Relativistic BCS–OHR Model

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ABSTRACT

We show that the simple one-dimensional BCS-Ohr model of fracture can be generalized to relativistic steady state motion and can serve as very simple and analytic zero order approximation for dynamic fracture and crack arrest problem.

KEYWORDS

BCS-OHR model; crack arrest; dislocation free zone; dislocation shielding; dynamic fracture; plastic zone size; steady state motion.

INTRODUCTION

Recent studies of crack arrest and other aspects of fast fracture as reviewed (Magata et al., 1987) in the continuum approximation have been based on time dependent creep constitutive laws which yield the crack tip fields of the inverse square root variety only when the power of the creep law is less than 3 (Hui and Riedel, 1981). In this work, the continuum approximation is used right down to the smallest dimensions at the crack tip. On the other hand, other dislocation-based models of fracture have emphasized the fact that dislocations are present in small numbers when viewed on an atomic level at the crack tip, and dislocation free zones (Chang and Ohr, 1981, Weertman et al., 1983) characterize the innermost crack tip regions. Although there exists some controversy about the presence of dislocation free zones, and the role they play in fracture, we believe they should not be ignored in modeling of the crack tip behavior. In particular, we believe that the fracture toughness laws for fast crack growth and arrest should incorporate ideas about the dislocation free zones in order to assure that the proper physics of the crack-dislocation interactions have been incorporated in the final models. In addition to the dislocation shielding at the crack tip, cleavage cracks also usually bifurcate onto neighboring parallel cleavage planes, with ligament formation, which also leads to a kind of shielding of the local crack tip stress intensity factor.

It is the purpose of this paper to develop the relativistic equations which govern crack shielding. In particular, we will generalize to steady state relativistic velocities the one dimensional BCS model (Bilby et al., 1963) (in the Chang and Ohr modification (1981), which serves as a zeroth order approximation for all dislocation-crack shielding modeling. In later investigations, we will pursue the consequences of dislocation free zones and ligament shielding for fast fracture and arrest, and compare such models with the continuum based case. Our argument focuses on mode-III moving crack and screw moving dislocations in the steady state condition because of the simplicity of that analysis, but the results are applicable in a qualitative way to the more physical mode-I problems.

FORCES ON ELASTIC MOVING SINGULARITIES

In this section we shall develop the necessary elastic analysis for the combined crack-dislocation configuration, which is in steady-state motion with velocity v along the x_1 axis. The wave equation in antiplane strain, characterized by displacement u_3,

$$\left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}\right) u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{1}$$

is satisfied by the substitution,

$$x_1 = X_1 - vt,\ x_2 = X_2,\quad \frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x_1^2}. \tag{2}$$

Here we have dropped the subscript 3, because we shall be concerned only with antiplane strain. Thus the solution for a set of uniformly moving singularities is also the solution of the Laplace equation,

$$\gamma^2 \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0, \tag{3}$$

with $\gamma^2 = 1 - v^2/c^2 = 1 - \beta^2$, where $\beta = v/c$.

In analogy to the standard mode-III time-independent analysis, we introduce the stress function $\eta(z)$ such that the displacements and stresses become

$$u = (2/\mu)\mathrm{Im}[\eta(z)],$$

$$\sigma = (1/\gamma)\sigma_{23} + i\sigma_{13} = \eta',$$

$$z = x_1 + i\gamma x_2, \tag{4}$$

where μ is the shear modulus, and η' is the derivative, $d\eta/dz$. The force on a moving singularity in 2-D elastic medium derived elsewhere (Lin and Thomson, 1986a) from the general Atkinson and Eshelby expression (1986) is

$$f_1 = (4\pi\gamma/\mu)\mathrm{Re}\{\mathrm{Res}[\eta'(z)]^2\}, \tag{5a}$$

$$f_2 = -(4\pi\gamma^2/\mu)\mathrm{Im}\{\mathrm{Res}[\eta'(z)]^2\}$$

$$-\frac{4\beta^2}{\mu}\oint [\mathrm{Im}\eta']^2 dx_1. \tag{5b}$$

The force components f_1 and f_2 on the moving singularity are calculated in terms of the residue at the singularity.

When applied to a crack, the f_2 force is not a physical motion for the crack because crack branching is not a self-similar translation of the crack (Lin and Thomson, 1986b), but (5b) can be applied to a dislocation and corresponds to a transverse dislocation force. The solution for a dislocation in steady-state motion in the x_1 direction is

$$2\eta' = \mu b/2\pi z + \sigma_{31}, \tag{6}$$

where σ_{31} is an externally applied stress. Direct application of (5b) gives the result

$$f_2 = -b\sigma_{31}, \tag{7}$$

which is the classic Peach-Koehler result (Hirth and Lothe, 1982).

When a moving crack under an applied load interacts with N moving dislocations at ζ_j, the field at z is known (Lin and Thomson, 1986a)

$$\eta'(z) = \frac{K(v)}{2\gamma\sqrt{2\pi z}} + \sum_j \frac{\mu b_j}{8\pi\sqrt{z}}\left(\frac{1}{\sqrt{z} - \sqrt{\zeta_j}} - \frac{1}{\sqrt{z} + \sqrt{\zeta_j}}\right). \tag{8}$$

where the applied dynamic stress intensity factor K(v) is related to the static one by

$$K(v) = \sqrt{1-\beta}\, K_{III}(0), \tag{9}$$

which was obtained by Eshelby (1969). The local stress intensity factor k(v) and the dislocation-induced stress intensity factor $K_d(v)$ associated with (8) are given as

$$k(v) = K(v) - K_d(v) \tag{10}$$

$$K_d(v) = -\gamma \lim_{z\to 0} 2\sqrt{2\pi z} \sum_j \frac{\mu b_j}{8\pi\sqrt{z}}\left(\frac{1}{\sqrt{z} - \sqrt{\zeta_j}} - \frac{1}{\sqrt{z} + \sqrt{\zeta_j}}\right),$$

$$= \sum_j \frac{\mu b_s \gamma}{2\sqrt{2\pi}}\left(\frac{1}{\sqrt{\zeta_j}} + \frac{1}{\sqrt{\zeta_j}}\right).$$

ONE DIMENSIONAL RELATIVISTIC BCS-OHR MODEL

In our one dimensional model, a moving crack under an applied load interacts with N moving dislocations, distributed on the x_1 axis. Substituting (8) into (5a) and setting $\zeta_j = x_j$, the force on the moving crack tip and the dislocation can be determined. For the crack,

$$f_c(v) = \frac{k^2(v)}{2\mu\gamma} - 2\gamma_s(v) \tag{11}$$

$$k(v) = K(v) - K_d(v) \tag{12}$$

$$K_d(v) = \sum_j \frac{\mu b \gamma}{\sqrt{2x_j}} . \tag{13}$$

For a dislocation at x_i

$$f_d(x_i,v) = \frac{b_i K(v)}{\sqrt{2\pi x_i}} - \frac{\gamma\mu b_i^2}{4\pi x_i} + \sum_j{}' \frac{\gamma\mu b_i b_j}{2\pi(x_i - x_j)}\sqrt{\frac{x_j}{x_i}} \tag{14}$$

$$-b_i \sigma_f(x_i, v),$$

where $K(v)$ is the applied stress intensity factor, b_i is the Burgers vector of a reference dislocation, and b_j is the Burgers vector of all other dislocations. The sum over j is a sum over the dislocation distribution. Σ' denotes a sum over all dislocations except that for which the force is being calculated. The local stress intensity factor, $k(v)$, for the moving crack is a shielded value relative to the external stress intensity when the Burgers vectors have a positive sign. The surface tension, $\gamma_s(v)$, and friction stress, $\sigma_f(v)$, to the dislocation motion determine the kinetic law for the crack tip velocity and the dynamic law for the moving dislocation. In steady state, the value of $f_c(v)$ for the crack and $f_d(x_i,v)$ for each dislocation goes to zero.

We are now in a position to derive a dislocation shielding theorem for the crack when all defects are in steady state motion with a velocity v. The force exerted on the total dislocation pileup is the sum of (14) over all the dislocations. Carrying out this sum with the help of (13) yields

$$F_d = \sum_{i=1}^{N} f_d\,(x_i,v) = \frac{K(v)\,K_d(v)}{\mu\gamma} - \frac{K_d^2(v)}{2\mu\gamma} \tag{15}$$

$$- \sum_{i=1}^{N} b_i\ \sigma_f(x_i,v).$$

Combining (12) and (15), we have the dislocation shielding theorem sought when $f_d\,(x_i,v) = 0$,

$$\frac{K^2(v) - k^2(v)}{2\mu\gamma} = \sum_{i=1}^{N} b_i\ \sigma_f\,(x_i,v). \tag{16}$$

Equation (16) is a general result in the sense that no assumptions are made regarding $\sigma_f(x_i,v)$ and in order to obtain more specific results the form of this function must be specified.

In the rest of this section we will replace the discrete distribution by a continuum distribution, $\alpha(x,v)dx$ derived by smearing the discrete dislocations in the pileup into continuous distribution of dislocation. Let W be the dislocation free zone, which is the distance between the moving crack tip and the nearest dislocation in the pileup. Let R be the plastic zone size, which is the distance between the moving crack tip and the furthest dislocation in the pileup.

In steady state, the force balance equation shown in (14) gives Cauchy principal-value integral

$$\sigma_f\ (x,v) = \frac{K(v)}{\sqrt{2\pi x}} + \int_W^R \frac{\gamma\mu}{2\pi}\sqrt{\frac{x'}{x}}\left[\frac{\alpha(x',v)}{x - x'}\right]dx' . \tag{17}$$

An alternative equation is characterized by the local-k when (12) and (17) are combined to eliminate $K(v)$,

$$\sigma_f(x,v) = \frac{k(v)}{\sqrt{2\pi x}} + \int_W^R \frac{\gamma\mu}{2\pi}\sqrt{\frac{x}{x'}}\left[\frac{\alpha(x'v)}{x - x'}\right]dx'. \tag{18}$$

Equations (17) and (18) are the standard singular integrals (Muskhelishvili, 1977) and the results have been summarized in a useful form (Head and Louat, 1955). The uniqueness relation (Head and Louat, 1955) for a distribution $\beta(x)$, which has zero values at W and R, is given by

$$\int_W^R \frac{P(x')\ dx'}{\sqrt{(x' - w)(R - x')}} = 0 \tag{19}$$

where $$p(x) = \frac{2\pi\sqrt{x}}{\mu\gamma}\left(\sigma_f(x,v) - \frac{K(v)}{\sqrt{2\pi x}}\right) \tag{20}$$

for (17) and

$$p(x) = \frac{2\pi}{\mu\gamma\sqrt{x}}\left(\sigma_f(x,v) - \frac{k(v)}{\sqrt{2\pi x}}\right) \tag{21}$$

for (18).

Integrating (19), the uniqueness relations become

$$K(v) = 2\sqrt{\frac{2}{\pi}}\ E\ (\kappa)\ \sigma_f(v)\ \sqrt{R} \tag{22}$$

$$k(v) = 2\sqrt{\frac{2}{\pi}}\ K(\kappa)\ \sigma_f(v)\ \sqrt{W} \tag{23}$$

where $K(\kappa)$ and $E(\kappa)$ are respectively the complete elliptic integral of the first and the second kind, with modulus $\kappa = \sqrt{1 - W/R}$. In these equations the reader will find a notational inconsistency, because we have used K to denote both the stress intensity factor and the complete elliptic integral of the first kind. We have done so to conform to the traditional notation. There is no difficulty, however, if readers will note that the argument of the stress intensity factor is always velocity, v, while, the argument of the elliptic integral is always κ. In the derivation of (22) and (23), we have assumed that the friction stress is a function of steady state velocity and independent of position. When $v \to 0$ (22) and (23) give the well-known static results (Chang and Ohr, 1981, Majumdar and Burns, 1983, Weertman et al., 1983). To complete our analysis, the integral in (17) is inverted and the dislocation density $\alpha(x,v)$ is obtained by the equation

$$\alpha(x,v) = \frac{1}{\pi^2}\sqrt{\frac{(x - 2)(R - x)}{x}}\int_W^R \frac{P(x')dx'}{\sqrt{x' - w)(R - x')}(x' - x)}, \tag{24}$$

where p(x) is given in (20). After the integrating (24), the dislocation density function becomes

$$\alpha(x) = \frac{4\sigma_f(v)}{\pi\mu\gamma} K(\kappa)\ Z(\phi,\ \kappa)$$

$$= \frac{4\sigma_f(v)}{\pi\mu\gamma}\ [K(\kappa)\ E(\phi,\kappa) - E(\kappa)\ F(\phi,\kappa)], \tag{25}$$

where $Z(\phi, \kappa)$ is Jacobi's zeta function with argument $\phi = \sin^{-1}(\sqrt{1 - w/x})/\kappa)$. $F(\phi, \kappa)$ and $E(\phi, \kappa)$ are incomplete elliptic integrals of the first and the second kind. The local stress intensity factor at the moving crack tip is

$$k(v) = K(v) - \frac{\gamma\mu}{\sqrt{2\pi}}\int_W^R \frac{\alpha(x',v)}{\sqrt{x'}}\ dx'. \tag{26}$$

Combining (25) and (26), we have

$$k(v) = K(v) - 2\sqrt{\frac{2}{\pi}}\ K(\kappa)\left[\frac{1}{\sqrt{1 - \kappa^2}}\ \frac{E(\kappa)}{K(\kappa)} - 1\right]\sigma_f(v)\sqrt{w}\ . \tag{27}$$

Substituting (22) into (27), we find the local stress intensity

$$k(v) = 2\sqrt{\frac{2}{\pi}}\ K(k)\ \sigma_f(v)\ \sqrt{v}\ . \tag{28}$$

This result is exactly the same as (23) derived from the uniqueness condition. The total Burgers vector of dislocations, B, is obtained by integrating the integral

$$B = \int_W^R \alpha(x',v)\ dx'$$

$$= \frac{4W\sigma_f(v)}{\pi\mu\gamma}\ K^2(k)\left[\frac{E^2(k)}{K^2(k)}\ \frac{1}{1-k^2} - 1\right]. \tag{29}$$

We note that equations (22), (23), and (29) derived for dislocation shielding in the relativistic steady state are not a complete set in terms of unknown variables, until the constitutive relation between $\sigma_f(v)$ and $\alpha(v)$ is specified. We will return to this part of the problem in a later paper.

In 2-D equation (16) becomes

$$\frac{k^2(v) - k^2(v)}{2\mu\gamma} = \iint b\sigma_f(x,y,v)\ \alpha\ (x,y,v)\ dxdy\ , \tag{30}$$

where $\alpha(x,y,v)$ is two dimensional dislocation density and $db = \alpha dxdy$. The strain rate is related to α by

$$b\alpha v = \dot{\epsilon} \tag{31}$$

Using (31), then (30) becomes

$$\frac{K^2(v) - k^2(v)}{2\mu\gamma} = \frac{1}{v}\iint \sigma_f(x,y,v)\ \dot{\epsilon}\ dxdy, \tag{32}$$

which was obtained by Freund and Hutchinson (1985) using a somewhat different approach.

One of the major conclusion of prior analysis of dynamic fracture is that for a given applied K(0) the plastic zone size, R, decreases with crack velocity. This result follows immediately from our simplified analysis, because (22) with (9) can be written as

$$\sqrt{1 - \beta}\ K(0) = 2\sqrt{\frac{2}{\pi}}\ E(\kappa)\ \sigma_f(v)\ \sqrt{R}\ . \tag{33}$$

Since $\sigma_f(v)$ is an increasing function of v, it follows that R must be a decreasing function of v to satisfy (33).

CONCLUSIONS

We have shown that the simple one-dimensional BCS-Ohr model of fracture can be generalized to relativistic steady state motion and can serve as a very simple and analytic zero order approximation for dynamic fracture and crack arrest problems.

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