Model of Fatigue Crack Growth Based on Energy Dissipated Within the Process Zone

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ABSTRACT

Viewing extension of crack by fatigue as a sequence of discreet growth steps of finite magnitude, we employ the concept of a cyclic R-curve to provide theoretical estimates of the fatigue crack growth rates for yielding and/or nonlinearly viscous materials. Influence of the material parameters such as ductility, tearing modulus and strain hardening exponent on the rate da/dN has been investigated. Two limiting cases of practical interest, 1) low cycle fatigue, and 2) near-critical fatigue, have been shown to obey certain simple laws such as a power and an exponential law. For the former case the rate of growth approaches zero as ε^m, while for the latter da/dN becomes unbounded as $\exp(1/\varepsilon)$, where m is a constant and ε denotes a small nondimensional quantity proportional to the intensity of the outer stress field.

Appropriate closed form expressions, valid for the entire range of fatigue process, excluding the near threshold propagation, are suggested.

KEYWORDS

Fatigue, crack extension, cyclic resistance curve, strain-hardening.

INTRODUCTION

The constitutive relations of a material dictate certain specific forms of the restraining stress within the structured end-zone associated with a sub-critical crack. In what follows the crack will be represented by an extended Dugdale model which incorporates

(1) strain hardening for stresses exceeding the yield stress, σ_0, and
(2) quasi-static crack extension governed by the Wnuk's final stretch (or CTOA) growth criterion.

Within the computational frame of such model, the displacements ahead of the crack tip are not zero, but they gradually approach zero over a certain finite length R, which measures the extent of the plastic zone. Thus the singular stress no longer exists at the crack tip, but it is replaced by the so-called restraining stress S defined over the distance x_1 (measured from the crack tip) which varies within the interval [0,R]. Variation of the restraining stress is deduced from the constitutive laws of elasto-plastic strain hardening solid and it is of the following form

$$S(x_1) = \begin{cases} \sigma_0 \left(\frac{R}{\Delta}\right)^\alpha = \text{const.} & 0 \le x_1 \le \Delta \\ \sigma_0 \left(\frac{R}{x_1}\right)^\alpha & \Delta \le x_1 \le R \end{cases} \qquad \text{(i)}$$

for the Ramberg-Osgood strain hardening material, in where the hardening index α is related to n and N as follows

$$\alpha = \begin{cases} \frac{1}{1+n} \\ \frac{N}{N+1} \end{cases} \qquad \text{(ii)}$$

The non-hardening problem of perfectly elasto-plastic solid is included in the Eq. (i) as a limiting case; when $\alpha \to N \to 0$ (or $n \to \infty$), we obtain $S(x_1) = \sigma_0$ within the entire end zone, which corresponds to the Dugdale model.

Symbol Δ denotes the extent of small region immediately adjacent to the crack tip, the so-called process zone, which enters into the theory through a criterion of quasi-static crack extension, such as Wnuk's final stretch criterion, Wnuk (1972, 1974, 1981) or Kfouri et al. concept of energy separation rate, Kfouri and Miller (1976), Kfouri and Rice (1977), or the CTOA criterion suggested by Shih and coworkers, cf. Kumar, German and Shih (1981), or criterion of Rice and Sorensen (1978) who employed a concept of a critical crack opening measured a fixed distance behind the tip of the crack. Smith (1980) has shown that all these criteria are equivalent among each other, and all of them can be reduced to McClintock's (1965) concept of critical strain generated a certain micro-structural distance (say, the process zone) ahead the crack tip.

Our next objective is to relate the energy dissipated within the entire end zone

$$J = -2 \int_0^R S(x_1)\, \delta u_y(x_1) \qquad \text{(iii)}$$

or, within the process zone only

$$J^\Delta = -2 \int_0^\Delta S(x_1)\, \delta u_y(x_1) \qquad \text{(iv)}$$

to the rate of fatigue crack growth, da/dN. The increment of the opening displacement $\delta u_y(x_1)$ is to be interpreted as follows

$$\delta u_y(x_1) = [u_y(x_1)]_{a+\delta a} - [u_y(x_1)]_a \qquad \text{(v)}$$

The details concerning evaluation of the quantities $\delta u_y(x_1)$ and J^Δ are presented in the next section.

The final results are somewhat different from those obtained by the earlier modeling procedures of Cherepanov (1967) and Wnuk (1971, 1973), based on an assumption of constant energy dissipated within the entire end zone, J = const. Law of fatigue crack propagation valid in the range of near-threshold loading conditions, as discussed by Radon (1986) and Miller (1988), will be the subject of a separate report.

MATHEMATICAL MODELING

Mathematical model employed here is compatible with Wnuk's final stretch criterion of quasi-static crack growth and with the CTOA criterion for continuing crack extension. The governing equation of a moving subcritical crack is derived from the requirement that the energy absorbed within the process zone just prior to fracture is a material property, invariant to the amount of slow stable cracking and to the geometrical configuration of cracked specimen. Wnuk (1972, 1974) has shown that setting the quantity

$$J^\Delta = -2 \int_0^\Delta S(x_1)\, \delta u_y(x_1) \qquad (1)$$

to a material constant $\hat{J} = \sigma_0 \hat{\delta}$, where $\hat{\delta}$ denotes the so-called final stretch* and σ_0 is the flow stress, results in the differential equation defining an R-curve for a non-hardening case

$$\frac{dR}{da} = M_1 - \frac{1}{2} \log (4e\, R/\Delta) \qquad (2)$$

Here, R denotes the length of the plastic zone associated with a quasi-static crack, R = R(a). Equation (2) is almost identical with the equation derived six years later by Rice and Sorensen in an entirely different way, cf. Rice and Sorensen (1978). In what follows we shall refer to Eq. (2) as Wnuk-Rice-Sorensen equation. Recently, this result has been generalized by Hunsacharoonroj (1987) who incorporated effects of the strain hardening on the shape of an R-curve. With $S(x_1) = \sigma_0 (R/x_1)^\alpha$ he obtained

$$\frac{dR}{da} = M_2 - \frac{1}{2} \left(\frac{R}{\Delta}\right)^\alpha \log^\kappa \left[4e \left(\frac{R}{\Delta}\right)^{1-\alpha}\right] \qquad (3)$$

in which $\kappa = 1 - 0.5\alpha$. Obviously Eq. (3) reduces to Eq. (2) when α approaches zero. Symbols M_1 and M_2 denote the tearing moduli for a non-hardening and hardening case, respectively. They are evaluated as certain multiples of the <u>minimum</u> moduli necessary for stable crack growth to occur, i.e.,

$$\begin{aligned} M_1 &= k\, M_{min}^{(1)} = k \left[\frac{1}{2} \log (4e\, R_{ini}/\Delta)\right] \\ M_2 &= k\, M_{min}^{(2)} = k \left\{\frac{1}{2} \left(\frac{R_{ini}}{\Delta}\right)^\alpha \log^\kappa \left[4e \left(\frac{R_{ini}}{\Delta}\right)^{1-\alpha}\right]\right\} \end{aligned} \qquad (4)$$

Again, for $\alpha = 0$, we obtain an identity $M_1 = M_2$. If we denote R_{ini}/Δ for non-hardening case by ρ_0 and use the symbol ρ_i for the hardening case, then these two quantities are related as follows

$$\rho_i = \rho_0 \,(1-2\alpha) \qquad (5)$$

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*The quantity $\hat{\delta}$ is proportional to the CTOA or to the Shih's tearing modulus, T_δ.

For a given nondimensional number k (k must be greater than one), given ductility parameter $\rho_0 = (R_{ini}/\Delta)_{\alpha=0}$, and for a given strain hardening index α, the tearing moduli M_1 and M_2 become determined, and equations (2) and (3) can be integrated numerically in order to generate the predictions of the fatigue crack growth rate. Following the procedure developed specifically for the fatigue process we obtain

$$\frac{da}{dN} = R_c \int_{J_{min}/J_c}^{J_{max}/J_c} \frac{dy}{M_1 - \frac{1}{2}\log[4e\,\rho_0\, y\delta]} \qquad (6)$$

for non-hardening material ($\delta = R_c/R_{ini} = J_c/J_{ini}$, $y = J/J_c$), and

$$\frac{da}{dN} = R_c^* \int_{J_{min}/J_c^*}^{J_{max}/J_c^*} \frac{dy}{M_2 - \frac{1}{2}\rho^\alpha \log^\kappa [4e\,\rho^{1-\alpha}]} \qquad (7)$$

for the hardening material. Here, $\rho = \rho_i\, y\delta^*$ and $\delta^* = R_c^*/R_{ini} = J_c^*/J_{ini}$. The integrals occurring in Eqs. (6) and (7) must be evaluated numerically. With lower limit J_{ini}/J_c^* replaced by $[r^2/(1 - r^2)]\ \Delta y$ and upper limit J_{max}/J_c^* replaced by $[1/(1 - r^2)]\ \Delta y$, one obtains the rate da/dN as a function of the parameter r, the r-ratio, and Δy (= $\Delta J/J_c$) as primary independent variable. Figures 1 and 2 show some examples of the relations described by Eqs. (6) and (7). All these results can be represented in the general form

$$\frac{da}{dN} \sim F(r, \Delta y, \rho_0, k, \alpha) \qquad (8)$$

For a given choice of the material parameters

ρ_0 ~ ductility
k ~ tearing modulus
α ~ strain hardening

the value of the upper plateau of the cyclic R-curve becomes known, and this value is used to normalize the range of the J-integral (or the resistance parameter R, which within the small scale yielding range is directly proportional to J-integral). Therefore, $J/J_c = R/R_c$, $\Delta J/J_c = \Delta R/R_c$, and $dR = R_c\,(dJ/J_c) = R_c dy$.

Within certain ranges of the parameter r and the dimensionless range of the J-integral, Δy, we attempted to curve-fit the data generated through the numerical integration of Eqs. (6) and (7). Here are some examples of a successful curvefitting process.

The normalizing constant for da/dN is the upper plateau of the R-curve, R_c^*. The value of this plateau can be computed exactly for the non-hardening case (α=0)

$$R_c = R_{ini}\,(4e\,\rho_0)^{k-1},\quad \rho_0 = R_{ini}/\Delta,\quad k = M/M_{min} \qquad (9)$$

and it can be estimated approximately for the hardening case ($\alpha \neq 0$), namely

$$R_c^* = R_{ini}^*\ RR_I(\alpha)\ [1 + A_0(\rho)\alpha + B_0(k)\alpha^2]$$

$$A_0(\rho) = a_{01} + a_{02}\,\rho + a_{03}\,\rho^2$$

$$B_0(k) = b_{01} + b_{02}\,k + b_{03}\,k^2 \qquad (10)$$

$a_{01} = 0.1386462$ $\quad a_{02} = -0.2845617$ $\quad a_{03} = 0.001734182$

$b_{01} = 293.9287$ $\quad b_{02} = -436.5258$ $\quad b_{03} = 176.642$

It is noted that the function $RR_I(\alpha)$ is defined by the expression standing in front of the square bracket in Eq. (12). Introducing a non-dimensional quantity $\delta = R_c/R_{ini}$ for the non-hardening case, and $\delta^* = R_c^*/R_{ini}^*$ for the hardening case, we can re-write equations (6) and (7) as follows

$$(da/dN)_{\alpha=0} = R_{ini}\ F(r, \Delta y, \rho_0, 0)\ \delta$$
$$(da/dN)_{\alpha\neq 0} = R_{ini}^*\ F(r, \Delta y, \rho_0, \alpha)\ \delta^* \qquad (11)$$

in which the symbol F is used to denote the integrals occurring in Eqs. (6) and (7). The ratios R_{ini}^*/R_{ini} and δ^*/δ are known functions of the hardening index α, namely

$$R_{ini}^*/R_{ini} = 1 - 2\alpha,\quad A = 0.6744,\quad B = 15.36945$$
$$\delta^*/\delta = (1 - 2\alpha)\,\frac{1 - 0.5\alpha}{1 - \alpha}\,\frac{1 + .2217\alpha}{1 + A\alpha + B\alpha^2}\,[1 + A_0(\rho_0)\alpha + B_0(k)\alpha^2] \qquad (12)$$

The nondimensional function F can be approximated through a curve-fitting technique. Considering two ranges of the independent variable Δy

<u>range I</u> $\quad 0.15 \le y \le 0.65$ $\quad$ (low cycle fatigue)

<u>range II</u> $\quad 0.65 \le y \le 0.90$ $\quad$ (near critical fatigue)

we obtain the following approximate closed-form expressions

$$\tilde{F}(r, \Delta y, \rho_0, \alpha) = \begin{cases} A_1(\rho_0)\ B_1(r)\ (\Delta y)^{C_1(\alpha)}, & \text{range I} \\ A_2(\rho_0)\ B_2(r)\ \exp[C_2(\alpha)\ \Delta y], & \text{range II} \end{cases} \qquad (13)$$

where

$$A_1(\rho_0) = a_{11} + a_{12}\rho_0 + a_{13}\rho_0^2 + a_{14}\rho_0^3 + a_{15}\rho_0^4 \qquad (14)$$

$$B_1(r) = b_{11} + b_{12}r + b_{13}r^2 + b_{14}r^3 + b_{15}r^4$$

$$C_1(\alpha) = c_1 + c_2\alpha$$

$a_{11} = 2.995075$ $\quad a_{12} = -.123357$ $\quad a_{13} = .002556864$

$a_{14} = -.000024687$ $\quad a_{15} = 8.89 \times 10^{-8}$

$b_{11} = .9991765$ $\quad b_{12} = -4.524384$ $\quad b_{13} = 72.86284$

$b_{14} = -299.4926$ $\quad b_{15} = 233.2185$

$c_1 = 1.681546$ $\quad c_2 = -0.5929918$

and

$$A_2(\rho_o) = a_{21} + a_{22}\rho_o + a_{23}\rho_o^2 + a_{24}\rho_o^3 + a_{25}\rho_o^4 \tag{15}$$

$$B_2(r) = b_{21} + b_{22}r + b_{23}r^2 + b_{24}r^3 + b_{25}r^4$$

$$C_2(\alpha) = d_1 + d_2\alpha + d_3\alpha^2 + d_4\alpha^3 + d_5\alpha^4$$

$a_{21} = .08563$ $\quad a_{22} = -.002233073$ $\quad a_{23} = -4.88602 \times 10^{-6}$

$a_{24} = 6.362336 \times 10^{-7}$ $\quad a_{25} = -4.3365 \times 10^{-9}$

$b_{21} = .999189$ $\quad b_{22} = 8.232657$ $\quad b_{23} = -196.7776$

$b_{24} = 1494.571$ $\quad b_{25} = -3632.513$

$d_1 = 4.296161$ $\quad d_2 = -19.96791$ $\quad d_3 = 323.0727$

$d_4 = -1416.393$ $\quad d_5 = 1830.035$

Maximum error in $\tilde{F}$ within range I is less than 1.11%, while for the range II the maximum error is less than 4%. As we can see, the low cycle fatigue is represented by a power law, cf. the first equation in (13), while the near-critical fatigue process is described by an exponential law, cf. the second equation in (13).

Using Eqs. (12) through (15) we can explain the dependence of the rate da/dN on

- the range, $\Delta y = \Delta J/J_c$
- R-ratio, $r = K_{min}/K_{max}$
- ductility parameter, ρ_o
- strain hardening index, α.

Figures 1, 2 and 3 show the results of such parametric studies. Figure 4 shows the reduction ratio

$$\frac{(da/dN)_{\alpha\neq 0}}{(da/dN)_{\alpha=0}} = RR_{II}(\alpha)\,(1 - 2\alpha)(\delta^*/\delta) \begin{cases} g_1(\Delta y,\alpha), & \text{range I} \\ g_2(\Delta y,\alpha), & \text{range II} \end{cases} \tag{16}$$

$$g_1(\Delta y,\alpha) = D_1(\alpha) + E_1(\alpha)\log(\Delta y)$$

$$g_2(\Delta y,\alpha) = D_2(\alpha) + E_2(\alpha)\,\Delta y$$

$$D_1(\alpha) = .99688 - .15280\alpha$$

$$E_1(\alpha) = -\ .00156 - .50401\alpha - .65052\alpha^2 \tag{16a}$$

$$D_2(\alpha) = 1.599501 - 0.299349\alpha - 37.56253\alpha^2 - 45.18395\alpha^3$$

$$E_2(\alpha) = -5.763548 + 92.73087\alpha - 402.4988\alpha^2 + 515.3295\alpha^3$$

The ratio δ^*/δ is defined by the second equation in (12). The reduction ratio, RR_{II}, is shown as a function of the strain hardening index, for fixed values of other pertinent parameters, and at two levels of the non-dimensional range of J-integral, $\Delta y = 0.1$ and 0.6 (for $\Delta y \geq 0.6$ there is hardly any noticeable dependence of the reduction ratio on Δy). The function $RR_I(\alpha)$ was defined previously by Eq. (12), namely,

$$RR_I(\alpha) = (1-2\alpha)\,\frac{1-0.5\alpha}{1-\alpha}\,\frac{1+0.2217\alpha}{1+0.6744\alpha + 15.36945\alpha^2}\,[1+A_o(\rho_o)\alpha + B_o(k)\alpha^2] \tag{16b}$$

CONCLUSIONS

Low cycle and near-critical fatigue processes were described analytically by a <u>power law</u> and by an <u>exponential law</u>, respectively, see Eqs. (13). The normalizing constant which multiplies the nondimensional rate of fatigue crack growth (F) has been represented as a product of three material characteristics (star is removed when $\alpha = 0$), i.e.,

$$R_c^* = (\Delta)\left(\frac{R_{ini}^*}{\Delta}\right)\,\delta^*\left(\frac{R_{ini}^*}{\Delta},\ \frac{M}{M_{min}},\ \alpha\right) \tag{17}$$

or, with n denoting the strain hardening exponent and with

$$\rho_o = R_{ini}^*/\Delta,\ k = M/M_{min},\ \alpha = 1/(n+1) \tag{18}$$

a more compact form follows

$$R_c^* = (\Delta)(\rho_o)\ (1-2\alpha)\ \delta^*(\rho_o,\ k,\ \alpha) \tag{19}$$

The analysis performed here reveals the numerical values for the δ-parameter, which equals the ratio of the upper plateau to the threshold level of the cyclic R-curve, see Fig. 5. Ratio R_{ini}^*/Δ may be identified with the ductility parameter, $\rho_i \sim \epsilon_{pf}/\epsilon_o$, where ϵ_{pf} denotes the plastic component of strain at failure. The most essential quantity, though, which is of fundamental nature in the studies of subcritical crack growth, namely the size of the process zone, Δ, within the framework of the present theory remains an "undetermined" parameter. It's specific value, being outside the range of the continuum mechanics, may be suggested by microstructural considerations, or it should be established by an experiment. In this way, the relations derived here and based on the principles underlying mechanics of fracture, can be used as a "bridge" between a continuum description of material response to fracture and the more fundamental, microstructural representation of material behavior.

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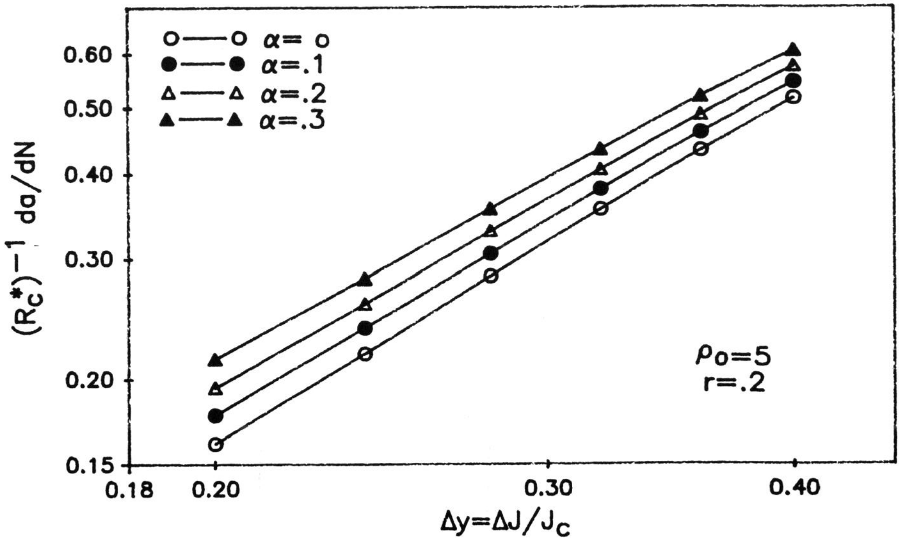


FIGURE 1. EFFECT OF STRAIN HARDENING (α) ON THE RATE OF FATIGUE CRACK GROWTH, da/dN

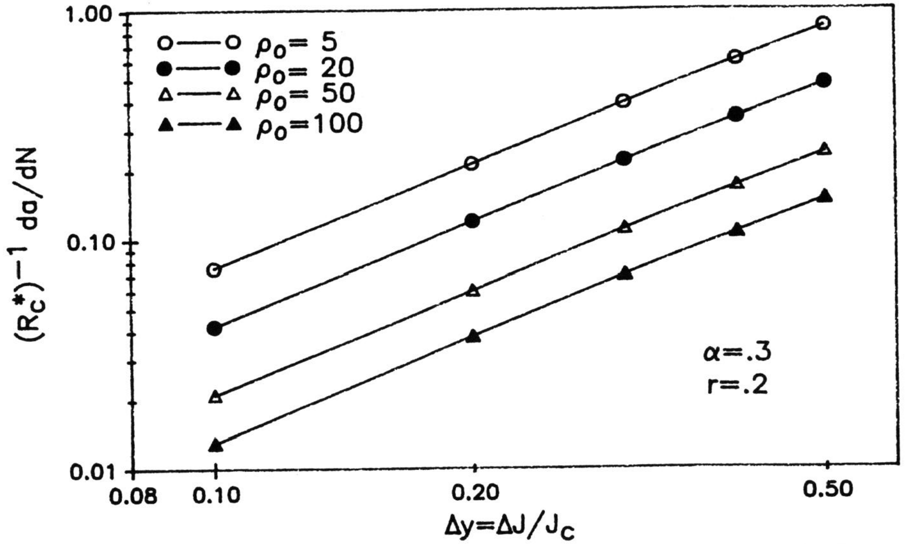
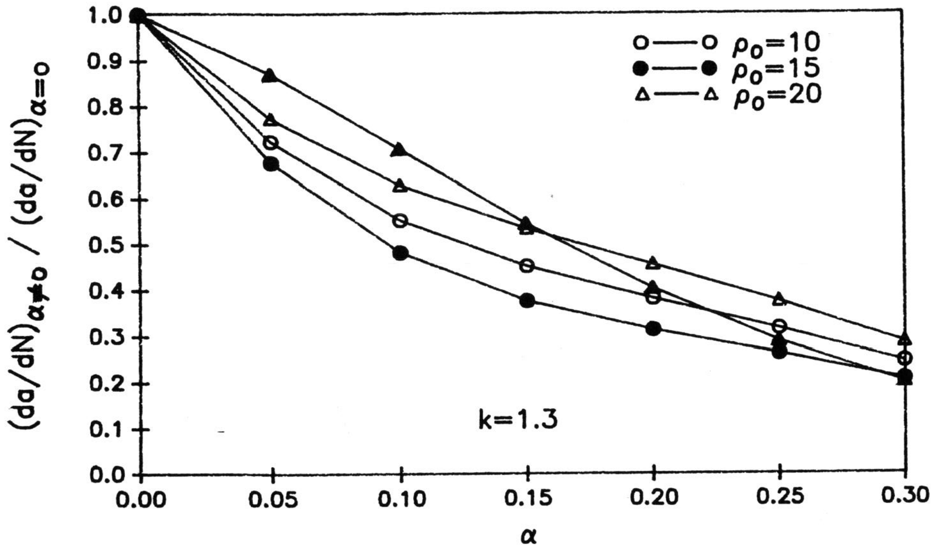
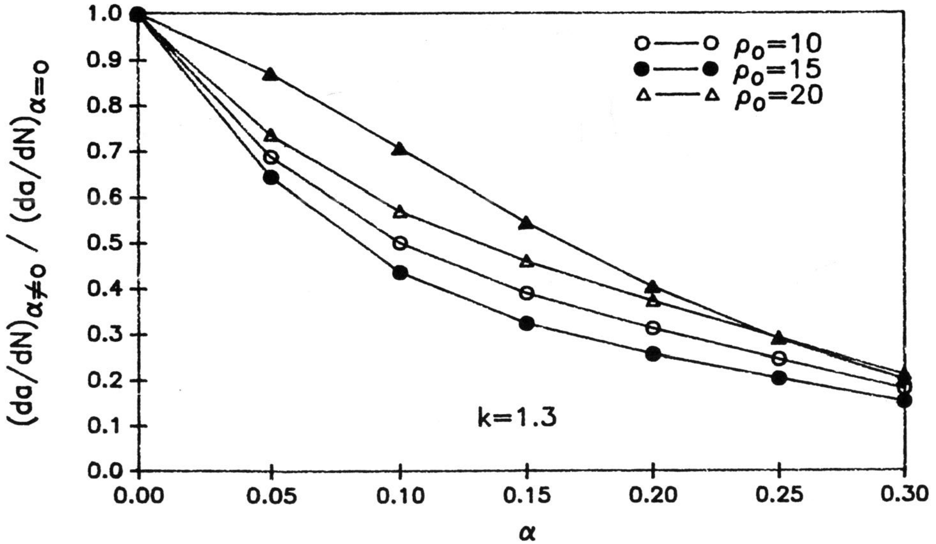


FIGURE 2. EFFECT OF DUCTILITY PARAMETER (ρ_o) ON THE RATE da/dN



(a). x=.1



(b). x=.6

FIGURE 4. REDUCTION OF THE RATE da/dN DUE TO INCREASING STRAIN HARDENING INDEX, $\alpha=1/(1+n)$

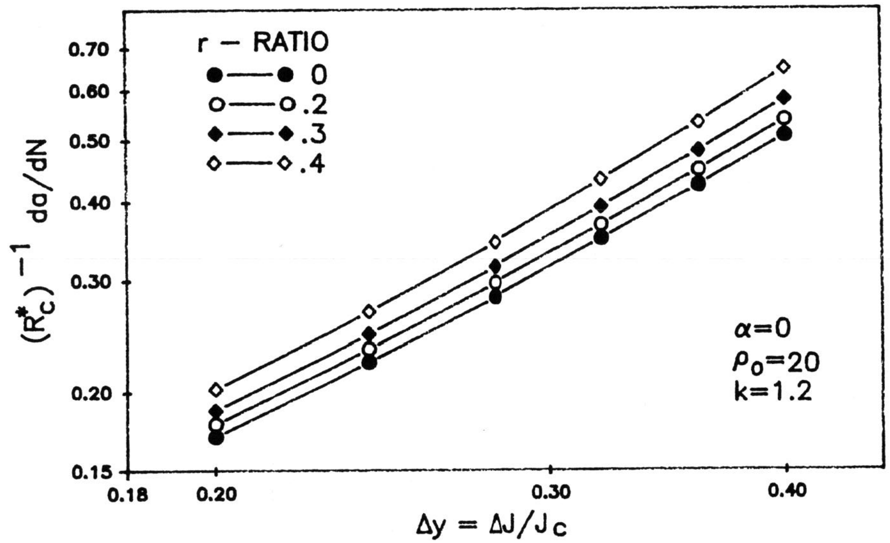


FIGURE 3. EFFECT OF r–RATIO ON THE RATE OF FATIGUE CRACK GROWTH, da/dN

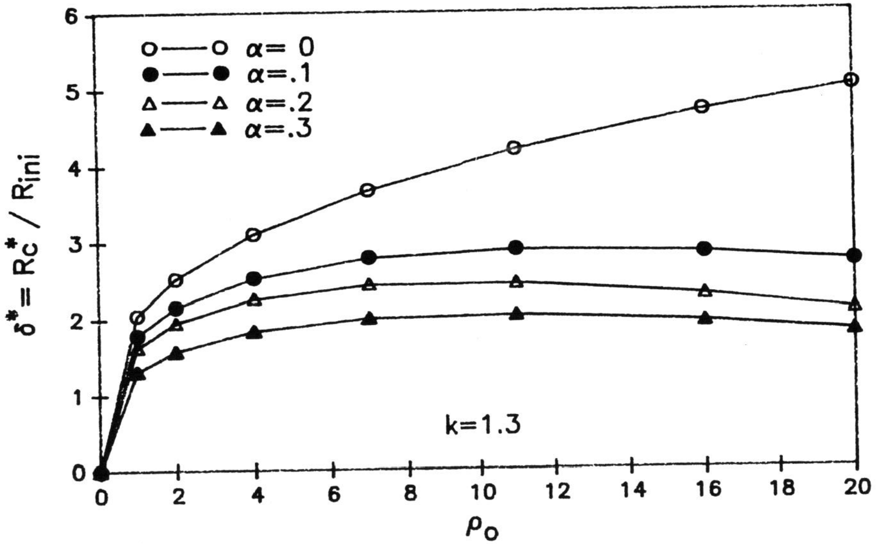


FIGURE 5. RATIO (δ^*) OF THE UPPER PLATEAU OF THE CYCLIC RESISTANCE CURVE TO ITS INITIATION (THRESHOLD) LEVEL