Stress Intensity Factor Variation of a Penny-shaped Crack Situated Close to the Free Surface of a Half-space

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ABSTRACT

Well known opening mode stress intensity factor variations for a penny-shaped crack situated close and orthogonaly to the surface (e.g. welding pores) of a half-space are the result of the "alternating-method" procedure and available since 1971. The comparison of those with solutions by the "singular-integral- equation-method" shows a remarkable discrepancy for points on the flaw-border nearest to the boundary. Applying the "singular-integral-equation-method" new results are obtained for cracks positioned very close to the free surface. They are computed by using new exact analytical formulae for required finite part integrals.

KEYWORDS

Penny-shaped crack, stress intensity factor, collocation method, singular integral equation, finite part integral, half-space.

INTRODUCTION

Flaws in pressure vessels, machine components and structural components are often approximated by circular, elliptical or semi-elliptical cracks. If the crack is situated in the neighbourhood of a stressfree surface, the theoretical analysis becomes difficult, since it involves additional geometric parameters describing the dimensions of the elastic solid. The local stresses are raising with decreasing distance between the free boundary and the crack. The study of embedded planar cracks near the free surface of a half-space subjected to various loadings has been the subject of special research in the past. A review of literature can be found by Panasyuk et al., (1981). Solutions for penny-shaped cracks situated parallel to the free surface of a half-space have been published by Kuzmin and Ufland, (1965), Srivastava and Singh, (1969), Low (1972), Lo (1979) and Guz and Nazarenko (1985).

Stress intensity factors for an embedded elliptical crack normal to the boundary in a halfspace approaching the free surface, are also available. Shah and Kobayashi, (1973), solved this problem using the "Schwarz-Neumann-alternating-method" as described by Kantorovich and Krylov, (1964). The analyses from Nisitani and Murakami, (1974), Isida and Noguchi, (1984), have been performed using the body force method. An approximative Mode-I-solution for a penny-shaped crack is available from Smith and Alavi, (1971), using the "alternating-method" too. Kaya, (1984), solved the same problem by a singular integral equation.
Both authors restricted their computations to the aspect-ratios h/a = 2,0; 1,5; 1,2; 1,1 (Fig.1: h ... distance from crack center to free surface, a ... crack radius).
Comparing Smith/Alavi's and Kaya's stress-intensity factor for the point on the crackfront nearest to the free surface (polarangle $\theta = 180°$) and for a ratio h/a = 1,1 a discrepancy of 10 % is found. Principally the finite element method (Nikishkov and Atluri, 1978; Banks-Sills and Sherman, 1986; Sham, 1987; Banks-Sills, 1988) or the boundary element method (Luo and Zhang, 1988) or a combination of both (Keat et al., 1988) can be applied. But for cracks very near to the surface a very fine element grid between the crack boundary and the free surface is necessary. This leads to very small elements in this region and therefore to a large number of elements for the whole system. On the other hand a strong coarsing outside of the interesting region gives rise for enormous element-stiffness differences and hence incorrect results.

The work reported is an extension of Kaya's method and contains new accurate k_I-results for penny-shaped cracks close to the free surface.

ALTERNATING METHOD (AM)

The "alternating-method" gives less accurate results than the "singular-integral-equation-method" due to the nature of its solution procedure. It can be explained by the following steps (Hartranft and Sih, 1972; Atluri, 1986):

Step 1: It is assumed that there exists no crack in the halfspace. The normal stresses at the location of the crack surfaces due to the applied load are calculated.

Step 2: Now the existence of the crack is taken into account. The normal stresses found in step 1 must then be removed by applying equal and opposite normal stresses to the crack surfaces. In order to do this, it is necessary to find a solution for a circular crack embedded in an infinite solid subjected to an arbitrary normal loading on the crack surface.

Step 3: These residual stresses on the surface of the halfspace are removed by applying opposite surface loadings on that surface of an uncracked semi-infinite solid. The normal stresses on the crack surface resulting from this removal of stresses from the free plane are then computed. Due to symmetry the shear stresses vanish.

Alavi, (1968), used for this "freeing-process" basic solutions for a semi-infinite solid when a small rectangular area of its surface is subjected to constant normal and shear stress.

Step 4: By applying opposite stresses on the crack area the computed stresses in step 3 are erased; but this will again cause some residual tractions on the surface of the half-space.

Step 5: Steps 3 and 4 are repeated until the residual stresses on the crack plane and on the surface of the half-space become negligible.
The final solution is obtained by superposing the results of each iteration step.

It's clear to see that this solution-procedure is useful only for "mild" ratios h/a and gives rise to low accuracy results.

SINGULAR INTEGRAL EQUATION METHOD (SIEM)

Now we are interested in reasonable results for ratios h/a smaller than 1,1. The only method to our knowledge which provides satisfactoring accuracy is the singular integral equation method. Thereby the complex boundary-value problem is reduced in an exact manner to one governing singular integral equation for the unknown crack opening displacement field $W(x_0, y_0)$, $x_0, y_0 \in \Omega$ (crack area). Solving this equation correctly one may not expect significant errors. Following the concept of Kaya, (1984), the singular integral equation for a semi-infinite solid with a penny-shaped crack perpendicular to the boundary (Fig.1) located on z = 0 plane and occupying a region specified by $(x,y) \in \Omega$ can be formulated as

$$\iint_{\Omega} \frac{W(x_0,y_0)\cdot dx_0\, dy_0}{\left[(x_0-x)^2+(y_0-y)^2\right]^{3/2}} + \iint_{\Omega} W(x_0,y_0)\cdot K(x_0,y_0;x,y)\, dx_0\, dy_0 = \frac{4\pi(1-\nu)}{\mu}\, p(x,y). \qquad (x,y)\in\Omega,\ \Omega: x^2+y^2\le 1 \qquad (1)$$

The regular kernel is expressed as

$$K(x_0,y_0;x,y) = \frac{1}{\left[(x_0+x+2q)^2+(y_0-y)^2\right]^{3/2}} + F(x_0,y_0;x,y)\ ; \qquad (2)$$

$$F(x_0,y_0;x,y) = 6\Big\{(1-2\nu)^2 \frac{1}{R(R+x_0+x+2q)^2} + \frac{4\nu}{R^3}\left(\frac{1}{3}-\nu\right) - (1-2\nu)(x_0+x+2q)\left[\frac{1}{R^3(R+x_0+x+2q)} + \frac{1}{R^2(R+x_0+x+2q)^2}\right] - \frac{1}{R^5}\left[(1-2\nu)2\nu(x_0+x+2q)^2 - 3(x+q)(x_0+q)\right]\Big\}\ ;$$

$$R = \left[(x_0+x+2q)^2+(y_0-y)^2\right]^{1/2}; \quad q = \frac{h}{a}\,.$$

The dimensions are normalized such by making the crack boundary a unit circle. The load is assumed to be symmetrically distributed with respect to the crack plane.

The unknown crack opening displacement $W(x_0, y_0)$ is represented in the form

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$$W(x_0,y_0) = g(x_0,y_0)\cdot\sqrt{1-x_0^2-y_0^2}\ . \tag{3}$$

The square root term ensures $W = 0$ on the crack boundary. The new unknown function $g(x_0, y_0)$ is expanded in terms of a finite double power series

$$g(x_0,y_0) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} a_{ij}\, x_0^i\, y_0^{2j}\ . \tag{4}$$

As mentioned the loading $p\ (x,y)$ is assumed to be an even function of y. Therefore, only even powers of y_0 are considered in $g\ (x_0, y_0)$. Inserting of (3), (4) in (1) gives

$$\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} a_{ij}\left[C_{ij}(x,y) + H_{ij}(x,y)\right] = \frac{4\pi(1-\nu)}{\mu}\, p(x,y)\ . \tag{5}$$

$C_{ij}(x,y)$ denoting the finite part integrals

$$C_{ij}(x,y) = \iint_\Omega \frac{x_0^i\, y_0^{2j}\sqrt{1-x_0^2-y_0^2}}{\left[(x_0-x)^2+(y_0-y)^2\right]^{3/2}}\, dx_0\, dy_0 \tag{6}$$

and

$$H_{ij}(x,y) = \iint_\Omega x_0^i\, y_0^{2j}\sqrt{1-x_0^2-y_0^2}\cdot K(x_0,y_0;x,y)\, dx_0\, dy_0 \tag{7}$$

the regular integrals.

To determine the unknown coefficients a_{ij} relation (5) is evaluated at symmetrically positioned collocation points $(x_K, y_K) \in \Omega$, $K=1,2\ \ldots\ M$:

$$x_K = r_s\cdot\cos(\varphi_t)\ ; \qquad y_K = r_s\cdot\sin(\varphi_t) \tag{8a}$$

with

$$\varphi_t = \frac{\pi}{60}\cdot t\ , \quad t = 1,2,\ldots 59\ . \tag{8b}$$

To represent the strong influence of the polar angle on the stress intensity factor k_1 r_p are selected as the roots of the Chebychev polynominal of the second kind, order n

$$n = 26:\ U_{26}(r_s) = 0;\quad r_s = \cos\left(\frac{s+1}{27}\pi\right),\ s = 0,1,\ldots 12 \tag{8c}$$

placing a concentration of collocation points near the crack border where the crack opening displacement rapidly increases. Totally 767 collocation points over the half circular area are used.

Finally a linear algebraic equation system is available to obtain the $(N_1+1)*(N_2+1)$ unknown coefficients a_{ij}:

$$\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} a_{ij}\left[C_{ij}(x_K,y_K) + H_{ij}(x_K,y_K)\right] = \frac{4\pi(1-\nu)}{\mu}\, p(x_K,y_K)\ . \tag{9}$$

The choice of N_1 and N_2 depends essentially on the disposability of solutions for the finite part integrals $C_{ij}(x,y)$. Therefore special consideration was focused on the derivation of exact formulae for $C_{ij}(x,y)$, $i=j=0(1)16$. The general solution is of the form (10)

$$C_{ij}(x,y) = -\pi^2\left(1+\frac{i+j}{2}\right)x^i y^j + \sum_{\substack{k=0\\(l\geq 2-k)}}^{i}\ \sum_{l=0}^{j}\ \sum_{\substack{q=0\\(k+l-q=\text{even})}}^{(k+l-2)}\ \sum_{\mu=0}^{(k+l-q)/2}\ \sum_{\substack{\nu=0\\(\nu+l=\text{even})}}^{k+l-2\mu}\ \sum_{m=0}^{\mu}\ \sum_{n=0}^{m} A\cdot a^{2(\mu-m)}\cdot x^{i+l-2\mu-\nu+2m-2n}\cdot y^{j-l+\nu+2n}\ , \tag{10a}$$

where

$$A = \binom{i}{k}\binom{j}{l} B\left(\frac{k+l-q-1}{2},\frac{3}{2}\right)\binom{k+l-2}{q}(-1)^{q+m}\binom{(k+l-q)/2}{\mu}\cdot \binom{k+l-2\mu}{\nu} B\left(\frac{2k+l-2\mu-\nu+1}{2},\frac{\nu+l+1}{2}\right)\binom{\mu}{m}\binom{m}{n}\ . \tag{10b}$$

B(a,b) denotes the Beta function:

$$B(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}\ . \tag{11a}$$

Using the well known relation for the Gamma function

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\ \sqrt{\pi}}{n!\ 2^{2n}} \tag{11b}$$

the Beta function is expressible as

$$m\geq n:\quad B\left(m+\frac{1}{2},n+\frac{1}{2}\right) = \frac{\pi}{2^{m+n+(m+n)/2}}\cdot\frac{\prod_{i=1+\left(\frac{m+n+1}{2}\right)}^{m}(2i-1)\cdot\prod_{i=1}^{n}(2i-1)}{\prod_{i=1}^{(m+n)/2} i}\ . \tag{11c}$$

This special formulation allows a standard FORTRAN DO - Loop without an integer-overflow. Following this manipulation equation (10b) is of the general form

$$A = \frac{\pi^2}{2^\alpha} \cdot A' \quad . \tag{12}$$

Now A' consists of products of binomials and is of INTEGER-Typ. Coding equation (10a,b) with the help of (11c) in a standard manner would again lead quickly to an INTEGER overflow (e.g.: i=1, j=5; i=2, j=4; i=3, j=3). To avoid such a difficulty special SUBROUTINEs, documented in a recent report (Mayrhofer and Fischer, 1988), were written with no restriction for the upper limits of i and j. The output of the FORTRAN program are the exact analytical formulae for the finite part integrals $C_{ij}(x,y)$; they are also listed in this special report for the values i=j=0(1)16.

In this paper $N_1 = 8$, $N_2 = 5$ are fixed. The regular integrals (7) were solved using a standard integration procedure. The overdetermined equation system (54*767) was solved using the method of least squares.

NUMERICAL RESULTS

In Fig. 2 the stress intensity factors k_I found by the AM are compared with those evaluated by the SIEM. For ratios h/a< 1,2 a significant deviation of the AM-results from the SIEM-results can be observed. Generally the AM-results are smaller for angles between θ = 100° and 180°. For ratios h/a< 1,1 satisfying k_I values can be found only by using SIEM.

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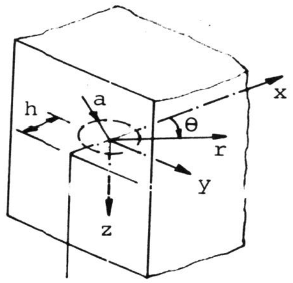

Fig. 1. Penny-shaped crack near the surface of a half-space.

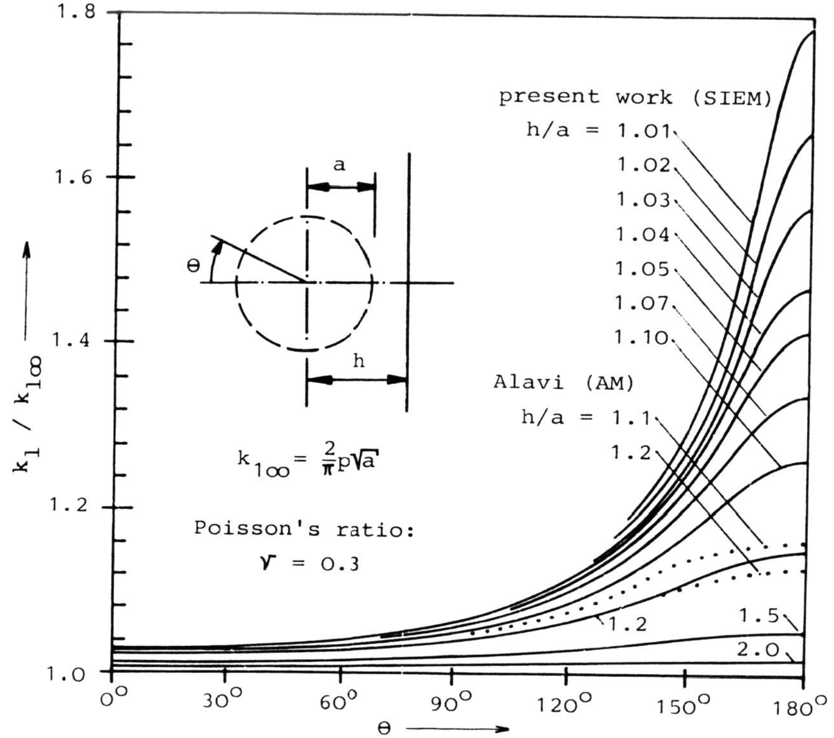

Fig. 2. Stress intensity factors for a penny-shaped crack approaching the surface of a half-space. Crack surfaces under constant pressure.