MPa})^2}{\sigma_{ii}} - 264.6\ (\mathrm{MPa}) + 2.5\sigma_{ii}\,. \tag{19}$$

ULTIMATE DAMAGE — FRACTURE

Now let us return to the general damage expression (9).

(g_1) If

$$\left.\begin{aligned}
a_o &= \left(\frac{EC}{1+\nu}\right)^{1/2} \\
a_2 &= -\frac{1}{2}a_o\frac{1-2\nu}{6EC}(3K)^2 \\
a_4 &= -\frac{1}{8}a_o\left(\frac{1-2\nu}{6EC}\right)^3(3K)^4 \\
a_6 &= -\frac{1}{16}a_o\left(\frac{1-2\nu}{6EC}\right)^3(3K)^6 \\
&\cdots\cdots\cdots \\
a_n &= 0 \text{ (for any other n)}
\end{aligned}\right\} \tag{20}$$

where E is the modulus of elasticity and C is the critical strain energy density, the general damage criterion (9) can be reduced to the following form:

$$S \geq \left(\frac{EC}{1+\nu}\right)^{1/2}\left\{1-\frac{1}{2}\cdot\left(\frac{1-2\nu}{6EC}\right)(\sigma_{ii})^2-\frac{1}{8}\left(\frac{1-2\nu}{6EC}\right)^2(\sigma_{ii})^4-\frac{1}{16}\left(\frac{1-2\nu}{6EC}\right)^3(\sigma_{ii})^6-\ldots\right\}$$

$$=\left(\frac{EC}{1+\nu}\right)^{1/2}\left[1-\left(\frac{1-2\nu}{6EC}\right)(\sigma_{ii})^2\right]^{1/2} \tag{21}$$

This is precisely the strain energy density criterion for failure under static loading condition (Sih, 1981).

(g_2) Under certain limitations, the above criterion can be reduced to Sih's strain energy density criterion as a special case as given below:

$$\frac{1}{2E}(\sigma_1^2+\sigma_2^2+\sigma_3^2) - \frac{\nu}{E}(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) = C\,. \tag{22}$$

(h_1) For fatigue failure, if σ_{ij} in the general damage criterion represent the individual maximum amplitudes of the reversible cyclic stress, then for ductile materials [$\tau_u \doteq 0.58\sigma_u$, where τ_u is the fatigue limit (shear strength) under reversed torsion stress, σ_u is the fatigue limit (normal strength) under reversed bending stress] (Cheng, 1986), and let $a_o = \tau_u$, $a_n = 0$ (if $n \neq 0$), then the general damage criterion is reducible to

$$S \geq \tau_u\,. \tag{23}$$

This is simply Hashin's criterion for ductile materials (Hashin, 1981):

$$\frac{I_1^2}{\sigma_u^2} - \frac{I_2}{\tau_u^2} = 1\,, \tag{24}$$

where

$$I_1 = \sigma_{11}+\sigma_{22}+\sigma_{33} = \sigma_1+\sigma_2+\sigma_3\,,$$

$$I_2 = \begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{vmatrix} + \begin{vmatrix}\sigma_{22} & \sigma_{23}\\ \sigma_{23} & \sigma_{33}\end{vmatrix} + \begin{vmatrix}\sigma_{33} & \sigma_{31}\\ \sigma_{31} & \sigma_{11}\end{vmatrix} = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\,.$$

(h_2) For brittle materials, if

$$\left.\begin{aligned}
a_0 &= \tau_u \\
a_2 &= -\frac{1}{2}a_0\left(\frac{1}{\sigma_u^2}-\frac{1}{3\tau_u^2}\right)(3K)^2 \\
a_4 &= -\frac{1}{8}a_0\left(\frac{1}{\sigma_u^2}-\frac{1}{3\tau_u^2}\right)^2(3K)^4 \\
a_6 &= -\frac{1}{16}a_0\left(\frac{1}{\sigma_u^2}-\frac{1}{3\tau_u^2}\right)^3(3K)^6 \\
&\cdots\cdots\cdots \\
a_n &= 0 \text{ (for any other n)}
\end{aligned}\right\} \tag{25}$$

then the general damage criterion is reduced to

$$\begin{aligned}
S \geq \tau_u &\left[1-\frac{1}{2}\left(\frac{1}{\sigma_u^2}-\frac{1}{3\tau_u^2}\right)(\sigma_{ii})^2-\frac{1}{8}\left(\frac{1}{\sigma_u^2}-\frac{1}{3\tau_u^2}\right)^2(\sigma_{ii})^4-\frac{1}{16}\left(\frac{1}{\sigma_u^2}-\frac{1}{3\tau_u^2}\right)^3(\sigma_{ii})^6-\ldots\right] \\
&=\tau_u\left[1-\left(\frac{1}{\sigma_u^2}-\frac{1}{3\tau_u^2}\right)(\sigma_{ii})^2\right]^{1/2}
\end{aligned} \tag{26}$$

which is exactly the criterion (Cheng, 1986; Hashin, 1981) given by Hashin earlier in the following form:

$$\frac{I_1^2}{\sigma_u^2}-\frac{I_2}{\tau_u^2}=1 .$$

(h_3) For the condition of a combined bending-torsion, if the axial stress σ_{11}, and the shear stress σ_{12} are the only nonvanishing stresses, then Hashin's criterion is reducible to

$$\frac{\sigma_{11}^2}{\sigma_u^2}+\frac{\sigma_{12}^2}{\tau_u^2}=1 . \tag{27}$$

Eq. (27) is the well known Gough's "ellipse quadrant" criterion (Gough and Pollard, 1935), and has been extensively applied to mechanical designs.

(h_4) In the case of biaxial cyclic stresses, i.e., σ_1 and σ_2 are the only nonvanishing stresses, Hashin's criterion is reducible to

$$\frac{(\sigma_1+\sigma_2)^2}{\sigma_u^2}-\frac{\sigma_1\sigma_2}{\tau_u^2}=1 . \tag{28}$$

which was found to be in good agreement with the experimental data by Rotvel (1970). Eq. (28) has been very useful for analysis and design in pressure vessels.

In fatigue failure research a problem of long standing is the establishment of criteria of failure where the state of cyclic stress is two or three dimensional. In 1981 Hashin derived an isotropic failure criterion for a general three dimensional state of cyclic stress which is claimed to be the most general quadratic approximation to the three dimensional isotropic failure condition for reversed cyclic stresses. However, Hashin's criterion is just a special case of the general damage criterion given in this report. It is seen that the present general damage criterion may well contain and govern the three-dimensional anisotropic nonsymmetrically cyclic stressed conditions.

Apparently this general criterion can give predictive information on various types of damages in spite of the utilization of only one unique free volume concept for the derivation of the criterion.

ACKNOWLEDGEMENT

This work was supported partially by an AFOSR Grant.

REFERENCES

Argon, A. S., (1973). Physical basis of distortional and dilational plastic flow in glass polymers, *J. Macromol. Sci.-Phys.* B**8** (3-4), 573- 596.
Argon, A. S. and J. G. Hannoosh, (1977). Initiation of crazes in polystyrene, *Philos. Mag.* **36** (5), 1195-1216.
Cheng, Y. S., (1986). Physical interpretation of Hashin's criterion of fatigue failure under multiaxial stress, *Eng. Fracture Mech.* **24** (2), 165-167.
Chern, S. S. and C. C. Hsiao, (1985). A generalized time-dependent theory on craze initiation in viscoelastic media, *J. Appl. Phys.* **57** (6), 1823-1834.
Gough, H. J. and H. V. Pollard, (1935). Strength of metals under combined alternating stress, *Proceedings of the Institute of Mechanical Engineers*, (London), **131**, 3.
Hashin, Z., (1981). Fatigue failure criteria for combined cyclic stress. *Int. J. Fracture*, **17** (2), 101-109.
Hsiao, C. C. (1987). Non-continuum craze-crack transition, in Damage Mechanics in Composites, ASME AD-12, 25-37.
Oxborough, R. J. and P. B. Bowden, (1973). A general critical-strain criterion for crazing in amorphous glassy polymers, *Philos. Mag.* **28** (3), 547-559.
Rotvel, F., (1970). Biaxial fatigue tests with zero mean stresses using tubular specimens, *Int. J. Mech. Sci.* **12** (7), 597-613.
Schwarzl, F. and A. J. Staverman, (1952). Time-temperature dependence of linear viscoelastic behavior, *J. Appl. Phys.* **23** (8), 838-843.

Sih, G. C., (1981). Experimental fracture mechanics: Strain energy density criterion, *Mechanics of Fracture 7*, edited by G. C. Sih, Noordhoff International Publishing, Leyden. 17-56.
Sternstein, S. S. and L. Ongchin, (1969). Yield criteria for plastic deformation of glassy high polymers in general stress fields, *Polym. Pre-prints, Amer. Chem. Soc.* **19** (2), 1117-1124.
Timoshenko, S. and J.N. Goodier, (1951). Theory of Elasticity, Chap. 4. McGraw-Hill Comp. New York.