Validity of Asymptotic Crack Tip Solutions for Plastic Materials

PETER GUDMUNDSON

Department of Strength of Materials and Solid Mechanics, The Royal Institute of Technology, S-100 44 Stockholm, Sweden

ABSTRACT

Upper bounds for the size of the zones in which asymptotic stress and strain fields dominate the crack tip fields have been estimated for rate independent plastic materials with low linear or power law hardening and a viscoplastic material with a high non-dimensional fluidity. Results are presented for steady state crack growth and stationary cracks in small scale yielding modus I and III loading conditions. It is shown that especially for steady state crack growth the zone of asymptotic dominance is extremely small also for moderately small hardenings. A conclusion which is drawn is that the perfectly plastic solutions describe the stress and strain fields at distances from the crack tip of the order of characteristic material dimensions also for materials with low hardening or high fluidity.

KEYWORDS

Asymptotic Solutions, Plasticity, Viscoplasticity, Stationary Crack, Growing Crack.

INTRODUCTION

For the determination of fracture criteria it is of importance to analyse the stress and strain fields close to a crack tip. Generally it is possible to derive asymptotic solutions which depend on some loading parameter and are valid as the distance from the crack tip approaches zero. Such solutions have been presented for a large class of material models and loading conditions (Amazigo and Hutchinson, 1977; Chitaley and McClintock, 1971; Drugan, Rice and Sham, 1982; Hult and McClintock, 1956; Hutchinson, 1968; Ponte Castaneda, 1987; Rice, 1968; Rice and Rosengren, 1968). These results are of interest concerning fracture criteria only if it can be shown that the asymptotic solutions describe the stress and strain fields over distances which are larger than characteristic material dimensions.

In the present investigation, the size of the zone in which the asymptotic solutions dominate is estimated. The analysis is limited to rate independent plasticity with low hardening and viscoplasticity with a large dimensionless fluidity parameter. These material models closely resemble a perfectly plastic material. It can therefore be expected that the stress–strain solution will be similar to a corresponding solution for a perfectly plastic material. Nevertheless, the hardening materials and the viscoplastic materials show contrary to a perfectly plastic material a stress singularity close to the crack tip (Amazigo and Hutchinson, 1977; Hutchinson, 1968; Ponte Castaneda, 1987; Rice, 1968; Rice and Rosengren, 1968). Well outside the singular zone, however, the perfectly plastic solution should give a good approximation to the solution for the considered material models. These facts have been utilized in the present analysis for the estimation of the region in which the perfectly plastic solution describes the stress and strain fields. The limit of this region should be an estimate of an upper limit to the zone in which the singular asymptotic solution dominates.

The analysis is based on an expansion of the solution in a small parameter which is related to hardening or fluidity respectively. The zeroth order solution is shown to be the perfectly plastic solution as can be expected. The criterion for dominance of the zeroth order solution is based on the size of the first order correction in effective stress. From the condition that this correction reaches a certain limit, a critical distance from the crack tip can be determined. Results are presented for stationary and steady state growing cracks in small scale yielding modus I and III loadings.

GOVERNING EQUATIONS

A stationary or a steady state quasistatically growing crack under small scale yielding conditions is considered, i. e. the stresses tend to the elastic solutions defined by the stress intensity factors for large distances in comparison to the size of the plastic zone.

In the analysis it is useful to introduce dimensionless variables. A characteristic length l is defined by $l = K^2/\sigma_0{}^2$, where σ_0 denotes the tensile yield stress and K the stress intensity factor (Modus I or III). The size of the plastic zone will be of order l. The time scale is denoted by T. For a growing crack in steady state, the characteristic time T can be defined as $T=l/v$, where v denotes the crack tip velocity. In the case of a stationary crack, the characteristic time must be defined by time constants of the external loading. If dimensionless variables are introduced as,

$$\bar{x}_i = l x_i, \qquad \bar{t} = Tt,$$
$$\bar{u}_i = u_i l \sigma_0/E \qquad \bar{\epsilon}_{ij} = \epsilon_{ij}\sigma_0/E \tag{1}$$
$$\bar{\sigma}_{ij} = \sigma_0 \sigma_{ij},$$

where the unbarred variables x_i, t, u_i, ϵ_{ij} and σ_{ij} denote dimensionless variables, the equilibrium equations and the constitutive law can be formulated as,

$$\sigma_{ji,j} = 0, \tag{2}$$
$$\dot{\epsilon}_{ij} = (1+\nu)\dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij} + \dot{\epsilon}^p_{ij}, \tag{3}$$

where the plastic strains, ϵ^p_{ij}, for rate independent plasticity are defined by,

$$\dot{\epsilon}^p_{ij} = \lambda s_{ij}, \tag{4}$$
$$\lambda = \frac{3}{2}\frac{1}{\sigma_e}\cdot\dot{\epsilon}^p_e, \tag{5}$$

and for a viscoplastic material according to the Perzyna model [9] by,

$$\dot{\epsilon}^p_{ij} = \gamma(\sigma_e - 1)^N\cdot\frac{s_{ij}}{\sigma_e}, \tag{6}$$
$$\gamma = \bar{\gamma} TE/\sigma_0, \tag{7}$$

where $\bar{\gamma}$ denotes a fluidity parameter (Perzyna, 1963). In the equations above, a comma denotes a differentiation with respect to a x- coordinate and a dot a differentiation with respect to t. The effective stress σ_e and effective plastic strain ϵ^p_e are determined from,

$$\sigma_e^2 = \frac{3}{2}s_{ij}s_{ij}, \qquad (\dot{\epsilon}^p_e)^2 = \frac{2}{3}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}, \tag{8}$$

where s_{ij} denotes the dimensionless stress deviator.

The small scale yielding problem is thus defined by eqs.(2,3), eqs.(4,5) or eqs.(6,7), the compatibility equation,

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{9}$$

and the boundary conditions which are expressed by traction free crack surfaces and

$$\sigma_{ji}n_j = \frac{1}{\sqrt{2\pi r}} f_{ji}(\varphi) n_j g(Tt) \tag{10}$$
for $r \to \infty$,

where $f_{ij}(\varphi)$ denotes the angle dependence of the linear elastic stress intensity solution, n_j a normal vector and $g(Tt)$ an eventual time dependent loading.

PERFECT PLASTICITY

The idea of the present analysis is to treat the effects of hardening or viscoplasticity as perturbations to the solution for perfect plasticity. Small scale yielding solutions for perfect plasticity are known for both stationary cracks (Hult and McClintock, 1956; Rice, 1968) and growing cracks (Chitaley and McClintock, 1971; Drugan *et al.*, 1982).

The perfectly plastic solution for a stationary crack in modus III has been derived by Hult and McClintock, 1956. The effective plastic strain is given by,

$$\epsilon^p_e = \frac{2(1+\nu)}{\pi}\cdot\frac{\cos(\varphi)}{r}, \tag{11}$$
$$\dot{\epsilon}^p_e = 2\cdot\frac{\dot{g}}{g}\cdot\epsilon^p. \tag{12}$$

For steady state crack growth in modus III the effective plastic strain and strain rate in front of the crack tip have been determined by Chitaley and McClintock, 1971,

$$\epsilon^p_e = \frac{2}{3}(1+\nu)\cdot[\ln(\frac{c_1}{r}) + \frac{1}{2}\ln^2(\frac{c_1}{r})], \tag{13}$$
$$\dot{\epsilon}^p_e = \frac{2}{3}(1+\nu)\cdot\frac{1}{r}\cdot[1 + \ln(\frac{c_1}{r})], \tag{14}$$

where $c_1 \simeq 1$ (Dean and Hutchinson, 1980).

A stationary crack in modus I has an effective plastic strain in the central fan field which is of the order of,

$$\epsilon_e^p = \frac{c_2}{r}, \qquad (15)$$

$$\dot{\epsilon}_e^p = 2 \cdot \frac{\dot{g}}{g} \cdot \epsilon^p, \qquad (16)$$

where $c_2 \simeq 0.2$ (Rice, 1968).

For a steady state growing crack in modus I the effective plastic strain in the centred fan field has been derived by Drugan, Rice and Sham, 1982, as,

$$\epsilon_e^p = \frac{5-4\nu}{3}[1 + \frac{1}{\sqrt{2}}\ln(\tan(\varphi/2)/\tan(\pi/8))] \cdot \ln(\frac{c_3}{r}), \qquad (17)$$

$$\dot{\epsilon}_e^p = \frac{2}{3} \cdot \frac{5-4\nu}{2\sqrt{2}} \cdot \frac{1}{r} \cdot \ln(\frac{c_3}{r}), \qquad (18)$$

where $c_3 \simeq 0.2$ (Drugan, Rice and Sham, 1982).

PERTURBATION ANALYSIS

Linear Strain Hardening

For linear strain hardening the relationship between effective stress and plastic strain is

$$\sigma_e = 1 + \frac{H}{E} \cdot \epsilon_e^p. \qquad (19)$$

If the hardening is small, i.e. $H/E = \delta << 1$, the solution can be expressed as an expansion in δ,

$$\sigma_e = \sigma_{e0} + \delta\sigma_{e1} + \ldots,$$
$$\epsilon_e^p = \epsilon_{e0}^p + \delta\epsilon_{e1}^p + \ldots, \qquad (20)$$

and similarly for the remaining variables. An introduction of eq.(20) into eqs.(2–5,8–10,19) yields that the zeroth order approximation is given by the perfectly plastic solution and that

$$\sigma_{e0} = 1, \qquad \sigma_{e1} = \epsilon_{e0}^p, \qquad (21)$$

where ϵ_{e0}^p is determined from eqs.(11,13,15,17) depending on the loading condition.

Power Law Hardening

The effective stress–strain relationship for a power law hardening material is defined by,

$$\epsilon_e^p = \sigma_e^n - \sigma_e, \qquad (22)$$

where n is a hardening parameter.

A small hardening is equivalent to a large n. If $1/n = \delta << 1$, the solution can be expanded in the same manner as in eq.(20). The first order approximation of eq.(22) then yields,

$$\sigma_{e0} = 1, \qquad \sigma_{e1} = \ln(1 + \epsilon_{e0}^p), \qquad (23)$$

where ϵ_{e0}^p is determined from eqs.(11,13,15,17) depending on the loading condition. The zeroth order solution is as for linear strain hardening defined by the perfectly plastic solution.

Viscoplasticity

The material law for viscoplasticity according to the Perzyna model is given in eqs.(6,7). If the dimensionless fluidity parameter γ is large, then the solution can be expanded in $1/\gamma = \delta << 1$. It turns out that the expansion in eq.(20) must be slightly modified in the viscoplastic case,

$$\sigma_e = \sigma_{e0} + \delta^s\sigma_{e1} + \ldots,$$
$$\epsilon_e^p = \epsilon_{e0}^p + \delta^s\epsilon_{e1}^p + \ldots, \qquad (24)$$

where s is a constant. The remaining variables are expanded in a similar way.

An introduction of eq.(24) in eq.(6) yields,

$$\sigma_{e0} = 1, \quad s = \frac{1}{N}, \quad \dot{\epsilon}_{ij0}^p = (\sigma_{e1})^N s_{ij0}. \qquad (25)$$

Equation (25) can be identified with the flow rule for a perfectly plastic deformation. Thus the zeroth order solution is defined by the perfectly plastic solution and

$$\sigma_{e1} = (\frac{3}{2}\dot{\epsilon}_{e0}^p)^{1/N}, \qquad (26)$$

where $\dot{\epsilon}_{e0}^p$ is given in eqs.(12,14,16,18) depending on the loading condition.

RESULTS

In the previous Section the first order correction to the effective stress was estimated for different material models. The perfectly plastic solution will be a good approximation if the correction is small. The condition that the first order correction in effective stress should be smaller than some tolerance Δ can be formulated as,

$$\delta^s \sigma_{el} < \Delta, \qquad (27)$$

where s is equal to 1/N for viscoplasticity and one for rate independent plasticity.

Since σ_{el} depends on the distance from the crack tip r, through eqs.(11–18,21,23,26), eq.(27) implies that a lower bound for r can be determined,

$$r > r_{cr} \qquad (28)$$

where r_{cr} will vary with material constants, loading conditions and the parameter Δ.

The critical distances for different loading conditions and material models have been determined and the results are presented in Tables 1–3.

Table 1. Non–dimensional critical distance for a linear strain hardening material as a function of hardening H/E and the tolerance Δ.

Mode III	Stationary	$r_{cr} = \frac{2(1+\nu)}{\pi} \cdot \frac{H}{E} \cdot \frac{1}{\Delta}$
Mode III	Growing	$r_{cr} = c_1 \cdot \exp(-(\frac{3}{1+\nu} \cdot \frac{E}{H} \cdot \Delta)^{1/2})$
Mode I	Stationary	$r_{cr} = c_2 \cdot \frac{H}{E} \cdot \frac{1}{\Delta}$
Mode I	Growing	$r_{cr} = c_3 \cdot \exp(-(\frac{3}{5-4\nu} \cdot \frac{E}{H} \cdot \Delta))$

Table 2. Non–dimensional critical distance for a power law hardening material as a function of the hardening n and the tolerance Δ.

Mode III	Stationary	$r_{cr} = \frac{2(1+\nu)}{\pi} \cdot (\exp(n\Delta) - 1)^{-1}$
Mode III	Growing	$r_{cr} = c_1 \cdot \exp(-[\frac{3}{1+\nu}(\exp(n\Delta) - 1)]^{1/2})$
Mode I	Stationary	$r_{cr} = c_2 \cdot (\exp(n\Delta) - 1)^{-1}$
Mode I	Growing	$r_{cr} = c_3 \cdot \exp(-[\frac{3}{5-4\nu}(\exp(n\Delta) - 1)])$

Table 3. Non–dimensional critical distance for a viscoplastic material as a function of the fluidity γ and the tolerance Δ.

Mode III	Stationary	$r_{cr} = \frac{6}{\pi} \cdot \frac{1+\nu}{\gamma} \cdot (\frac{1}{\Delta})^N \cdot \frac{\dot{g}}{g}$
Mode III	Growing	$r_{cr} = \frac{1+\nu}{\gamma} \cdot (\frac{1}{\Delta})^N \cdot \ln(\frac{\gamma}{1+\nu} \cdot \Delta^N c_1)$
Mode I	Stationary	$r_{cr} = c_2 \cdot \frac{3}{\gamma} \cdot (\frac{1}{\Delta})^N \cdot \frac{\dot{g}}{g}$
Mode I	Growing	$r_{cr} = \frac{5-4\nu}{2\sqrt{2}} \cdot \frac{1}{\gamma} \cdot (\frac{1}{\Delta})^N \cdot \ln(\frac{2\sqrt{2}}{5-4\nu} \cdot \gamma \cdot \Delta^N c_3)$

The results in Tables 1–3 can be compared to alternative estimates presented in the literature. Dunayevsky and Achenbach, 1982, determined a critical distance for a linearly hardening material in steady state modus III crack growth. They found that it was of the order of $O(\exp[-\sqrt{(E/H)}])$. As can be observed from Table 1, the E/H– dependence is in agreement with the present analysis. Nilsson and Ståhle, 1988, have also derived critical distances for various material models and loading conditions. Their results were based on the size of the second order term compared to the first order term in the asymptotic expansion. This is a different criterion compared to the present analysis, where the primary analysis concerned the dominance of a perfectly plastic solution. The conclusions by Nilsson and Ståhle, 1988, are however in agreement with the results presented here.

As an example a material with a linear strain hardening of $H/E = 0.01$ and a power law hardening material with the same initial hardening ($n = 101$) have been evaluated for mode I loadings. The tolerance Δ, see eq.(27), was selected to be 0.1. For the linear hardening material, the critical distance will be $2 \cdot 10^{-2}$ and $7 \cdot 10^{-5}$ for a stationary and a growing crack respectively. The power law hardening material has for a stationary crack a critical distance of $8 \cdot 10^{-6}$ and for a growing crack 10^{-8000}. These results indicate that the perfectly plastic solution should define a sufficiently accurate solution at relevant distances from the crack tip with the possible exception of a stationary crack in a linearly hardening material.

CONCLUSIONS

The final results presented in Tables 1–3 show that the asymptotic solutions dominate in a very small region for small hardenings or high fluidities, especially for steady state crack growth. It is also observed that the zone is much smaller for a power law hardening in comparison to a linear hardening. A conclusion which can be drawn is that for hardenings below a certain limit and for fluidities above some other limit, the asymptotic solutions have no practical significance. Instead, the perfectly plastic solution determines the stress and strain fields at distances of the order of characteristic material dimensions.

REFERENCES

Amazigo, J.C. and J.W. Hutchinson (1977). Crack–Tip Fields in Steady Crack–Growth with Linear Strain–Hardening. *J. Mech. Phys. Solids*, **25**, 81–97.

Chitaley, A.D. and F.A. McClintock (1971). Elastic–Plastic Mechanics of Steady Crack Growth under Anti–Plane Shear. *J. Mech. Phys. Solids*, **19**, 147–163.

Dean, R.H. and J.W. Hutchinson (1980). Quasi-Static Steady Crack Growth in Small-Scale Yielding. In: *Fracture Mechanics: Twelfth Conference*, pp. 383–405. ASTM-STP 700, American Society for Testing and Materials, Philadelphia.
Drugan, W.J., J.R. Rice and T-L Sham (1982). Asymptotic Analysis of Growing Plane Strain Tensile Cracks in Elastic-Ideally Plastic Solids. *J. Mech. Phys. Solids*, **30**, 447–473.
Dunayevsky V. and J.D. Achenbach (1982). Radial Nonuniformity of the Fields Near a Moving Crack Tip in a Material With Linear Strain Hardening. *J. Appl. Mech.*, **49**, 646–647.
Hult J.A.H. and F.A. McClintock (1956). Elastic-Plastic Stress and Strain Distributions Around Sharp Notches Under Repeated Shear. In: *Proceedings of the 9th International Congress of Applied Mechanics*, pp. 51–58. Brussels.
Hutchinson J.W. (1968). Singular Behaviour at the End of a Tensile Crack in a Hardening Material. *J. Mech. Phys. Solids*, **16**, 13–31.
Nilsson F. and P. Ståhle (1988). Crack Growth Criteria and Crack Tip Models. To be published in *Solid Mech. Arch.*
Perzyna P. (1963). The Constitutive Equations for Rate Sensitive Plastic Materials. *Quarterly of Appl. Math.*, **20**, 321–332.
Ponte Castaneda P. (1987). Asymptotic Fields in Steady Crack Growth with Linear Strain-Hardening. *J. Mech. Phys. Solids*, **35**, 227–268.
Rice J.R. (1968). Mathematical Analysis in the Mechanics of Fracture. In: *FRACTURE: An Advanced Treatice* (H. Liebowitz, ed), Vol. 2, pp. 191–311. Academic Press, New York, London.
Rice J.R. and G.F. Rosengren (1968). Plane Strain Deformation Near a Crack Tip in a Power-Law Hardening Material. *J. Mech. Phys. Solids*, **16**, 1–12.