Studies on a Three-dimensional Semi-infinite Crack

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ABSTRACT

The purpose of this paper is to report on a new approach to deal with the analytical evaluation for the solution of a three dimensional problem of a semi-infinite stress-free crack in an unbounded, linearly, elastic, isotropic medium.

The fundamental solution of a Green's function for a point load acting in the interior of an unbounded, homogeneous, isotropic, thick layer is used in connection with the boundary integral equations technique in order to obtain a formulation for the stress and displacement fields around the crack. This new formulation is based in the Green function obtained by the author and defined elsewhere. The formulation may analogously be applied to other three-dimensional cracked finite or infinite configurations.

KEYWORDS

Plate solution, 3-D Crack, Boundary Element Method.

1. REVIEW OF THE BOUNDARY INTEGRAL EQUATION METHOD IN THREE-DIMENSIONAL CRACKED CONFIGURATIONS

The boundary-integral equation method for elastic stress analysis is now well established as a complementary tool to finite element methods (Banerjee, 1981; Brebbia, 1978). Some of the advantages over other numerical solution procedures are well known. Foremost is the reduction of the dimensionality of the problem by one. Secondly, it is specially well suited to problems extending to infinity. Thirdly, since the formulation is based on fundamental solutions that satisfy the governing differential equations, approximation of the variables is required only on the boundary in the numerical solution of the equations. Once the boundary solution has been numerically obtained, interior values may easily determined. These features are particularly advantageous for modeling regions with high stress gradients with great accuracy and efficiency, making this technique an appealing tool for numerical solution of problems in linear-elastic fracture mechanics.

In a number of papers, various boundary integral formulation have been shown to be useful for particular classes of boundary-value problems. The main difference among them focusses on the use of diverse Green's functions which are more appropriate for dealing with the geometry of the problem considered. Thus, for generic two-dimensional and three-dimensional elastostatic problems, Kelvin's fundamental solution of a concentrated load in an infinite medium has been widely used (Risso, 1967; Cruse, 1969). Also, for problems involving a free–surface the solution presented by Melan (1932), for the stress distribution due to point loads applied within the isotropic half–space, or the one given by Mindlin (1936), for the half–space are of an utter interest. These fundamental solutions have been applied to the boundary element technique (BEM) by Telles & Brebbia (1981).

Further, for elastostatic problems containing cutouts, Kelvin's solutions in 2D and 3D have been profusely used. On this lane rests the pioneering work of Cruse and VanBuren (1971). Since then a huge stack of papers involving cracks, tackled by the boundary integral technique, have been published. Among them, the early study by Snyder and Cruse (1973) is worth to be mentioned. In their report several finite anisotropic two–dimensional plates were analised by using the Green function corresponding to a concentrated load applied to an infinite plane containing a semi–infinite stress free crack. Thus, two–dimensional crack problems with finite geometry are then solved by the BEM with no crack modeling required; the crack presence being accounted for by the new fundamental solutions.

Several numerical formulations have been devised for the application of the boundary integral equation method to fracture mechanics problems in three–dimensions. Pioneering work, in this field, was that of Cruse (1970, 1972, 1973, 1974, 1975, 1977, 1971, 1977) and Tan & Fenner (1978), among others (1985).

It is the purpose of this paper to report on the application of a new analytical method to the solution of the three–dimensional through crack on an infinite plate, as well as its use to other 3–D cracked, finite or infinite configurations.

2. INTEGRAL REPRESENTATIONS

This section deals with the integral formulation of the elastostatic problem for the layer–space defined for the case of a semi–infinite three–dimensional crack. The following developments are based on previous progress, by the author (1985, 1987, 1987), in the development of the necessary theoretical background for the determination of stresses and displacements in three–dimensional plates subjected to concentrated point loads.

Consider a linear, elastic, isotropic, homogeneous medium occupying a space described by an infinite three–dimensional thick layer subjected to a concentrated body force acting at a point ξ in a direction indicated by the unit vector $\mathbf{e}^j$ and having the form

$$F_i(\mathbf{x}) = \delta(\mathbf{x} - \xi)\delta_{ij}\mathbf{e}^j,$$

where $\mathbf{x}$ is a point of the body, $\delta(\cdot)$ is the Dirac delta function and δ_{ij} is Kronecker's delta. The response of this body, obtained as the solution of the equations of elastostatics is given by the displacement vector and the stress tensor

$$u_i(\mathbf{x}) = u_i^j(\mathbf{x};\xi)\mathbf{e}^j,$$
$$\sigma_{ik}(\mathbf{x}) = \sigma_{ik}^j(\mathbf{x};\xi)\mathbf{e}^j,$$

where the second–order symmetric tensor $u_i^j(\mathbf{x})$ and the third–order tensor σ_{ik}^j (fundamental singular solution) express the displacement component in the i–direction and the stress ij–component, respectively, at the point $\mathbf{x}$ due to a concentrated force of magnitude $\mathbf{F}$ acting at the point ξ in the j–direction.

The fundamental solution pair $[\mathbf{u}^j(\mathbf{x};\xi), \mathbf{s}^j(\mathbf{x};\xi)]$, defined for all points $\mathbf{x}$ except ξ, is called the three–dimensional Layer–state and is characterised by the properties established elsewhere.

The boundary integral equation technique is based on the Betti reciprocal work theorem, which provides an integral relationship between two elastostatic states. If one of them corresponds to a fundamental singular solution, Somigliana's identity is obtained. Depending on the fundamental solution used, various formulations may be available and more suitable for solving certain problems.

Consider now a linear, elastic, isotropic, homogeneous body R, with surface ∂R, subjected to body forces $\mathbf{F}(\mathbf{x})$, surface tractions $\mathbf{s}(\mathbf{x})$ and boundary displacements $\mathbf{u}(\mathbf{x})$ (boundary conditions of Newmann, Dirichlet or mixed type). Using the three–dimensional Layer–state, one can derive Somigliana's integral representation for the displacement and stress field, in the form

$$u_j(\xi) = -\int_{\partial R} s_i^j(\mathbf{x};\xi)u_i(\mathbf{x})\,dA_x + \int_{\partial R} u_i^j(\mathbf{x};\xi)s_i(\mathbf{x})\,dA_x + \int_R u_i^j(\mathbf{x};\xi)F_i(\mathbf{x})\,dV_x, \tag{1}$$

$$\sigma_{ij}(\xi) = -\int_{\partial R} \bar{s}_k^{ij}(\mathbf{x};\xi)u_k(\mathbf{x})\,dA_x + \int_{\partial R} \bar{u}_k^{ij}(\mathbf{x};\xi)s_k(\mathbf{x})\,dA_x + \int_R \bar{u}_k^{ij}(\mathbf{x};\xi)F_k(\mathbf{x})\,dV_x, \tag{2}$$

where ξ is a point of the body, $\mathbf{x}$ a point in the body or on the surface; $\mathbf{u}$, $\mathbf{s}$, $\mathbf{F}$ are the displacement, traction and body force vectors, respectively; $\mathbf{u}^j$, $\mathbf{s}^j$, $\bar{\mathbf{u}}^{ij}$, $\bar{\mathbf{s}}^{ij}$ are the Green tensors as described elsewhere.

The solution of the elastostatic problem can be inferred from the boundary equation, derived from (1)

$$c_{ji}(\xi)u_i(\xi) + \int_{\partial R} s_i^j(\mathbf{x};\xi)u_i(\mathbf{x})\,dA_x = \int_{\partial R} u_i^j(\mathbf{x};\xi)s_i(\mathbf{x})\,dA_x + \int_R u_i^j(\mathbf{x};\xi)F_i(\mathbf{x})\,dV_x, \tag{3}$$

where the tensor $c_{ji}(\xi)$ is equal to 1/2 if ξ is at a smooth surface.

In problems with zero body forces, the last integral in equations (1),(2) and (3) vanish. Thus, equation (3) becomes

$$c_{ji}(\xi)u_i(\xi) + \int_{\partial R} s_i^j(\mathbf{x};\xi)u_i(\mathbf{x})\,dA_x = \int_{\partial R} u_i^j(\mathbf{x};\xi)s_i(\mathbf{x})\,dA_x, \tag{4}$$

which relates the boundary displacements $\mathbf{u}$ and tractions $\mathbf{s}$.

For arbitrary geometries and complicated variations of the related functions, equation (3) or (4) has to be treated numerically.

3. BOUNDARY EQUATION FOR A SEMI-INFINITE THREE-DIMENSIONAL THROUGH-CRACK

Consider a three-dimensional infinite layer R containing a traction free crack L. Assume that the region R is embedded in an infinite plate and that a unit load $\mathbf{F}(\boldsymbol{\theta})$ is applied at some point $\boldsymbol{\theta}$ in the region R but not on the boundary ∂R. The system under consideration and the coordinate axes are depicted in Fig.1.

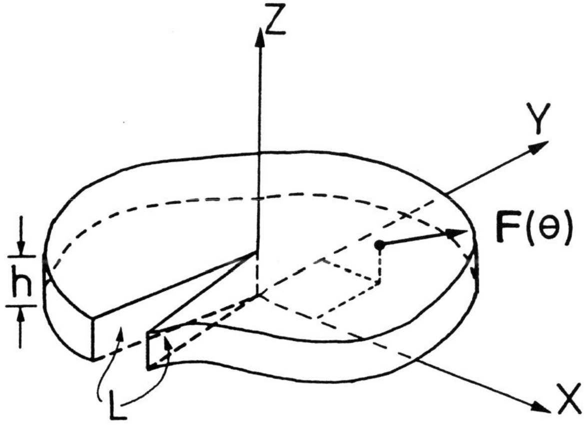

Fig. 1. Three-dimensional infinite layer containing a semi-infinite through-crack subjected to a unit load applied at point $\boldsymbol{\theta}$.

If the load point is represented by a unit force in the k-direction,

$$F_i(\mathbf{x}) = \delta(\mathbf{x} - \boldsymbol{\theta})\delta_{ik}\mathbf{e}^k, \tag{5}$$

the displacement and traction field will be given by

$$u_i(\mathbf{x}) = \hat{u}_i^k(\mathbf{x};\boldsymbol{\theta})\mathbf{e}^k, \tag{6}$$

$$s_i(\mathbf{x}) = \hat{s}_i^k(\mathbf{x};\boldsymbol{\theta})\mathbf{e}^k, \tag{7}$$

where $\hat{\mathbf{u}}^k(\mathbf{x};\boldsymbol{\theta})$ and $\hat{\mathbf{s}}^k(\mathbf{x};\boldsymbol{\theta})$ represent the corresponding solution vectors for displacement and traction, respectively, for a concentrated load parallel to the k-axis.

Let S_u, S_l denote the upper and lower planar surfaces of the plate, respectively, and L the crack surface; thus $\partial R = S_u \cup S_l \cup L$. Since the two planar surfaces are traction free, then

$$\int_{S_u \cup S_l} u_i^j(\mathbf{x};\boldsymbol{\xi}) s_i(\mathbf{x})\, dA_x = 0,$$

and the second integral in (3) is exclusively extended to the surface crack.

Also, the first integral in (3) provides

$$\int_{S_u \cup S_l} s_i^j(\mathbf{x};\boldsymbol{\xi}) u_i(\mathbf{x})\, dA_x = 0,$$

as s^j vanishes there (traction free condition of the analytical point load solution).

The third integral in (3), taking into account (5), has the form

$$\int_R u_i^j(\mathbf{x};\boldsymbol{\xi})\delta(\mathbf{x} - \boldsymbol{\theta})\delta_{ik}\mathbf{e}^k\, dV_x = u_k^j(\boldsymbol{\theta};\boldsymbol{\xi})\mathbf{e}^k.$$

From the above and (4.2), expression (3) results in

$$c_{ji}(\boldsymbol{\xi})\hat{u}_i^k(\boldsymbol{\xi};\boldsymbol{\theta}) = -\int_L s_i^j(\mathbf{x};\boldsymbol{\xi})\hat{u}_i^k(\mathbf{x};\boldsymbol{\theta})\, dA_x + u_k^j(\boldsymbol{\theta};\boldsymbol{\xi}),$$

where summation is implied on the index i.

REFERENCES

ASME. (1977).*Boiler and Pressure Vessel Code. Section XI, Rules for in Service Inspection of Nuclear Power Plant Components, Appendix A-Analysis of Flow Indications.* 175-183.

Atluri, S. N. (ed.) (1986). *Computational Methods in the Mechanics of Fracture.* Vol.2 in *Computational Methods in Mechanics.* Elsevier Science Publishers.

Atluri, S. N., K. Kathiresan and A. S. Kobayashi (1975). Three-dimensional linear elastic fracture mechanics analysis by a displacement-hybrid finite element model. Paper No.L-7/3. *Trans. 3rd Int. Conf. on Struct. Mech. in Reactor Tech.* London.

Banerjee, P. K. and R. Butterfield (1981). *Boundary Element Methods in Engineering Science.* McGraw-Hill.

Bazant, Z. P. and L. F. Estenssoro (1977). General numerical method for three-dimensional singularities in cracked or notched elastic solids. *Proc. 4th Int. Conf. on Fracture.* Uni. of Waterloo. *Fracture* 3,371-385.

Benitez, F. G. and A. J. Rosakis (1985). *Three Dimensional Elastostatics of a Layer and a Layered Medium.* Division of Engineering and Applied Sciences SM85-21, California Inst. of Technology,Pasadena,CA.

Benitez, F. G. and A. J. Rosakis (1987). Three dimensional elastostatics of a layer and a layered medium. *J. Elasticity* 18, 3-50.

Benitez, F. G. and A. J. Rosakis (1987). Force at a point in the interior of a three dimensional elastic layer. *Proceedings of the International Conference on Computational Engineering Mechanics.* Science Press, Beijing, China, 211-224.

Benthem, J. P. (1977). State of stress at the vertex of a quarter-infinite crack in a half space. *Int. J. Solids Struc.*. 13, 479-492.

Brebbia, C. A. (1978).*The Boundary Element Method for Engineers.* Pentech Press, Plymouth.

Cartwright, D. J. and D. P. Rooke (1975). Evaluation of stress intensity factors. *J. Strain Analysis* 10, 217-224.

Cruse, T. A. (1969). Numerical solutions in three-dimensional elastostatics. *Int. J. Solids Struc.* 5, 1259-1274.

Cruse, T. A. (1970). Lateral constraint in a cracked, three-dimensional elastic body. *Int. J. Fract. Mech.* 6, 326-328.

Cruse, T. A. (1972). Numerical evaluation of elastic stress intensity factors by the boundary-integral equation method. In *The Surface Crack : Physical Problems and Computational Solutions.* (J. L. Swedlow ed.) ASME, N. Y. 153-170.

Cruse, T. A. (1973). Application of the boundary-integral equation method to three-dimensional stress analysis. *J. Computer & Structures* 3, 509-527.

Cruse, T. A. (1974). An improved boundary integral equation method for three-dimensional elastic stress analysis. *J. Computer & Structures* **4**, 741–757.

Cruse, T. A. (1975). *Boundary-Integral Equation Method for Three-Dimensional Elastic Fracture Mechanics Analysis*. AFOSR-TR-75-0813.

Cruse, T. A. and G. J. Meyers (1977). Three-dimensional fracture mechanics analysis. *J. Structural Div. ASCE* **103**, 309–320.

Cruse, T. A. and W. VanBuren (1971). Three-dimensional elastic stress analysis of a fracture specimen with an edge crack. *Int. J. Fract. Mech.* **7**, 1–16.

Cruse, T. A. and R. B. Wilson (1977). *Boundary-Integral Equation Method for Elastic Fracture Mechanics Analysis*. AFOSR-TR-78-0355.

Folias, E. S. (1975). On the three-dimensional theory of cracked plates. *J. Appl. Mech.* ASME **42**, 663–674.

Green, A. E. and I. N. Sneddon (1950). The stress distribution in the neighborhood of a flat elliptical crack in an elastic solid. *Proceedings Cambridge Philosophical Soc.* **46**, 159–164.

Hartranft R. J. and G. C. Sih (1970). An approximate three-dimensional theory of plates with applications to crack problems. *Int. J. Engng. Science* **8**, 711–729.

Hellen, T. K. and W. S. Blackburn (1975). *The Calculation of Stress Intensity Factors in Two and Three Dimensions Using Finite Elements*. CEGB Report RD/B/N3378.

Hilton, P. D. and L. N. Gifford (1984). *Computational Fracture Mechanics - Nonlinear and 3-D Problems*. ASME AMD-61.

Hilton, P. D., B. V. Kiefer and G. C. Sih (1978). Specialised finite element procedures for 3D crack problems. In *Conference on Numerical Methods in Fracture Mechanics*. Swansea.

Irwin, G. R. (1962). The crack extension force for a part through crack in a plate. *J. Appl. Mech.* **29**, 651–654.

Kassir, M. K. and G. C. Sih (1966). Three-dimensional stress distribution around an elliptical crack under arbitrary loadings. *J. Appl. Mech.* **88**, 601–611.

Kellogg, O. D. (1958). *Foundations of Potential Theory*. Dover Publ.,N.Y.

Knowles, J. K. and N. M. Wang (1960). *On the Bending of an Elastic Plate Containing a Crack*. GALCIT SM60-11, California Inst. of Technology, Pasadena, CA.

Kobayashi, A. S. (1973). *Experimental Techniques in Fracture Mechanics*. SESA, Connecticut.

Kobayashi, A. S. (1976). Crack-opening displacement in a surface flawed plate subjected to tension or plate bending. *Proc. 2nd Int. Conf. on Mechanical Behavior of Materials, ASM*, 1073–1077.

Kobayashi, A. S. and A. N. Enetanya (1976). Stress intensity factor of a corner crack. In *Mechanics of Crack Growth*. ASTM STP 590, Philadelphia, ASTM, 477–495.

Labbens, R. C., J. Helipt and A. Pellissier-Tanon (1976). Weight functions for three-dimensional symmetrical crack problems. In *Cracks and Fracture*. ASTM STP 601, Philadelphia, ASTM, 448–470.

Liebowitz, H. (ed.) (1968–70). *Fracture* (in 7 volumes), Vol.4, Academic Press, N.Y.

Liu, A. F. and H. P. Kan (1977). *A Survey of Stress Intensity Equations for Corner Cracks at a Hole*. Northrop Corporation Aircraft Division Report NOR 77-112, Structural Life Assurance Research Dept. 3853/82.

Lowengrub, M. (1961). Stress in the vicinity of a crack in a thick elastic plate. *Quart. Appl. Math.* **XIX 119**.

Luxmore, A. R. and D. R. J. Owen (eds.) (1978). *Numerical Methods in Fracture Mechanics*. University College, Swansea.

Maddox, S. J. (1975). An analysis of fatigue cracks in fillet welded joints. *Int. J. Fracture* **11**, 221–243.

Marcal, P. V. (1972). Three-dimensional finite element analysis for fracture mechanics. In *The Surface Crack : Physical Problems and Computational Solutions*. (J. L. Swedlow ed.), ASME, N.Y.

McGowan, J. J. (ed.) (1980). *A Critical Evaluation of Benchmark Surface Flaw Problems*. Soc. for Experimental Stress Analysis.

Melan, E. (1932). Der Spannungszustand der durch eine Einzelkraft in innenn Beanspruchten Halbscheibe. *Z. angw Math. Mech.* **12**, 343–346.

Mindlin, R. D. (1936). Force at a point in the interior of a semi-infinite solid. *Physics* **7**, 195–202.

Moriya, K. (1977). *Hybrid Crack Elements fot Three-Dimensional Solids and Plate Bending*. Doctoral Dissertation, Massachusetts Inst. of Technology.

Muki, R. (1960).*Progress in Solid Mechanics*. Vol.1, North-Holland Publishing Co., Amsterdam, 401.

Newman, J. C. and I. S Raju (1979). *Analyses of Surface Cracks in Finite Plates Under Tension or Bending Loads*. NASA TP-1578.

Perucchio, R. and A. R. Ingraffea (1985). An integrated boundary element analysis system with interactive computer graphics for three-dimensional linear-elastic fracture mechanics. *J. Computer & Structures* bf 20, 1–3, 157–171.

Pian, T. H. H. and K. Moriya (1978). Three dimensional fracture analysis by assumed stress hybrid elements. In *Numerical Methods in Fracture Mechanics*. (A. R. Luxmore and D. R. J. Owen eds.), University College, Swansea, 364–374.

Pickard, A. C. (1986). *The Application of 3-Dimensional Finite Element Methods to Fracture Mechanics and Fatigue Life Prediction*. Engineering Materials Advisory Services LTD, Warley, U.K.

Raju, I. S. and J. C. Newman (1977). *Improved Stress Intensity Factors for Semi-Elliptical Surface Cracks in Finite-Thickness Plates*. NASA TM X-72825.

Raju, I. S. and J. C. Newman (1979). Stress-intensity factors for a wide range of semi-elliptical surface cracks in finite-thickness plates. *J. Engng. Fract. Mech.* **11**, 817–829.

Rhee,H. C., S. N. Atluri, K. Moriya and T. H. H. Pian (1977). Hybrid finite element procedures for analyzing through flaws in plates in bending. *Proc. 4th Int. Conf. on Struc. Mech. in Reactor Tech.* SMiRT Paper No.M-2/4, San Francisco, CA.

Rizzo, F. J. (1967). An integral equation approach to boundary value problems of classical elastostatics. *Quart. Appl. Math.* **25 1**, 83–95.

Sach, R. A. (1946). Extension of Griffith theory of rupture to three dimensions. *Proc. Phys. Soc.* **58**, 729–736.

Sadowsky, M. A. and E. G. Sternberg (1949). Stress concentration around a triaxial ellipsoidal cavity. *J. Appl. Mech.* ASME.

Sih, G. C. (ed.) (1973). *Mechanics of Fracture*. (6 volumes), Vol.1, Noordhoff, Leyden.

Smith, F. W. and M. J. Alavi (1971). Stress intensity factors for a penny-shaped crack in a halfspace. *J. Engng. Fract. Mech.* **3**, 241.

Smith, F. W. and D. R. Sorensen (1976). The semi-elliptical surface crack - A solution by the alternating method. *Int. J. Fracture* **12**, 47–57.

Sneddon, I. N. (1946). The distribution of stress in the neighborhood of a crack in an elastic solid. *Proc. Royal Soc. London* A-187.

Snyder, M. D. and T. A. Cruse (1973). *Crack Tip Stress Intensity Factors in Finite Anisotropic Plates*. Technical Report. AFML-TR-73-209.

Sokolnikoff, I. S. (1952). *Mathematical Theory of Elasticity*. McGraw-Hill Book Co., 2nd ed.

Sternberg, E. and R. A. Eubanks (1955). On the concept of concentrated loads and an extension of the uniqueness theorem in the linear theory of elasticity. *Arch. Rational Mech. Anal.* **4 1**, 135–168.

Tan, C. L. and R. T. Fenner (1978). Three-dimensional stress analysis by the boundary integral equation method. *J. Strain Analysis* **13**, 213–219.

Telles, J. C. F. and C. A. Brebbia (1981). Boundary element solution for half-plane problems. *Int. J. Solids Struc.* **17 12**, 1149–1158.

Thompson, W. (Lord Kelvin) (1848). On the equations of equilibrium of an elastic solid. *Cambridge and Dublin Mathematical Journal* **3**, 87–89.

Tracey, D. M. (1973). 3D elastic singularity element for evaluation of K along an arbitrary crack front. *Int. J. Fracture* **9**, 340–343.

Tracey, D. M. (1974). Finite elements for three-dimensional elastic crack analysis. *Nuclear Engng and Design* **26**, 282–290.

Turteltaub, M. J. and E. Sternberg (1968). On concentrated loads and Green's functions in elastostatics. *Arch. Rational Mech. Anal.* **29** 3, 193–240.

Williams, M. L. (1951). Surface stress singularities resulting from various boundary conditions in angular corners of plates under bending. *Proc. First U.S. National Congress of Applied Mechanics.*

Williams, M. L. (1961). The bending stress distribution at the base of a stationary crack. *J. App. Mech.*, ASME.

Williams, M. L. and R. H. Owens (1954). Stress singularities in angular corners of plates having linear flexural rigidities for various boundary conditions. *Proc. Second U.S. National Congress of Applied Mechanics.*

Wheeler, L. T. and E. Sternberg (1968). Some theorems in classical elastodynamics. *Arch. Rational Mech. Anal.* **31** 1, 51–90.

Yagawa, G. and T. Nishioka (1978). Three-dimensional finite element analysis of a through-wall crack in a thick plate. *Int. J. Numer. Meth. in Engng.* **12**, 1295–1310.

Yamamoto, Y. and Y. Sumi (1978). Stress intensity factors for three-dimensional cracks. *Int. J. Fracture* **14**, 17–38.

Yamamoto, Y. and N. Tokuda (1973). Determination of stress-intensity factors in cracked plates by the finite element method. *Int. J. Numer. Meth. in Engng.* **6**, 427–439.