Semi-weight Function Method in Fracture Mechanics — Use of a Reciprocal Theorem

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ABSTRACT

In this paper authors propose a new useful method, semi-weight function method, to calculate the stress intensity factor in fracture mechanics. This method has sound theoretic foundation, and can be very easy used in several problems, difference from the weight functions, the semi-weight functions are independent to boundary conditions, It only need to know displacements and tractions along any of the paths around the crack tip.

KEYWORDS

semi-weight function; fracture mechanics; stress intensity factor; reciprocal theorem

INTRODUCTION

The stress intensity factor (SIF) is an important parameter in fracture mechanics, but to find it in a cracked body is not and easy work, recently huge amount of papers in this area were published. About twenty years ago, finite element method (FEM) achieved a lot in many fields, and was introduced to solve fracture mechanical problems, and many trpical engineering problemshad been solved.FEM is the method which needs great amount of computartion, especially in fracture mechanical problems, in which large amount of elements should be concentrated near the crack tip. Lately singular element which shape functions have the same stress singularity as stress field near the crack tip was developed, but to construct a singular element is neither a simple work, and the size of the singular element is hard to determine.

In seventy years, Bueckner proposed the weight function concept (Buechner, 1973), since then many papers have been published (Zhang, 1981). The weight functions process many interesting characters, such as, one can obtain stress intensity factor by simply calculating an integral along the boundary of the cracked gbody, but to find weight functions in also a hard work.

The semi-weight function concept is proposed in this paper. The most important difference from the weight functions are that the semi-weight functions are independent to boundary conditions. It is not only needed to calculate the integral along the loaded boundary, but needed to calculate the integral along all of the boundary or any of the paths around the crack tip . In this method the displacements and tractions along the boundary or a path around the crack tip is needed to know. To obtain displacements and tractions along boundary is much easier than to obtain weight functions or to compute stress field near the crack tip. If one uses FEM to calculate displa-cements along boundary, one needs not to highly concentrate elements near the crack tip. and a lot of complex work is saved.

SEMI-WEIGHT FUNCTION

In 1970 Bueckner, first proposed the concept of weight functions. By the weight function method, the SIF K_I can be expressed as

$$k_I = \int_{\Gamma_\sigma} \bar{p}_i \; u_i \; ds \qquad (1)$$

where u_i (i=1,2) is the weight function and Γ_σ is the loaded boundary. The weight functions possess following properties:

(1) The u_i satisfies the equilibrium equation.

(2) Lim $u_i = O(r^{-1/2})$, near the crack tip.

(3) The u_i satisfies the traction free conditions in the loaded boundary

$$P_i(u_i)|_{\Gamma_\sigma} = 0$$

In order to satisfy the third property, it is difficult to find the weight functions.If also implies that the weight functions are different for different plane problems. They may be vary with the shape of the crack body and boundary conditions. Oftem FEM or other method is employed to find the weight functions.

Semi-weight weight functions are the functions that satisfy the condition (1)and (2), near the crack tip they can be written as

$$u_i^{(s)} = Ar^{-1/2} f_i(0) \qquad (2)$$

These functions satisfy the conditions of free traction along the crack surface. Similar to the weight functions, because they result in infinite energy, ther do not stand for the displacements of a real status of a cracked body.

Suppose a cracked body V, showed in figure 1, is subjected load f_i, p_i, and u_i, σ_i are the corresponding displacements and stresses.

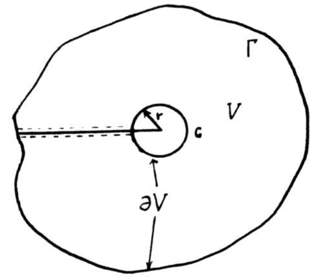

Fig. 1

The well known reciprocal theorem

$$\int\int_V f_i^{(s)} u_i \, dv + \int_{\partial V} p_i^{(s)} u_i \, ds = \int\int_V f_i u_i^{(s)} dv + \int_{\partial V} p_i u_i^{(s)} ds \qquad (3)$$

in the theory of elasticity is still stand although u_i, p_i are not a real status of an elasticity, because proving this theorem only assumes that the matrix of elasticity coefficient is symmetric.

Let us select an integral path C, encircling the crack tip, showed in figure 1. The equation (3) can be wrtten as

$$\int_c (p_i^{(s)} u_i - p_i u_i^{(s)})\,ds = \int\int_v f_i u_i^{(s)}\,dv + \int_\Gamma (p_i u_i^{(s)} - p_i^{(s)} u_i)\,ds \qquad (4)$$

where the p_i $(i=1,2)$ is the traction corresponding to u_i.

Let r (ref.Fig.1) tend to zero. The displacement and stress near the crack tip can be written as

$$\begin{aligned} u &= (k_I/8G)\sqrt{2/\pi}\, r^{1/2}[(2K-1)\cos(\theta/2)-\cos(3\theta/2)] \\ v &= (k_I/8G)\sqrt{2/\pi}\, r^{1/2}[(2k+1)\sin(\theta/2)-\sin(3\theta/2)] \\ \sigma_x &= (k_I/\sqrt{2\pi})\, r^{-1/2}\cos(\theta/2)[1-\sin(\theta/2)-\sin(3\theta/2)] \\ \sigma_y &= (k_I/\sqrt{2\pi})\, r^{-1/2}\cos(\theta/2)[1+\sin(\theta/2)-\sin(3\theta/2)] \\ \tau_{xy} &= (k_I/\sqrt{2\pi})\, r^{-1/2}\cos(\theta/2)\sin(\theta/2)\cos(3\theta/2) \end{aligned} \qquad (5)$$

where

$$k = \begin{cases} 3-4\nu & \text{for plane strain} \\ (3-\nu)/(1+\nu) & \text{for plane stress} \end{cases} \qquad (6)$$

The left side of the equation (4) is CAK_I,

$$\begin{aligned} CAK_I &= \int_c (p_1^{(s)} u_1 + p_2^{(s)} u_2 - p_1 u_1^{(s)} - p_2 u_2^{(s)})\,ds \\ p_1 &= \sigma_x \cos\theta + \tau_{xy}\sin\theta \\ p_2 &= 2_{xy}\cos\theta + \sigma_y \sin\theta \end{aligned} \qquad (7)$$

let $A=1/C$, from equation (4), one can get

$$K_I = \int\int_v f_i u_i^{(s)}\,dv + \int_\Gamma (p_i u_i^{(s)} - p_i^{(s)} u_i)\,ds \qquad (8)$$

of course the integral is independent to the integral path, as a special case it can be the boundary of the body.

Equation (8) shows that to find the stress intensity factor, one only to find the boundary tractions and displacements, or to calculate the integral along any path around the crack tip. By this method the complex numeral analysis and computation near the crack tip can be avoided.

Unlike to the weight functions, the semi-weight functions are independent to boundary conditions, so one need not to find the semi-weight functions from one plane problem to another.

It can be seen that the semi-weight functions u_i is not unique.

For the two dimensional problems, the semi-weight functions can be expressed as

$$\begin{aligned} u^{(s)} &= -(A/2G)\, r^{-1/2}[(k-3/2)\cos(\theta/2)+(1/2)\cos(5\theta/2)] \\ v^{(s)} &= (A/2G)\, r^{-1/2}[(k+3/2)\sin(\theta/2)-(1/2)\sin(5\theta/2)] \\ \sigma_x^{(s)} &= Ar^{-3/2}[(1/4)\cos(3\theta/2)+(3/4)\cos(7\theta/2)] \\ \sigma_y^{(s)} &= Ar^{-3/2}[(7/4)\cos(3\theta/2)-(3/4)\cos(7\theta/2)] \\ \tau_{xy} &= (3/4)Ar^{-3/2}[-\sin(3\theta/2)+\sin(7\theta/2)] \end{aligned} \qquad (9)$$

and the corresponding A is

$$A = -\frac{2G}{\sqrt{2\pi}\,(k+1)} \qquad (10)$$

NUMERICAL COMPUTATION

As an example of this method, let us consider a classical problems, that is the single edge cracked plate, with the width W, length L, and crack length a, subjected uniform tension p showed in figure 2.

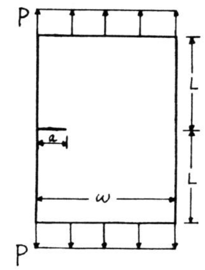

Fig. 2

To find the displacemenr along the boundary, we simply adopt Williams' expansions,through the energy method.

$$\begin{aligned} u &= \Sigma\, c_i u_i \\ v &= \Sigma\, c_i v_i \end{aligned} \qquad (11)$$

where u_i, v_i is the ith term of Williams' expansions.

Because every term of the Williams'expansions satisfies the equilibrium equation, the potential energy may be written as

$$E = 1/2\ \{a\}^T [K] \{a\} - \{a\}^T \{b\} \qquad (12)$$

where

$$[K]=[\int_\Gamma (p_{xi}u_j+p_{yi}v_j)ds] \quad (13)$$
$$\{a\}=[c_1, c_2, ...c_n]$$
$$\{b\}=\{\int_\Gamma \bar{p}_y v_i ds\}$$

and p_{xi}, p_{yi} are two tractions corresponding to u_i and v_i, Γ is the boundary of the plane. Performance of the minimisation respect to the (a) results om

$$[K]\{a\}=\{b\}$$

and
$$\{a\}=[K]^{-1}\{b\}$$
$$\{u\}=[U]\{a\}$$

By formula (8), one reach K.

The value of K_I's change with the a/W are listed in th table 1

Table 1.

a/w	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Fan,(1981)	2.11	2.43	2.94	3.76	5.01	7.15	11.3	21.3
SWF	2.09	2.32	2.77	3.52	4.71	6.74	10.7	20.0
Deviation	1.0%	4.5%	5.8%	6.4%	6.0%	6.0%	5.3%	6.0%

Note: SWF stands for Semi-weight function.

It can be seen that this method is very simple, but the results can be used in engineering designing. To reach the results only a few minutes of CPU time is needed in micro-computer. If more accurate method is adopted to calculate the boundary displacements and tractions, more accurate results will be expected.

CONCLUSION

The semi-weight function method is a convenient method for calculationg the stress intensity factors in various problems. It only need to know the tractions and displacements along any of the paths around the crack tip. The stress intensity factor can be calculate from the formula (8). The semi-weight function is independent to the boundary conditions. By using this method the complex analysis near the crack tip can be avoided. This method can also combine easily with other method to reach good results.

REFERENCE

Bueckner, H. F.(1973). Field singularities and related integral expressions. Mechanics of Fracture 1. Chapter 5. edited by G. C. Sih.

Fan Tian-you. (1981). Basic of fracture mechanics.(In Chinese)

Zhang Xiao-ti. (1981). The computation of the weight function for a cracked body by superposing the finite element solution on the strong singular term. Acta Mechanica Sinica. Special Issue. (in Chinese)