Statistical Properties of Generalized Strain Criterion for Multiaxial Random Fatigue

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ABSTRACT

Statistical properties of generalized criterion of the maximum shear and normal strains on the fracture plane have been presented. Functions of probability distribution and spectral density of the equivalent strain have been analysed on the assumption that a random tensor of strain state is a six-dimensional stationary and ergodic Gaussian process. The expected value and variance of the equivalent strain have been determined as well. From spectral analysis a new limitation has been derived for extension of some multiaxial cyclic fatigue criteria to random loadings. It is connected with the fact that in some cases the frequency band of the equivalent strain is greater than that for the random tensor of strain state.

KEYWORDS

Multiaxial fatigue, fatigue criteria, random loadings, triaxial strain state, spectral analysis.

INTRODUCTION

Investigations of multiaxial random fatigue were started in 1976 and at present they are at their preliminary stage (Macha, 1976). Some theoretical results have been obtained so far, for instance formulation of:

- generalized fatigue stress criterion for long life time (Macha, 1984, 1985) ,
- generalized fatigue strain criterion for long and short life time with regard to plastic strains (Macha, 1988),

- three methods of prediction of the expected fatigue fracture plane positions: method of weight functions, method of the variance, method of damage cumulation (Będkowski, Macha, 1987 , Macha, 1985).

The mentioned criteria of multiaxial random fatigue were formulated by extension of some multiaxial cyclic fatigue criteria. For some criteria, however, there are theoretical limitations making it impossible to apply them under random loadings (Macha, 1985).

In this paper statistical properties of the generalized strain fatigue criterion under multiaxial random loadings are presented. Assuming a random strain tensor to be a six-dimensional stationary and ergodic Gaussian process the probability distribution function and power spectral density function of the equivalent strain are analysed. From spectral analysis of the criterion a new limitation for extension of the criteria of multiaxial cyclic fatigue has been determined. The limitation results from the fact that in some cases frequency bands of the equivalent strain show up as wider than frequency bands of the random strain tensor.

GENERALIZED CRITERION OF THE MAXIMUM SHEAR AND NORMAL STRAINS IN THE FRACTURE PLANE

It is assumed that:

1. Fatigue fracture is caused by the normal strain $\varepsilon_\eta(t)$ and the shear strain $\varepsilon_{\eta s}(t)$ in the direction $\bar{s}$ in the fracture plane with the normal $\bar{\eta}$ (Fig.1).
2. The direction $\bar{s}$ on the fracture plane agrees with a mean direction of the maximum shear strain $\varepsilon_{\eta s\,max}(t)$.
3. For a given fatigue life the maximum value of a linear combination of strains $\varepsilon_{\eta s}(t)$ and $\varepsilon_\eta(t)$ under multiaxial random loadings satisfies the equation

$$\max_t \left\{ b\varepsilon_{\eta s}(t) + k\,\varepsilon_\eta(t) \right\} = q \qquad (1)$$

where b,k,q - constants for selection of a particular form of equation (1) .

For the established expected fatigue fracture plane position,

strains $\varepsilon_\eta(t)$ and $\varepsilon_{\eta s}(t)$ are calculated according to the transformation rule for components of the strain tensor

$$\varepsilon_\eta(t) = \hat{\alpha}_{\eta i}\,\hat{\alpha}_{\eta j}\,\varepsilon_{ij}(t) \ ,$$
$$\varepsilon_{\eta s}(t) = \hat{\beta}_{\eta i}\,\hat{\beta}_{sj}\,\varepsilon_{ij}(t) \ , (i,j=x,y,z) \qquad (2)$$

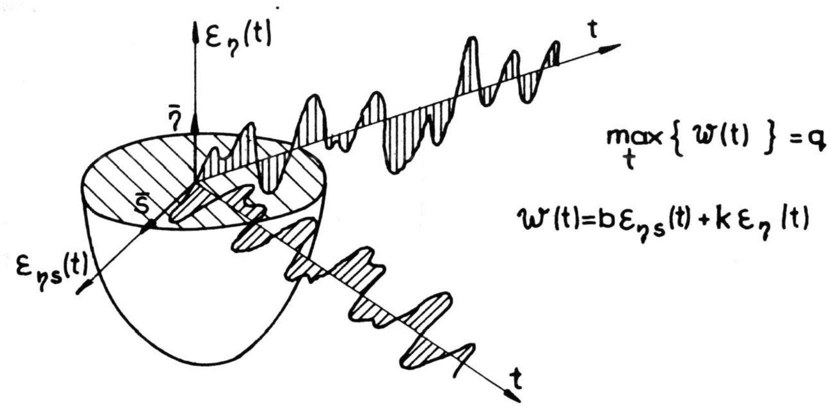

Fig.1 Shear and normal strains - $\varepsilon_{\eta s}(t)$ and $\varepsilon_\eta(t)$ respectively - on the fracture plane. $\max_t\{w(t)\}$ is 100% quantile of the random variable w.

In (2) $\hat{\alpha}$ and $\hat{\beta}$ are respectively cosines of angles between directions of unit vectors $\bar{\eta}$ and $\bar{s}$ and directions of the fixed coordinate system Oxyz.

The cosines $\hat{\alpha}$ and $\hat{\beta}$ are usually expressed by the mean direction cosines $\hat{l}_n, \hat{m}_n, \hat{n}_n$ (n=1,2,3) of axes of principal strains $\varepsilon_1(t) \geqslant \varepsilon_2(t) \geqslant \varepsilon_3(t)$ (Będkowski, Macha, 1987, Macha, 1986).

Assuming the determined fatigue fracture plane positions and values of b,k,q we obtain different particular forms of equation (1) (Macha, 1985, 1988) . From (1) and (2) it results that the equivalent strain $\varepsilon_{red}(t)$ is always a certain

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function $f[\cdot]$ linearly dependent on strains $\varepsilon_{ij}(t)$ and constants b, k, ν (Poisson's ratio) and nonlinearly dependent on the mean direction cosines $\hat{l}_n, \hat{m}_n, \hat{n}_n$ (n=1,2,3)

$$\varepsilon_{red}(t) = f[\varepsilon_{ij}(t), \hat{l}_n, \hat{m}_n, \hat{n}_n, b, k, \nu] \quad (3)$$

PROBABILITY DISTRIBUTION OF $\varepsilon_{red}(t)$ AND ITS PARAMETERS

The equivalent strain (3) linearly dependent on components of strain state $\varepsilon_{ij}(t)$ can be expressed as a sum of six components

$$\varepsilon_{red}(t) = \sum_{k=1}^{6} a_k y_k(t) \quad (4)$$

where the constant coefficients a_k depend on a particular form of equation (1) assumed and the expected fracture plane position and $y_k(t)$ are components of strain $\varepsilon_{ij}(t)$ respectively.

The expected value of equivalent strain is equal to

$$\hat{\varepsilon}_{red} = \sum_{k=1}^{6} a_k \hat{y}_k \quad (5)$$

where $\hat{y}_k$ are the expected values of strains $\varepsilon_{ij}(t)$.

The variance of equivalent strain is equal to

$$\mu_{\varepsilon\, red} = \sum_{s=1}^{6} \sum_{t=1}^{6} a_s a_t \mu_{yst} \quad (6)$$

where a_s and a_t are the same coefficients as a_k in (4) and are suitably chosen for elements μ_{yst} of the covariance matrix of the random strain tensor.

From equation (4) it results that probability distribution function of the equivalent strain is of the same type as that for the random tensor of strain state. In a particular case when the joint probability distribution function of random strain tensor is of a normal type $N(\hat{\underline{y}}, \underline{\underline{\mu}}_y)$ the probability distribution function of the reduced strain is of a normal type $N(\hat{\varepsilon}_{red}, \mu_{\varepsilon red})$, too.

From equation (5) it results that if strain state components have their expected values equal to zero, $\hat{\underline{y}} = \underline{0}$, then the expected value of equivalent strain is equal to zero as well, i.e. $\hat{\varepsilon}_{red} = 0$. From the physical point of view it is an important and expected result. Such a result cannot be obtained if the equivalent strain is calculated according to other criteria, for example the criterion of maximum principal strain $\varepsilon_1(t)$, the criterion of maximum principal shear strain $\gamma_1(t)$ or the criterion of maximum octahedral shear strain $\gamma_{oct}(t)$, according to the following formulas:

$$\varepsilon_{red}(t) = \varepsilon_1(t) = f_1[\varepsilon_{ij}(t)] \quad (7)$$

$$\varepsilon_{red}(t) = \frac{1}{(1+\nu)} \gamma_1(t) = \frac{1}{(1+\nu)}[\varepsilon_1(t) - \varepsilon_3(t)] = f_2[\varepsilon_{ij}(t), \nu] \quad (8)$$

$$\varepsilon_{red}(t) = \frac{1}{(1+\nu)\sqrt{2}} \gamma_{oct}(t) = \frac{1}{(1+\nu)\sqrt{2}} \left\{ [\varepsilon_1(t) - \varepsilon_2(t)]^2 + [\varepsilon_2(t) - \varepsilon_3(t)]^2 + [\varepsilon_3(t) - \varepsilon_1(t)]^2 \right\}^{1/2} = f_3[\varepsilon_{ij}(t), \nu] \quad (9)$$

because $f_1[\cdot]$, $f_2[\cdot]$, $f_3[\cdot]$ are nonlinear functions with respect to $\varepsilon_{ij}(t)$.

SPECTRAL ANALYSIS OF $\varepsilon_{red}(t)$

While calculating fatigue life, frequency structure of the equivalent strain is very important. There is an important question of frequency band width of $\varepsilon_{red}(t)$ when frequency bands of strain state components $\varepsilon_{ij}(t)$ are given. The equivalent strain $\varepsilon_{red}(t)$ according to (4) can be understood as an output signal from a linear physical system with 6 in-puts for which signals $\varepsilon_{ij}(t)$ were applied. In this case spectral transmittances of the system $H_k(f)$ are not dependent on frequency f and they are equal to constant coefficients i.e. $H_k(f) = a_k$, $(k=1,\dots,6)$.

On the assumption that random components of strain state are correlated, the power spectral density of equivalent strain can be determined from the following equation

$$G_{\varepsilon red}(f) = \sum_{i=1}^{6} \sum_{j=1}^{6} H_i^*(f)\, H_j(f)\, G_{ij}(f) =$$

$$\sum_{i=1}^{6} \sum_{j=1}^{6} a_i a_j G_{ij}(f) = \sum_{i=1}^{6} a_i^2 G_{ii}(f) + 2\sum_{i<j} a_i a_j \,\mathrm{Re}\,[G_{ij}(f)] \tag{10}$$

where $H_k(f)$ - spectral transmittance of physical system for the input signal $y_k(t)$, $H_k^*(f)$ - conjugate function to $H_k(f)$, $G_{ij}(f)$ - one-sided functions of power spectral density of strain $y_i(t)$ and $y_j(t)$, $\mathrm{Re}\,[G_{ij}(f)]$ - real part of complex function $G_{ij}(f)$.

If random components of the strain state are uncorrelated the second term in (10) will be equal to 0 and the frequency band of equivalent strain will be determined only by the natural frequency bands of the power spectral density function $G_{ii}(f)$ of strain state components $\varepsilon_{ij}(t)$. For any frequency f the following inequality holds:

$$|G_{ij}(f)|^2 \leq G_{ii}(f)\, G_{jj}(f) \tag{11}$$

So it results that the value of expression $|G_{ij}(f)|$ is limited for each value of frequency f by corresponding values of $G_{ii}(f)$ and $G_{jj}(f)$. From (10) and (11) an important inequality concerning frequency structure of the equivalent strain results:

$$f_{max\,\varepsilon red} \leqslant \max_{ij} \{ f_{max\,ij} \}, \quad (i,j=1,\dots,6) \tag{12}$$

Inequality (12) means that the maximum frequency of equivalent strain, $f_{max\,\varepsilon red}$, is lower or equal to the maximum frequency from among all frequencies appearing in components of the strain state, $f_{max\,ij}$. This favourable result is due to a linear form of equation (1) and it further results in the fact that on reduction of the triaxial loading to the equivalent uniaxial one, frequency bands of the strain state components transform to a frequency band of the equivalent strain without increasing its width.

The equivalent strain calculated according to (7) - (9), owing to nonlinearity of functions $f_1[\cdot]$, $f_2[\cdot]$, $f_3[\cdot]$, will have the maximum frequency $f_{max\,\varepsilon red}$ higher than the maximum frequency of all frequencies occurring in strain state components and the following inequality opposite to (12) is valid

$$f_{max\,\varepsilon red} > \max_{ij} \{ f_{max\,ij} \}, \quad (i,j=1,\dots,6) \tag{13}$$

Such unfavourable increase in frequency band width of the equivalent strain causes serious difficulties in calculating fatigue life, mainly in cycle counting and fatigue damage cumulating.

Frequency increase will result in counting a greater number of cycles in a time unit and overestimating fatigue damage. Another disadvantage is that some cycles will have a shorter period and the effect of load frequency on fatigue life of a given material should be considered while calculating damage.

For example, a non-linear operation of squaring a sinusoidal signal with frequency f_0, present in (7) - (9), increases its basic frequency twice and leads to the occurrence of harmonic components with frequencies $4f_0$, $6f_0$ etc. Thus, from spectral analysis of the equivalent strain it results that there is a new important limitation for extension of multiaxial cyclic fatigue criteria to random loadings.

Mathematical models of fatigue criteria in which components of the strain state $\mathcal{E}_{ij}(t)$ form nonlinear functions are not applicable under multiaxial random loadings due to increase in the frequency band of the equivalent strain as compared to the frequency band of the random strain tensor. Thus, formulation of the multiaxial random fatigue with various invariants of strain state is not correct, except for the first linear invariant.

CONCLUSIONS

1. The generalized criterion of maximum normal and shear strains on the fracture plane, owing to its linear combination of strain state components, has got good statistical properties which make it possible to calculate fatigue life under multiaxial random loadings.
2. Probability distribution function of the equivalent strain is of the same type as joint probability distribution function of the random tensor of strain state.
3. If strain state components have the expected values equal to zero, then the expected value of equivalent strain is equal to zero as well.
4. The variance of the equivalent strain depends linearly on the variance of strain state components and nonlinearly on the mean direction cosines of principal strain axes which are used for determining the fatigue fracture plane position.
5. While reducing the multiaxial state of loading to the uniaxial one with the discussed criterion, the frequency bands of strain state components are transformed to the frequency band of equivalent strain without increase of its width.
6. From spectral analysis it results that mathematical models of fatigue criteria in which strain state components form nonlinear functions can be hardly adopted under multiaxial random loading due to an unacceptable increase in the frequency band of the equivalent strain as compared with the frequency band of the random strain tensor. This new important limitation prevents application of various nonlinear invariants of strain state to formulation of multiaxial random fatigue criteria.

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