Stress Intensity Factors for a Cut Over an Ellipsoidal Cap in a 3-D Elastic Medium

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ABSTRACT

The paper calculates first the stress field in an infinite elastic medium containing a cut over an ellipsoidal cap by solving a singular integral equation for the displacement discontinuity across the cut. This field is then used to calculate the stress intensity factors along the edge of the cut.

KEYWORDS

Ellipsoidal cut; displacement discontinuity; stress intensity factors.

INTRODUCTION

The problem under consideration arises in the study of the elastic field and fracture parameters of an ellipsoidal inclusion or inhomogeneity which has debonded over a part of its boundary from the surrounding medium. The inclusion refers to a region of the medium which has undergone an internal deformation (Eshelby, 1957) whereas an inhomogeneity refers to a region of material whose elastic properties are different from those of the medium. In this paper we will only consider a cut over an ellipsoidal cap in an infinite elastic body and will calculate first the stress field in the body by solving a singular integral equation for the displacement discontinuity across the cut. We will also indicate how these results can be immediately generalised to a partially debonded ellipsoidal inclusion. The latter problem is of considerable interest in the transformation toughening of ceramics (Evans and Cannon, 1986). The stress field will then be used to calculate the stress intensity factors along the edge of the cut. Solutions of two-dimensional problems of partially debonded elliptic inhomogeneities under anti-plane and plane strain conditions were recently reported (Karihaloo and Viswanathan, 1985 a,b,c).

MATHEMATICAL FORMULATION

Consider an infinite elastic medium of Lamé constants λ and μ occupying the volume V and subject to the polynomial remote stress field

$$\sigma_{ij}^{0}(\vec{x}) = \sum_{\alpha,\beta,\gamma=0}^{N} A_{\alpha\beta\gamma ij} x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \tag{1}$$

There is a cut S in the medium over an ellipsoidal cap such that the edge ∂S of the cut described by

$$\left(\frac{x_2^2}{b^2}\right) + \left(\frac{x_3^2}{c^2}\right) = \left(1 - \frac{L^2}{a^2}\right) \equiv \rho_0^2 \tag{2}$$

lies in the plane $x_1 = L$ (Fig 1). The origin of the co-ordinates has been chosen to coincide with the centre of an imaginary ellipsoid with semiaxes a, b, c. In subsequent generalisation to a partially debonded inclusion, the region occupied by the latter will be identified with this ellipsoid.

In the absence of the cut, the stress field everywhere in the medium is equal to the remotely imposed field (1). In order to maintain this solution in the presence of the cut, surface forces must be applied to the two sides of S. This will produce a relative displacement $\gamma_k(\vec{x})$ over the surface S and a stress field, say, $\sigma_{ij}^{+}(\vec{x})$ everywhere $(i, j = x_1, x_2, x_3)$. Superposition of σ_{ij}^{0} and σ_{ij}^{+} gives the field in the medium.

The surface forces which are applied to produce $\gamma_k(\vec{x})$ are equal and opposite and therefore generate a Somigliana dislocation over S whose field can be written down at once when $\gamma_k(\vec{x})$ is given. When $\sigma_{ij}^{+}(\vec{x})$, which will have discontinuities across S, is added to $\sigma_{ij}^{0}(\vec{x})$ the final field will correspond to the solution being sought if, over S,

$$\left(\sigma_{ij}^{0} + \sigma_{ij}^{+}\right) n_j = 0 \tag{3}$$

where n_j is the outward normal to S.

Since $\sigma_{ij}^{+}(\vec{x})$ is available as an integral representation depending on $\gamma_k(\vec{x})$, (3) provides a (singular) integral equation for $\gamma_k(\vec{x})$. A convenient expression for $\sigma_{ij}^{+}(\vec{x})$ is (Eshelby, 1961)

$$\sigma_{ij}^{+}(\vec{x}) = C_{ijk\ell}(\lambda, \mu) \frac{\partial u_k^{+}}{\partial x_\ell} \tag{4}$$

where

$$u_k^{+}(\vec{x}) = \frac{1}{8\pi(1-\nu)} \Theta_{kpq} I_{pq} \tag{5}$$

$$C_{ijk\ell}(\lambda, \mu) = \lambda \delta_{ij} \delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \tag{6}$$

Here δ_{ij} is the Kronecker delta, λ and μ are Lamé constants of the medium with $\nu = \lambda/[2(\lambda + \mu)]$, and

$$I_{pq} = \int_S \gamma_p(\vec{x}')|\vec{x} - \vec{x}'| n_q(\vec{x}') dS(\vec{x}') \tag{7}$$

$$\Theta_{kpq}(\vec{x}) = \frac{\partial^3}{\partial x_k \partial x_p \partial x_q} - \left[\nu \delta_{pq} \frac{\partial}{\partial x_k}\right] \nabla^2 - (1-\nu)\left[\delta_{kp}\frac{\partial}{\partial x_q} + \delta_{kq}\frac{\partial}{\partial x_p}\right] \nabla^2 \tag{8}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial x_q^2} = \frac{\partial^2}{\partial \bar{x}_k^2} + \frac{\partial^2}{\partial \bar{x}_p^2} + \frac{\partial^2}{\partial \bar{x}_q^2} \tag{9}$$

In (9), and elsewhere, $\bar{x}_i = x_i - x_i'$, latin indices take the values $1, 2, 3$ and the usual summation convention for the repeated indices is implied. The singular integral (7) is to be understood in the sense of Cauchy principal value. It can be shown that (4) reduces to

$$\sigma_{ij}^{+}(\vec{x}) = -\int_S \gamma_p(\vec{x}') n_q(\vec{x}') H_{ijpq}(\vec{x}, \vec{x}') dS(\vec{x}') \tag{10}$$

where

$$H_{ijpq}(\vec{x}, \vec{x}') = C_{ijk\ell}(\lambda, \mu) \frac{\partial S_{pkq}}{\partial x_\ell} \tag{11}$$

Here

$$S_{pkq}(\vec{x}, \vec{x}') = C_{pqrs}(\lambda, \mu) \frac{\partial G_{kr}}{\partial x_s'} \tag{12}$$

and $G_{kr}(\vec{x}, \vec{x}')$ is the elastic Green's function. For an infinite isotropic medium

$$G_{kr}(\vec{x}, \vec{x}') = \frac{1}{16\pi\mu(1-\nu)} \left(\frac{\bar{x}_k \bar{x}_r}{\bar{R}^3} + \frac{(3-4\nu)}{\bar{R}} \delta_{kr}\right) \tag{13}$$

where $\bar{R} = |\vec{x} - \vec{x}'|$ and $\vec{x}$ is the position vector with components $x_i (i = 1, 2, 3)$.

With the notation of (1) the (singular) integral equation for $\gamma_k(\vec{x})$ is

$$n_j(\vec{x}) \int_S \gamma_p(\vec{x}') n_q(\vec{x}') H_{ijpq}(\vec{x}, \vec{x}') dS(\vec{x}') = n_j(\vec{x}) \sum_{\alpha,\beta,\gamma=0}^{N} A_{\alpha\beta\gamma ij} x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \tag{14}$$

The solution of this equation, and the calculation of the stress intensity factors along the edge of S is given in the next section. Here we indicate the generalisation of (14) to the case when S denotes the surface over which an ellipsoidal inclusion has debonded from the surrounding medium. The ellipsoidal region has undergone a stress free transformation strain e_{ij}^{T}. Eshelby (1961) has shown that if e_{ij}^{T} is a polynomial in $\vec{x}$, then inside the transformed region so also is the induced strain e_{ij}^{c} measured from the untransformed, unstressed state. Using Eshelby's notation for the stresses corresponding to the strains e_{ij}^{T} and e_{ij}^{c},

$$\sigma_{ij}^{T}(\vec{x}) = \sum_{\alpha,\beta,\gamma=0}^{N} B_{\alpha\beta\gamma ij} x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \tag{15}$$

$$\sigma_{ij}^{c}(\vec{x}) = \sum_{\alpha,\beta,\gamma=0}^{N} C_{\alpha\beta\gamma ij} x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \tag{16}$$

where $C_{\alpha\beta\gamma ij}$ can be expressed via $B_{\alpha\beta\gamma ij}$ using the potentials of ellipsoids (Eshelby, 1957). We note *en passant* that $\sigma_{ij}^{c}(\vec{x})$ is discontinuous across the boundary of the transformed region and

$$\left[\sigma_{ij}^{c}(out) - \sigma_{ij}^{c}(in)\right] n_j = -\sigma_{ij}^{T} n_j \tag{17}$$

With the additional stress fields (15), (16) the traction-free condition over S (3) now becomes

$$\left[\sigma_{ij}^{c}(out) + \sigma_{ij}^{0} + \sigma_{ij}^{+}(out)\right] n_j = \left[\sigma_{ij}^{c}(in) + \sigma_{ij}^{0} + \sigma_{ij}^{+}(in) - \sigma_{ij}^{T}\right] n_j = 0 \tag{18}$$

Although the field $\sigma_{ij}^{c}(out)$ is very complicated the field $\sigma_{ij}^{c}(in)$ is the polynomial (16) so that use of (18) with the form of the tractions available on the *inner* surface of S greatly facilitates the derivation of the integral equation in place of (14)

$$-\sigma_{ij}^{+}(\vec{x}) n_j(\vec{x}) = n_j(\vec{x}) \sum_{\alpha,\beta,\gamma=0}^{N} \left(C_{\alpha\beta\gamma ij} - B_{\alpha\beta\gamma ij} + A_{\alpha\beta\gamma ij}\right) x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \tag{19}$$

SOLUTION

Without going into detail, it can be shown that

$$\begin{aligned} -\sigma_{ij}^{+}(\vec{x}) &= \int_S \gamma_p(\vec{x}') n_q(\vec{x}') H_{ijpq}(\vec{x}, \vec{x}') dS(\vec{x}') \\ &= -2(\lambda - \mu)\alpha_0 \delta_{ij} T_{pp} + 2(3\lambda\alpha_0 + \mu\alpha_1)\delta_{ij} T_{pq,pq} \\ &\quad -2\mu\alpha_0 (T_{ij} + T_{ji}) - 6\mu\alpha_0 T_{pp,ij} - 10\mu\alpha_1 T_{pq,pqij} \\ &\quad +\mu(\alpha_1 + 3\alpha_0)(T_{iq,jq} + T_{pi,pj} + T_{jq,iq} + T_{pj,ip}) \end{aligned} \tag{20}$$

where

$$\alpha_0 = (2\nu - 1)/[8\pi(1-\nu)], \qquad \alpha_1 = 3/[8\pi(1-\nu)] \tag{21}$$

and

$$\begin{aligned} T_{k\ell} &= \int_S \frac{\gamma_k(\vec{x}') n_\ell(\vec{x}')}{\bar{R}^3} dS(\vec{x}') \\ T_{k\ell,ij} &= \int_S \frac{\gamma_k(\vec{x}') n_\ell(\vec{x}')}{\bar{R}^5} \bar{x}_i \bar{x}_j dS(\vec{x}') \\ T_{k\ell,imnj} &= \int_S \frac{\gamma_k(\vec{x}') n_\ell(\vec{x}')}{\bar{R}^7} \bar{x}_i \bar{x}_j \bar{x}_m \bar{x}_n dS(\vec{x}') \end{aligned} \tag{22}$$

Substituting (20) into (14) gives a (singular) integral equation for $\gamma_k(\vec{x})$ which will reduce to an identity in only two of the three co-ordinates x_i, say in x_2 and x_3 on $\vec{x} \subset S$. In analogy with crack problems, we represent the displacement discontinuity $\gamma_k(\vec{x})$ in the following form:

$$\gamma_k(\vec{x}) = \sqrt{(\rho_0^2 - \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2})} \sum_{m,n=0}^{N} \gamma_{mn}^{(k)} x_2^m x_3^n \tag{23}$$

Assuming the integrals (22) have been evaluated and equating the coefficients of like terms on either side of (14) leads to a system of equations for determining $\gamma_{mn}^{(k)}$. The stress field $\sigma_{ij}^{+}(\vec{x})$ is then calculated from (20). Next we calculate the strength of stress singularity along the edge of the cut whence fracture is likely to initiate. It is customary in the mechanics of crack propagation to express this strength through stress intensity factors or displacement discontinuity across the cut. This will be done in the next section.

STRESS INTENSITY FACTORS AND DISPLACEMENT JUMP

The stress intensity factors (SIF) along the edge ∂S of the cut S (Fig 1) are related to the stresses at the point $\vec{x}$ immediately outside of ∂S, i.e. at $\vec{x} \subset \partial S_+$. In particular the singular stresses σ_{ij}^{+} (20) control the SIF. At $\vec{x} \subset \partial S_+$ the singular behaviour of the integrals (22) is described by the dominant residue of the respective integrand as $r \to 0$, where r is defined such that $x_2' = x_2 + \ell_2 r$ and similarly x_3' with $\ell_2 = \cos\phi$ and $\ell_3 = \sin\phi$. Without going into detail, we find that

$$\begin{aligned} T_{k\ell} &\approx -\pi W_{k\ell}(\vec{x})/D_0(\vec{x}) \\ T_{k\ell,ij} &\approx -\pi W_{k\ell,ij}(\vec{x})/D_0(\vec{x}) \\ T_{k\ell,imnj} &\approx -\pi W_{k\ell,imnj}(\vec{x})/D_0(\vec{x}) \end{aligned} \tag{24}$$

where

$$D_0(\vec{x}) = \sqrt{(\frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} - \rho_0^2)} \tag{25}$$

and $W_{k\ell}, W_{k\ell,ij}$ and $W_{k\ell,imnj}$ depend on the expansion coefficients $\gamma_{ij}^{(k)}$ and involve complicated elliptic integrals. The tractions at $\vec{x} \subset \partial S_+$ are

$$\tau_i(\vec{x}) \equiv \sigma_{ij}^{+} n_j(\vec{x}) = -\pi g_i(\vec{x})/D_0(\vec{x}) \tag{26}$$

where $g_i(\vec{x})$ represent the scalar product of σ_{ij}^{+} (20) and n_j with the integrals $T_{k\ell}, T_{k\ell,ij}$ and $T_{k\ell,imnj}$ in (20) replaced by $W_{k\ell}, W_{k\ell,ij}$ and $W_{k\ell,imnj}$, respectively. We now resolve the cartesian components (26) at the point $\vec{x} \subset \partial S_+$ along the normal $\vec{n}$ and tangential directions $\vec{t_1}, \vec{t_2}$ at this point

$$\begin{aligned} \tau^{(\vec{n})} &= -\pi n_i g_i(\vec{x})/D_0(\vec{x}) \\ \tau^{(\vec{t_1})} &= -\pi t_{1i} g_i(\vec{x})/D_0(\vec{x}) \\ \tau^{(\vec{t_2})} &= -\pi t_{2i} g_i(\vec{x})/D_0(\vec{x}) \end{aligned} \tag{27}$$

Finally, the stress intensity factors are given by

$$[K_n, K_{t_1}, K_{t_2}]^T = \left[\tau^{(\vec{n})}, \tau^{(\vec{t_1})}, \tau^{(\vec{t_2})}\right]^T \sqrt{2\pi d_0} \tag{28}$$

where superscript T refers to transpose, and d_0 is the distance between points $\vec{X}$ and $\vec{x}$ (Fig 1) which is related to $D_0(\vec{x})$ as follows:

$$D_0^2(\vec{x}) \approx 2 d_0 A(\phi)/\sqrt{1 + a^2 A^2(\phi)} \tag{29}$$

Here $A(\phi) = (x_2 \cos\phi/b^2) + (x_3 \sin\phi/c^2)$, and ϕ is the polar angle between the points (X_2, X_3) and (x_2, x_3) in the plane containing the latter, i.e. ∂S (Fig 1). In a non-dimensional form, the stress intensity factors are

$$\left[\bar{K}_n, \bar{K}_{t_1}, \bar{K}_{t_2}\right]^T = [K_n, K_{t_1}, K_{t_2}]^T / (\mu\sqrt{L}) \tag{30}$$

where $x_1 = L$ defines the plane containing the edge of cut ∂S.

The displacement discontinuity across the cut is given by (5) which may be reduced to

$$u_k^{+}(\vec{x}) = -\int_S \gamma_i(\vec{x}') n_j(\vec{x}') S_{ikj}(\vec{x}, \vec{x}') dS(\vec{x}') \tag{31}$$

where (*vide* (12) and (13))

$$S_{ikj}(\vec{x},\vec{x}') = -\alpha_0\frac{\bar{x}_k}{\bar{R}^3}\delta_{ij} + \alpha_0\frac{\bar{x}_i}{\bar{R}^3}\delta_{kj} + \alpha_0\frac{\bar{x}_j}{\bar{R}^3}\delta_{ki} - \alpha_1\frac{\bar{x}_k\bar{x}_i\bar{x}_j}{\bar{R}^5} \tag{32}$$

Let $\Delta[f]$ denote the jump experienced by a function f across the cut S. It is easily shown that

$$\Delta[S_{ikj}(\vec{x},\vec{x}')n_j(\vec{x}')] = -\delta_{ik}\delta(\vec{\xi}-\vec{\xi}') \tag{33}$$

where $\vec{\xi}$ and $\vec{\xi}'$ are the tangential co-ordinate vectors of points on the tangent plane at $\vec{x}$, and $\delta(\cdots)$ is the Dirac delta function. From (31) and (33) we arrive at the obvious result that $\Delta[u_k^+(\vec{x})] = \gamma_k(\vec{x}), \vec{x} \subset S$, with $\gamma_k(\vec{x})$ being given by (23).

The analytical expressions for the SIF and $\gamma_k(\vec{x})$, like the expression for the stress field due to the cut, are therefore available as (multiple) sums for ease of numerical computation. Several numerical examples will be given during the presentation. Full details will be published elsewhere.

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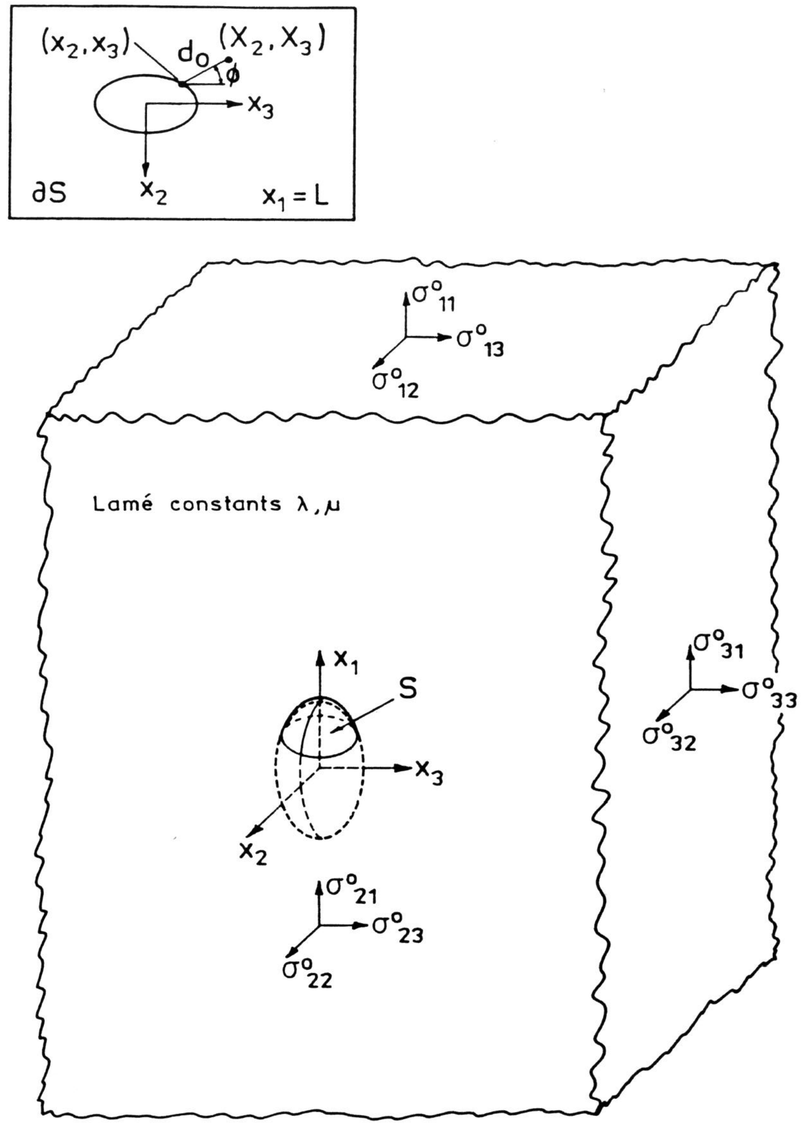

Fig 1. A cut S over an ellipsoidal cap in an infinite elastic medium showing the origin of co-ordinates at the centre of an imaginary ellipsoid. The inset shows the plane $x_1 = L$ containing the elliptical edge of cut ∂S.