Steady Crack Growth in Elastic–Plastic Work-hardening Solids: Near-tip Fields and Stress Discontinuity

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ABSTRACT

The near-tip stress and deformation fields of a steadily growing crack are studied for an elastic-plastic solid with power-law work-hardening. It is shown that the same set of governing differential equations for the field quantities applies whether or not the plastic strain rate is assumed to be coaxial with the stress deviator, i.e, whether or not the usual plastic normality rule is used. Assuming continuous displacement and velocity fields, conditions of discontinuity are analyzed, and by means of numerical results it is shown that stress discontinuities do indeed exist when the plastic normality rule is relaxed.

KEYWORDS

Growing crack; normality rule; discontinuity surface.

1. INTRODUCTION

The near-tip stress and strain fields of a dynamically growing rectilinear crack in an elastic-plastic solid with power-lawer work-hardening have been discussed by Gao and Nemat-Nasser (1983), for the elastically and plastically incompressible plane strain case. The assumption of incompressibility reduces the number of independent strain rate components (as well as strain components) from three to two, for plane strain problems. This has farreaching consequences for the nature of the solution to the corresponding governing equations. In particular, it allows the formulation of the near-tip field quantities for a broad class of elastic-plastic materials, with or without the assumption of the normality rule, in terms of exactly the same differential equations, by slight modification and interpretation of a single parameter, namely the Mach number. For elastic perfectly-plastic solids, this has been recently shown and exemplified by Nemat-Nasser and Obata (1988). The purpose of this research note is to show that, for elastic-plastic solids with power-law work-hardening, essentially similar conclusions can be obtained by proper modification and redefinition of the Mach number.

One particular consequence of our analysis is that the near-field quantities can indeed admit discontinuities for plasticity models which include the vertex structure of the yield surface associated with slip-induced crystal plasticity. Conditions associated with discontinuity surfaces are briefly discussed, and discontinuity relations are obtained for incompressible materials in plane strain. Then, assuming a continuous velocity field, numerical results are obtained, which illustrate the existence of discontinuous stress fields. We also briefly examine more general properties of discontinuity surfaces when the velocity is discontinuous.

2. POWER-LAW HARDENING WITH NONCOAXIALITY

We denote the strain by $\underset{\sim}{\varepsilon}$, the stress by $\underset{\sim}{\sigma}$, and the stress deviator by $\underset{\sim}{s}$, and let $\underset{\sim}{\mu}$ and $\underset{\sim}{\nu}$ be a pair of orthogonal unit deviatoric second-order tensors defined by

$$\underset{\sim}{\mu} = \frac{\underset{\sim}{s}}{|\underset{\sim}{s}|}, \qquad |\underset{\sim}{s}| = (\underset{\sim}{s}:\underset{\sim}{s})^{1/2}, \tag{2.1}$$

$$\underset{\sim}{\mu}:\underset{\sim}{\nu} = 0, \qquad |\underset{\sim}{\mu}| = |\underset{\sim}{\nu}| = 1, \qquad \mathrm{tr}\,\underset{\sim}{\mu} = \mathrm{tr}\,\underset{\sim}{\nu} = 0. \tag{2.2}$$

The plastic strain rate $\dot{\underset{\sim}{\varepsilon}}^p$ is then defined by

$$\dot{\underset{\sim}{\varepsilon}}^p = \hat{\lambda}\,\underset{\sim}{\mu} + \hat{\eta}\,\underset{\sim}{\nu}. \tag{2.3}$$

As is seen, the first term in the right-hand side is coaxial with the stress deviator and corresponds to the plastic strain rate in the usual J_2-plasticity theory. The second term gives a plastic strain rate component which is normal to the stress deviator. This term, therefore, does not contribute to the rate of plastic work which is given by

$$\dot{\xi} = \underset{\sim}{\sigma}:\dot{\underset{\sim}{\varepsilon}}^p = \underset{\sim}{s}:\dot{\underset{\sim}{\varepsilon}}^p = \hat{\lambda}\,|\underset{\sim}{s}|. \tag{2.4}$$

We consider a power-law work-hardening defined by

$$\xi = \int \dot{\xi}\,dt = \xi_o\,|\underset{\sim}{s}|^n. \tag{2.5}$$

Then it follows from (2.4) and (2.5) that

$$\hat{\lambda} = n\,\xi_o\,|\underset{\sim}{s}|^{n-2}\frac{\partial}{\partial t}|\underset{\sim}{s}|. \tag{2.6}$$

The total strain rate now is given by the sum of the elastic and plastic parts, as

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}}^e + \dot{\underset{\sim}{\varepsilon}}^p = \frac{3}{2E}\dot{\underset{\sim}{s}} + \hat{\lambda}\,\underset{\sim}{\mu} + \hat{\eta}\,\underset{\sim}{\nu} \qquad (\mathrm{tr}\,\dot{\underset{\sim}{\varepsilon}}^e = \mathrm{tr}\,\dot{\underset{\sim}{\varepsilon}}^p = 0), \tag{2.7}$$

where $3/2E = 1/2\mu$ for elastically incompressible problems, with E and μ being the Young modulus and shear modulus, respectively. In the sequel, we examine the relevant field equations, using the notation of Gao and Nemat-Nasser (1983). Our objective is to show that the same basic differential equations apply to a solid with constitutive relations given by (2.7), whether $\hat{\eta}$ is identically zero or is non-zero, i.e., whether or not the plastic strain rate is normal to the J_2-yield surface.

3. FIELD EQUATIONS

Consider a straight crack which propagates steadily under plane strain conditions. The crack growth speed V is constant, measured in the spatially fixed Cartesian coordinates (x,y). Let (r,θ) be the polar coordinates moving with the crack tip, as shown in Fig. 1. For steady state deformations, the time derivative becomes

$$\frac{\partial}{\partial t} = V\left(\sin\theta\frac{\partial}{r\partial\theta} - \cos\theta\frac{\partial}{\partial r}\right), \tag{3.1}$$

and is denoted by superposed dot.

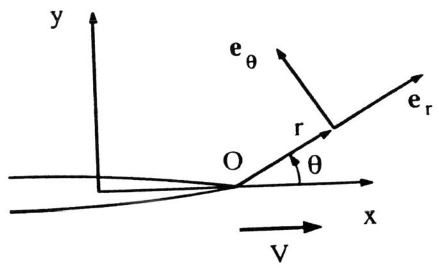

Fig. 1: Crack Growth with stationary coordinates (x,y) and moving polar coodintaes (r,θ)

Denote the mass density by ρ, the velocity by $\underset{\sim}{v}$, and the acceleration by $\underset{\sim}{w}$. The equations of motion are

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{r\theta}}{r\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = \rho\,w_r, \qquad \frac{\partial\sigma_{\theta\theta}}{r\partial\theta} + \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = \rho\,w_\theta, \qquad \sigma_{r\theta} = \sigma_{\theta r}, \tag{3.2}$$

where components in the moving r,θ-coordinates are used. The boundary conditions on the crack surfaces are

$$\sigma_{\theta r} = \sigma_{\theta\theta} = 0 \text{ on } \theta = \pm\pi. \tag{3.3}$$

The strain rate $\dot{\underset{\sim}{\varepsilon}}$ is defined in terms of $\underset{\sim}{v}$ as

$$\dot{\varepsilon}_{rr} = \frac{\partial v_r}{\partial r}, \qquad \dot{\varepsilon}_{\theta\theta} = \frac{\partial v_\theta}{r\partial\theta} + \frac{v_r}{r}, \qquad \dot{\varepsilon}_{\theta r} = \dot{\varepsilon}_{r\theta} = \frac{1}{2}\left(\frac{\partial v_r}{r\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right). \tag{3.4}$$

The incompressibility of the material requires $\mathrm{tr}\dot{\underset{\sim}{\varepsilon}} = \dot{\varepsilon}_{rr} + \dot{\varepsilon}_{\theta\theta} = 0$, where the strain rate is given by constitutive relations (2.7).

3.1 Order-of-Magnitude Estimates

From the incompressibility condition, Gao and Nemat-Nasser (1983) suggest the displacement potential

$$U = r^2(ln r)^\delta\{(ln r)f(\theta) + g(\theta)\} + \dots . \tag{3.5}$$

From this it follows that

$$O(\underset{\sim}{\varepsilon}) = O((ln r)^{\delta+1}), \tag{3.6a}$$

and from the power-law work-hardening condition, we have

$$O(\xi) = O(\underset{\sim}{\varepsilon}:\underset{\sim}{\sigma}) = O(|\underset{\sim}{\sigma}|^{n}). \tag{3.6b}$$

Hence,

$$O(\underset{\sim}{\sigma}) = O(\underset{\sim}{\varepsilon}^{1/(n-1)}) = O(\,(lnr)^{(\delta+1)/(n-1)}\,). \tag{3.6c}$$

From the equations of motion, on the other hand, we have

$$O(\underset{\sim}{\nabla}\cdot\underset{\sim}{\sigma}) = O(\frac{1}{r}(lnr)^{(\delta+1)/(n-1)}) = O(\underset{\sim}{w}). \tag{3.6d}$$

Hence, in view of (3.6a), we set

$$O(\underset{\sim}{w}) = O(\frac{1}{r}(lnr)^{\delta}), \tag{3.6e}$$

and obtain

$$\delta = \frac{1}{n-2}. \tag{3.7}$$

Thus, the order of $\hat{\lambda}$ and $\hat{\eta}$ becomes

$$O(\hat{\lambda}) = O(\hat{\eta}) = O(\dot{\underset{\sim}{\varepsilon}}^{p}) = O(\frac{1}{r}(lnr)^{\delta}). \tag{3.6f}$$

Now, since $\hat{\lambda}\,\underset{\sim}{\mu} = \hat{\lambda}\,\underset{\sim}{s}/|\underset{\sim}{s}|$, we set

$$\frac{\hat{\lambda}}{|\underset{\sim}{s}|} = \lambda + \ldots\,, \qquad \frac{\hat{\eta}}{|\underset{\sim}{s}|} = \eta + \ldots\,, \tag{3.8}$$

note that $O(\lambda) = O(\eta) = O(1/r)$, and rewrite (2.3) as

$$\dot{\underset{\sim}{\varepsilon}}^{p} = \lambda\,\underset{\sim}{s} + \eta\,|\underset{\sim}{s}|\,\underset{\sim}{\nu} + \ldots\,. \tag{3.9}$$

Moreover, since the leading term of $|\underset{\sim}{s}|$ becomes independent of θ as shown by Gao and Nemat-Nasser (1983), the leading terms of $\dot{\underset{\sim}{s}}$ and $|\underset{\sim}{s}|\dot{\underset{\sim}{\mu}}$ are equal. Then, from $\underset{\sim}{\mu}:\underset{\sim}{\mu} = 1$, we get $\underset{\sim}{\mu}:\dot{\underset{\sim}{\mu}} = 0$, and hence, $\dot{\underset{\sim}{\mu}}$ is parallel to $\underset{\sim}{\nu}$. Indeed, if we set

$$\mu_{rr} = -\mu_{\theta\theta} = -\cos\psi/\sqrt{2}, \qquad \mu_{r\theta} = \mu_{\theta r} = \sin\psi/\sqrt{2}, \tag{3.10}$$

$$\nu_{rr} = -\nu_{\theta\theta} = \sin\psi/\sqrt{2}, \qquad \nu_{r\theta} = \nu_{\theta r} = \cos\psi/\sqrt{2},$$

with $\psi = \psi(\theta)$ and $|\underset{\sim}{\mu}| = |\underset{\sim}{\nu}| = 1$, it follows that

$$\dot{\underset{\sim}{\mu}} = \frac{V}{r}\sin\theta\,\psi'\,\underset{\sim}{\nu} + \ldots\,. \tag{3.11}$$

Thus,

$$\dot{\underset{\sim}{s}} = |\underset{\sim}{s}|\,\dot{\underset{\sim}{\mu}} + \ldots = \frac{V}{r}\sin\theta\,\psi'\,|\underset{\sim}{s}|\,\underset{\sim}{\nu} + \ldots\,. \tag{3.12}$$

Therefore, (3.9) becomes

$$\dot{\underset{\sim}{\varepsilon}}^{p} = \lambda\,\underset{\sim}{s} + (\frac{V}{r}\sin\theta\,\psi')^{-1}\,\eta\,\dot{\underset{\sim}{s}} + \ldots\,. \tag{3.13}$$

Since $O(\eta) = O(1/r)$, we have $O(\,(V\sin\theta\psi'/r)^{-1}\,\eta\,) = O(1)$. Hence, we set

$$(\frac{V}{r}\sin\theta\,\psi')^{-1}\,\eta = \bar{\eta}, \qquad \frac{1}{2\mu^{*}} = \frac{1}{2\mu} + \bar{\eta}, \tag{3.14}$$

and define the Mach number by

$$M^{*} = V\left(\frac{\rho}{\mu^{*}}\right)^{1/2}. \tag{3.15}$$

3.2 Governing Differential Equations

With the modification of the Mach number, (3.15), it can easily be shown that the near-tip field equations are exactly the same as those reported by Gao and Nemat-Nasser (1983), except for the fact that M must be replaced by M^{*}. In particular, if we define the leading terms of the near-tip field quantities by

$$U = r^{2}(lnr)^{\delta}\{\,f\,lnr + g\,\}, \tag{3.16}$$

$$\sigma_{rr} = (lnr)^{\delta}\{\sigma - \tau\cos\psi\}, \qquad \sigma_{\theta\theta} = (lnr)^{\delta}\{\sigma + \tau\cos\psi\}, \qquad \sigma_{r\theta} = (lnr)^{\delta}\,\tau\sin\psi,$$

where f, g, σ, ψ, and τ are functions of θ, we arrive at

$$f = \bar{A}\sin 2\theta + \bar{B}\cos 2\theta + \bar{C}, \qquad \tau = K\ (= \text{constant}). \tag{3.17}$$

Normalizing the field parameters as

$$\{A,B,C\} = \frac{\mu^{*}}{K}(1+\delta)\,\{\bar{A},\bar{B},\bar{C}\}, \qquad \Lambda = \frac{1}{r\rho V}\lambda, \qquad \Sigma = \frac{\sigma}{K}, \tag{3.18}$$

we obtain a set of governing differential equations for ψ, Λ, and Σ:

$$(\psi' - 2)\,F - 2M^{*2}\,G = 0, \qquad \Sigma' F - 2M^{*2}\{G\sin\psi - (B+C)F\} = 0, \tag{3.19}$$

$$\Lambda = \frac{M^{*2}}{F}\,[\,2\cos\psi\{A\cos\theta - (B+C)\sin\theta\}\} + M^{*2}\sin\theta\{A\sin(\psi-2\theta) - 2(B+C)\sin(\psi-\theta)\}\,],$$

where F and G are given by

$$F = \cos^{2}\psi - M^{*2}\sin^{2}\theta, \qquad G = A\cos(\psi-2\theta) - 2\,(B+C)\sin\theta\cos(\psi-\theta). \tag{3.20}$$

These equations are identical with those reported by Gao and Nemat-Nasser (1983).

4. DISCONTINUITY SURFACE

Suppose that there exists a discontinuity surface, S, at $\theta = \theta_{S}$ as shown in Fig. 2; across S, the displacement is continuous, but the velocity, strain,

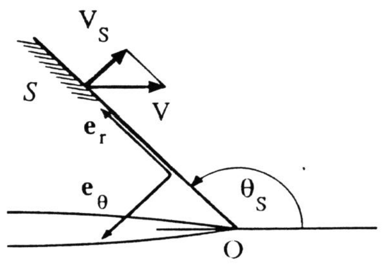

Fig.2 : Dicontinuity surface

and stress fields may be discontinuous. The strain tensor, $\underset{\sim}{\varepsilon}$, is given by the symmetric part of the displacement gradient. Denoting the jump across S by [ ], we have the kinematic conditions:

$$[\varepsilon_{rr}] = 0, \quad V_S\,[\varepsilon_{r\theta}] = [v_r], \quad V_S\,[\varepsilon_{\theta\theta}] = [v_\theta], \tag{4.1}$$

and the continuity and equilibrium conditions:

$$[m] = 0, \quad m\,[v_r] + [\sigma_{r\theta}] = 0, \quad m\,[v_\theta] + [\sigma_{\theta\theta}] = 0, \tag{4.2}$$

where $m = \rho(V_S - v_\theta)$ and $V_S = V\sin\theta_S$. Since the material is incompressible, $[\mathrm{tr}\underset{\sim}{\varepsilon}] = 0$, and hence (4.1) and (4.2) lead to

$$[\varepsilon_{\theta\theta}] = 0, \quad [v_\theta] = 0, \quad [\sigma_{\theta\theta}] = 0. \tag{4.3}$$

Therefore, $\varepsilon_{\theta\theta}$, v_θ, and $\sigma_{\theta\theta}$ are continuous across S. Only v_r, $\varepsilon_{r\theta}$, $\sigma_{r\theta}$, and σ_{rr} may be discontinuous across S.

Assume that the radial velocity component is continuous across S, i.e, $[v_r] = 0$. Then, from (4.1) and (4.2), it follows that the strains and tractions are continuous across S. However, the stress component σ_{rr} may be discontinuous. Substituting the functions ψ and Σ from (3.18) into the continuity of the tractions, i.e., $\underset{\sim}{e}_\theta \cdot [\underset{\sim}{\sigma}] = \underset{\sim}{0}$, we obtain

$$[\sin\psi] = 0 \text{ and } [\Sigma] + \cos\psi = 0 \text{ at } \theta = \theta_S. \tag{4.4}$$

Hence, the discontinuous stress component is given by

$$\frac{1}{K}[\sigma_{rr}] = [\Sigma] = -\cos\psi. \tag{4.5}$$

Using the condition $\Lambda \geq 0$ for the positive rate of plastic work, from the discontinuity condition (4.4) we obtain a unique solution for Σ and ψ in terms of parameters A, B, C, and M^*. Figure 3 shows the variation of Σ and ψ with respect to θ for $M^* = 0.3$, $A = 0.15$, and $B+C = 0.12$. At $\theta_S \simeq -0.61$ and $\theta_S \simeq 0.69$, there exist two discontinuity surfaces on which the stress field

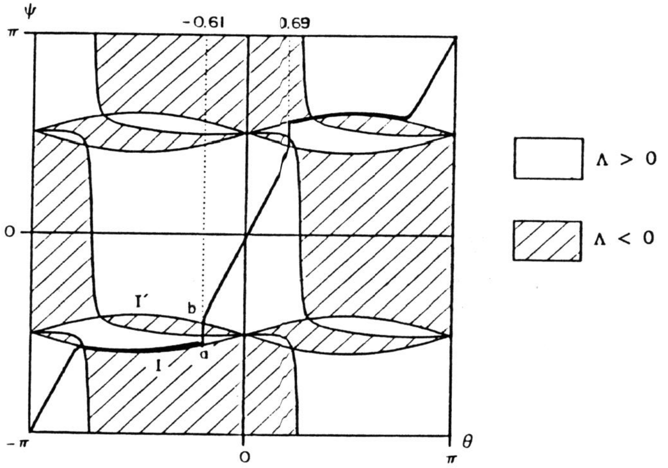

Fig. 3: (a) Variation of ψ with respect to the polar angle θ

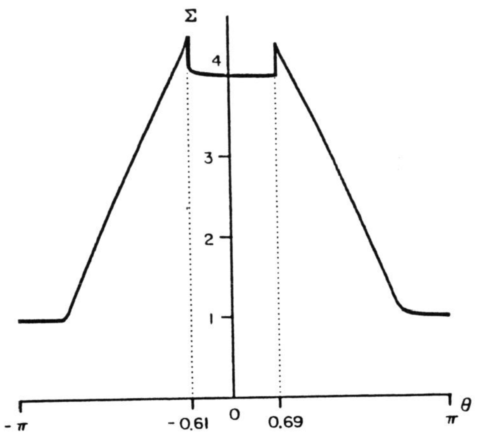

Fig. 3: (b) Variation of Σ with respect to the polar angle θ

is discontinuous, even though the velocity and strain fields are continuous. When the velocity field is continuous, then the internal energy density e (defined by $\rho\dot{e} = \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}}$), is also continuous across S, since the conservation of energy requires (all thermal effects are neglected)

$$m\,[\,e + \frac{1}{2}|\underset{\sim}{v}|^2\,] + [\,\underset{\sim}{v}\cdot(\underset{\sim}{e}_\theta\cdot\underset{\sim}{\sigma})\,] = m\,[\,e + \frac{1}{2}v_r^2\,] + [v_r\sigma_{r\theta}] = 0. \tag{4.6}$$

Then, $[\underset{\sim}{v}] = \underset{\sim}{0}$ and its consequence $[\sigma_{rr}] = [\sigma_{r\theta}] = 0$, lead to $[e] = 0$. Therefore, we conclude that the continuity of the displacement and velocity across S leads to the continuity of the strains, tractions, and internal energy on S, except for the stress component σ_{rr}, i.e.,

$$[\underset{\sim}{u}] = [\underset{\sim}{v}] = \underset{\sim}{0} \rightarrow [\underset{\sim}{\varepsilon}] = \underset{\sim}{0}, \ \underset{\sim}{e}_\theta\cdot[\underset{\sim}{\sigma}] = \underset{\sim}{0}, \text{ and } [e] = 0. \tag{4.7}$$

On the other hand, if it happens that the velocity is discontinuous, then these field quantities may not remain continuous across S. Indeed, although $[v_\theta]$ is zero due to material incompressibility, nonzero $[v_r]$ leads to

$$V_S\,[\varepsilon_{\theta r}] = [v_r], \quad [\sigma_{\theta r}] = -m\,[v_r]. \tag{4.8}$$

It should be noted that for (4.8) to hold, beside discontinuity in ψ or Σ, different values for parameters of U (i.e., A, B, and C) or of $|\underset{\sim}{s}|$ (i.e, K) may have to be admitted across S.

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