J-integral Calculation by Using Collocation Method and Elasto–Viscoplastic Theory

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ABSTRACT

A combination of collocation method and elasto-viscoplastic theory is used to calculate J-integral of internal crack specimens. The results obtained compare very favorably with those obtained by finite element method, and less computer time is required.

INTRODUCTION

The elasto-viscoplastic theory can be used as a unified numerical solution approach for visco-plasticity, plasticity and creep problems in solids[1], and it was found that the steady-state solution of the viscoplastic problem is identical to the corresponding conventional static elasto-plastic solution, for instance. On the other hand, boundary collocation method has been found to be a simple, effective and accurate tool of analysis for calculating the stress intensity factors of fracture specimens. It has been shown that the combination of the elasto-viscoplastic theory and the collocation method provides a useful approach to the solution of a series of elasto-plastic fracture problems. In this paper, the proposed method is used to calculate the J-integral value of center-cracked and slant-cracked plates. Numerical examples show that this method offers good accuracy and requires less computation time.

BASIC FORMULATION

(1) Theory of elasto-viscoplasticity

For elasto-plastic solids, the total strain, ε, can be separated into elastic, ε_e, and viscoplastic, ε_{vp}, components. The relation is expressed as[2,3]

$$\varepsilon = \varepsilon_e + \varepsilon_{vp} \tag{1}$$

The total stress depends on the elastic strain component according to

$$\sigma = D\varepsilon_e = D(\varepsilon - \varepsilon_{vp}) \tag{2}$$

where D is the elastic matrix.

The viscoplastic flow rule is

$$\dot{\varepsilon}_{vp} = \gamma < \Phi(F) > \frac{\partial F}{\partial \sigma} \tag{3}$$

where (·) represents differentiation with respect to time, γ is a fluidity parameter controlling the plastic flow rate, and $\Phi(F)$ is a positive monotonically increasing function

$$< \Phi(F) > = \begin{cases} \Phi(F) & \text{for } F > 0 \\ 0 & F \le 0 \end{cases} \tag{4}$$

A simple form of $\Phi(F)$ adopted in this paper is

$$\Phi(F) = \frac{F}{\sigma_s} \tag{5}$$

where F is the Mises yield function, and σ_s is the uniaxial yield stress. The strain increment $\Delta(\varepsilon_{vp})_n$ occurring in a time interval $\Delta t_n = t_{n+1} - t_n$ can be expressed by difference formula

$$\Delta(\varepsilon_{vp})_n = (\dot{\varepsilon}_{vp})_n \Delta t_n \tag{6}$$

According to eq. (2), the stress increment $\Delta\sigma_n$ is

$$\begin{aligned} \Delta\sigma_n &= D[\Delta\varepsilon_n - \Delta(\varepsilon_{vp})_n] \\ &= \Delta(\sigma_e)_n - \Delta(\sigma_{vp})_n \end{aligned} \tag{7}$$

and the total stress at time t_{n+1} is

$$\sigma_{n+1} = \sigma_n + \Delta\sigma_n \tag{8}$$

From eq. (6), the viscoplastic strain increment can be calculated. When plastic problems are to be analyzed by this method, the time factor is not important, and γ can be taken as 1, Δt can be any value but must satisfy the stability limit according to Ref.[4].

(2) Collocation Method

Using eq. (2), stresses in plastic condition can be written in tensor form as

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij} - \sigma_{ij}^{vp} \tag{9}$$

where

$$\sigma_{ij}^{vp} = \lambda\delta_{ij}\varepsilon_{kk}^{vp} + 2G\varepsilon_{ij}^{vp} \tag{10}$$

Sutstituting eq. (9) into equilibrium equation, yields

$$(\lambda+G)u_{j,ji} + Gu_{i,jj} + f_i = 0 \tag{11}$$

where

$$f_i = X_i - \sigma_{ij,j}^{vp} \tag{12}$$

Eq. (11) has the same form as that of elasticity, and the only difference is that f_i here depends not only on the body force, but also on the plastic deformation.

Similarly, by substituting eq. (9) into the stress boundary equation, then

$$[\lambda\delta_{ij}u_{k,k} + G(u_{i,j} + u_{j,i})]\, n_j = F_i \tag{13}$$

where

$$F_i = P_i + \sigma_{ij}^{vp} n_j \tag{14}$$

Here F_i is dependent on both the loading and the plastic deformation.

Suppose the displacement u_i has the approximate expression

$$u_i \simeq \tilde{u}_i = \sum_{k=1}^{n} c_k \psi_k \tag{15}$$

where $\psi_1, \ldots, \psi_k$ are k appropriately chosen functions, the coefficients c_k are to be determined by collocation method.

Substituting eq. (15) into eqs. (11), (13) and displacement conditon, they will result in the residuals for the equilibrium equation, stress and displacement boundary conditions, and they are given by

$$\begin{cases} R_d = (\lambda + G)\tilde{u}_{j,ji} + G\tilde{u}_{i,jj} + f_i & \text{in domain} \\ R_s = [\lambda\delta_{ij}\tilde{u}_{k,k} + G(\tilde{u}_{i,j} + \tilde{u}_{j,i})]n_j - F_i & \text{on stress boundary} \\ R_u = \tilde{u}_i - \bar{u}_i & \text{on displacement boundary} \end{cases}$$

or in matrix form

$$\{R\} = [K]\{C\} - \{F'\} \tag{16}$$

By using a least square concept, a set of equations can be obtained by minimizing the square of the residuals with respect to the parameters {C}. Thereafter, the coefficients of displacement function can be determined.

(3) Displacement Functions

The first step in the solution by the elasto-viscoplastic method is the calculation of elastic stress field. If some collocation points are found to have been yielded, stress relaxation can then be carried out. It is therefore necessary to select suitable displacement function ψ_k in eq. (15). For center-cracked plate problems, some stress functions which satisfy the harmonic equation in the domain are used to calculate the stress intensity factor by the boundary collocation method[5,6]. The corresponding displacement functions may then be taken as the basic part for ψ_k in eq. (15). Taking into account that stress relaxation must be carried out on both the domain and the boundary, a complementary part of the displacement functions in the form of a power series of the x and y coordinates have to be added. This is used for equilibrium corrections when a point is under plastic condition.

According to Ref. [6], the complex stress funcitons $\phi(z)$ and $\omega(z)$ may be given as

$$\begin{cases} \phi(z) = \sqrt{z^2 - a^2} \sum_{k=1}^{M} E_k z^{2k-2} + \sum_{k=1}^{M} F_k z^{2k-1} \\ \omega(z) = \sqrt{z^2 - a^2} \sum_{k=1}^{M} E_k z^{2k-2} - \sum_{k=1}^{M} F_k z^{2k-1} \end{cases} \quad (17)$$

where a is half crack length, and E_k, F_k are undetermined coefficients. They are real for symmetric problems, and are complex for general cases, i.e.

$$\left.\begin{aligned} E_k &= A_k + i B_k \\ F_k &= C_k + i D_k \end{aligned}\right\} \quad (18)$$

The solution of the linear elastic plane problems may be obtained by the formulae[7]

$$\begin{cases} \sigma_x + \sigma_y = 4\mathrm{Re}\Phi(z) \\ \sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z-\bar{z})\overline{\Phi'(z)} \\ u + iv = \frac{1}{2G}\left[\kappa\phi(z) - \omega(\bar{z}) - (z-\bar{z})\overline{\Phi(z)}\ \right] \end{cases} \quad (19)$$

where

$$\Phi(z) = \phi'(z), \quad \Omega(z) = \omega'(z)$$

From eq. (17), $\Phi(z)$, $\Omega(z)$ and $\Phi'(z)$ can be determined, and the corresponding stresses and displacements can be calculated by eq. (19).

The displacement functions calculated by eq. (19) are now taken as the basic part of the approximate expression in eq. (15). It is obvious that the relevant stresses have a $1/\sqrt{r}$ singularity, and satisfy the harmonic equation and symmetric condition.

The complementary part of the displacements, i.e. the non-harmonic functions, is assumed as follows

$$\begin{cases} u = \sum_{i=1}^{N} \sum_{j=1}^{L} c_{ij}\, x^{2i}\, y^{2j-1} \\ v = \sum_{i=1}^{N} \sum_{j=1}^{L} d_{ij}\, x^{2i-1}\, y^{2j} \end{cases} \quad (20)$$

The combined form of displacement functions are taken as $\tilde{u}$ in eq. (15). They are then substituted into the equilibrium equations or boundary conditions at the collocation points, then the residual equation can be obtained. By using least square method, a set of equations including the coefficients of displacement functions can be formed and can be written in the following matrix form

$$[K]\{C\} = \{F'\} \quad (21)$$

where [K] matrix can be determined from the coordinates of collocation

points, {C} is the coefficient vector $\{A_k, B_k, C_k, D_k, c_{ij}, d_{ij}\}^T$ which remains to be solved, {F'} is the vector depending on boundary conditions and plastic deformations, and can be expressed as

$$\{F'\} = \begin{Bmatrix} -\sigma^{vp}_{ij,j} \\ P_i + \sigma^{vp}_{ij} \cdot n_j \\ \bar{u}_i \end{Bmatrix} \quad (22)$$

During the first calculation step, the viscoplastic stresses and their derivatives are assumed to be zero, {F'} can be determined by the boundary conditions by eq. (22). If the effective stresses σ_i at some points are greater than the yield stress σ_s, σ^{vp}_{ij} are calculated according to eqs. (6) - (8), and $\sigma^{vp}_{ij,j}$ by finite difference method. Hence {F'} is obtained by eq. (22). The iterative procedure is repeated until the effective stresses at all points are no greater than σ_s. The whole iterative procedure is repeated again with each load increment until the final loading is reached.

(4) J-Integral

It has been shown by Rice[8] that the following integral quantity is path independent:

$$J = \int_{\Gamma} \left(W\, dy - T \frac{\partial u}{\partial x}\, ds \right) \quad (23)$$

where T is the traction vector on a plane defined by the outward drawn normal, u is the displacement vector, W is the strain energy density and the y direction is taken normal to the crack line.

When the stresses, strains and displacements have been solved, J-integral can be calculated along a given path. For a general elasto-plastic case, W can be separated into its elastic and plastic components

$$W = W^e + W^p \quad (24)$$

where W^e is given by

$$W^e = \frac{1}{2} \sigma_{ij}\ (\varepsilon_{ij})^e \quad (25)$$

in which $(\varepsilon_{ij})^e$ denotes the elastic components of strain. The plastic work contribution is given by[3]

$$W^p = \int_0^{\varepsilon^p_i} \sigma_i\, d\varepsilon^p_i \quad (26)$$

where σ_i is the effective stress, where σ_{ij} is the deviatoric stress components, and ε^p_i is effective plastic strain and can be calculated by the components determined by eq. (6).

CENTER-CRACKED PLATE

In two-dimension crack problem, J-integral in eq. (23) can be written as

$$J = \int_{\Gamma} \left\{ W\,dy - [(\sigma_x n_x + \tau_{xy} n_y)\frac{\partial u}{\partial x} + (\tau_{xy} n_x + \sigma_y n_y)\frac{\partial v}{\partial y}]\,ds \right\} \tag{27}$$

For plane stress case, $\sigma_z = \tau_{yz} = \tau_{zx} = 0$,

$$W^e = \frac{1}{2E}[(\sigma_x^2 + \sigma_y^2) - 2\nu\sigma_x\sigma_y + 2(1+\nu)\,\tau_{xy}^2] \tag{28}$$

and for plane strain case, $\sigma_z = \nu(\sigma_x + \sigma_y)$, $\tau_{yz} = \tau_{zx} = 0$,

$$W^e = \frac{1}{2E}[(1-\nu^2)(\sigma_x^2 + \sigma_y^2) - 2\nu(1+\nu)\sigma_x\sigma_y + 2(1+\nu)\,\tau_{xy}^2] \tag{29}$$

The effective or equivalent plastic strain for plane problems is given by the following expression

$$d\varepsilon_i^P = \frac{\sqrt{2}}{3}\sqrt{(d\varepsilon_x^P - d\varepsilon_y^P)^2 + (d\varepsilon_y^P)^2 + (d\varepsilon_x^P)^2 + \frac{3}{2}(d\gamma^P{}_{xy})^2} \tag{30}$$

At any plastic point $d\varepsilon_i^P$ can be calculated by the stress components.

Based on the above analysis, the calculation of J-integral can be performed along a given path. Because of the characterization of path-independence, it is convenience to chose the path which composed by several line segments which are parallel to the coordinates axes as shown in Fig. 1. In the center-cracked plate, the symmetry leads to choose half integral path and the total J-integral is twice the value.

Along line segment AB in the above figure, $n_x = 1$, $n_y = 0$, $ds = dy$, from eq. (27) the increment of J-integral can be calculated by

$$J_{AB} = \int_0^{L_y} [(W^e + W^P) + (\sigma_x \frac{\partial u}{\partial x} + \tau_{xy}\frac{\partial v}{\partial x})]\,dy \tag{31}$$

Alone BC, $n_x = 0$, $n_y = 1$, $dy = 0$, $ds = -dx$, then the increment becomes

$$J_{BC} = \int_{L_x}^{-L_x} (\tau_{xy}\frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial x})dy \tag{32}$$

Alone CD, $n_x = -1$, $n_y = 0$, $ds = -dy$, the increment is

$$J_{CD} = \int_{L_Y}^{0} (W^e + W^P) - (\sigma_x \frac{\partial u}{\partial x} + \tau_{xy}\frac{\partial v}{\partial x})]\,dy \tag{33}$$

When the coefficients of the complex stress functions have been determined, the stresses and the displacements at any point of the plate can be calculated by the related formulae. Then $\partial u/\partial x$ and $\partial v/\partial x$ can be calculated by finite difference method, and the integral can be evaluated by the numerical quadrature formulae.

In order to compare with the existing results in Ref. [9] where FEM is used, same dimensions, material properties and applied loads of the plane stress specimen are assumed in the solution. There are altogether 85 collocation points with 47 points in the domain and 38 points on the boundary. In the calculation, every side of the integral path shown in Fig. 1 is divided into a certain sections, for example, 10 sections.

In order to compare with each other, the distances of the segments L_x and L_y are equal to the radial distance in calculation by FEM. For different integral path and loading, the J values by collocation method (CM) and by FEM are listed in Table 1 and shown graphically in Fig. 2 where r/a is the ratio of the path radial distance to the half crack length.

From Table 1 and Fig. 2 it can be seen that the J-values by the two methods coincide with each other generally. The difference of the average value according to a certain loading is about 5%. For the path near the crack tip, J-integral values by the two methods are far less than the average value. The reason is that the deformation near the crack tip is very violent. This must influence on the conditions of J-integral. Sumpter and Turner[10] have performed elastic-plastic finite element computations (using an incremental flow rule) on a series of center-cracked plates. The J contour integral calculated in the usual way was found to be path independent for intermediate and large contours (path length 0.5a to 10a). According to the above conclusion, it is not surprised that the J-integral values of the small radial distance in Table 1 or Fig. 2 are not satisfactory. Except these cases, the stabiity of calculation and the coincidence of the two methods will be better.

In the above calculation by the collocation method, every side of the contour is divided into 10 segments. Similar calculations have also been performed for 5, 15 or 20 segments, results for different segments also coincide very well and the difference is less than 1%. This shows that the stability of the calculation by CM and elasto-viscoplastic theory is satisfactory.

Table 1. J ($x\sigma_s^2 bx10^{-4}$) values by CM and FEM

σ/σ_s	r/a	0.1712	0.3248	0.4331	0.6102	0.7480	0.9744	average value
0.4	CM	0.2397	0.2402	0.2401	0.2396	0.2390	0.2382	0.2394
	FEM	0.2474	0.2556	0.2548	0.2558	0.2565	0.2579	0.2545
0.5	CM	0.4536	0.5109	0.5205	0.5107	0.5018	0.4874	0.4975
	FEM*							
0.6	CM	0.6094	07.197	0.7593	0.7840	0.7822	0.7614	0.7361
	FEM	0.7136	0.7645	0.7697	0.7837	0.7831	0.7904	0.7675

* When σ/σ_s = 0.5 there are no calculated data listed in Ref. [8].

SLANT-CRACKED PLATE

The calculation of J-value for mixed mode problem can be carried out as for center-cracked plate. According to the definition of J-integral and the angular period of π with respect to the center of the crack, J-value can be calculated along two half integral path as shown in Fig. 3, the total J-integral is

$$J = J_1 + J_2 = (J_{AB} + J_{BC} + J_{CD}) + (J_{A'B'} + J_{B'C'} + J_{C'D'}) \tag{34}$$

where J_{AB}, J_{BC} and J_{CD} can be calculated by eqs. (31) - (33), $J_{A'B'}$, $J_{B'C'}$ and $J_{C'D'}$ can be calculated in the same way, the difference is only that along B'C', ds = dx.

For the mixed mode examples shown in Fig. 3, the collocation points and the plastic zones are distributed in Fig. 4. The material constants are: $E=2\times10^4$, $\nu=0.3$, $\sigma_s=1$. The related dimensions are: h/b=2, a/b=0.2475, $\alpha=45°$. J-integral values have been calculated along different paths. The radial distances of the path, r/a, are taken as 0.05, 0.075, 0.1, 0.125, 0.15, 0.175, 0.2 respectively.

(1) Plane Stress Case

In the first step, the stress intensity factors are calculated. In elastic condition, the relation between the J-integral and the stress intensity factors of the mixed mode fracture problem is[11]

$$J = \frac{1}{E'}(K_I^2 + K_{II}^2) \qquad (35)$$

where E' = E for plane stress case, and $E/(1-\nu^2)$ for plane strain case.

By boundary collocation method, K_I and K_{II} for the case have been calculated, $K_I/(\sigma\sqrt{\pi a}) = 0.6090$, $K_{II}/(\sigma\sqrt{\pi a}) = 0.5424$. Inserting them into eq. (35), $J = 0.5236\times10^{-5}$. The calculated J-values in the different loading and paths are shown in Table 2.

In Table 2 there are two groups of J-integral when $\sigma/\sigma_s = 0.45$, one is at the first step and the deformation is assumed elastic, the other is calculated by stress relaxation using viscoplastic theory. At the first step the average J-value is 0.5108×10^{-5}. This differs from that calculated by eq. (35) by 2.5%. From Table 2 it can be also seen that the difference among the J-values in a certain loading is not large. This means that the path independence of J-integral for mixed mode fracture problem can be shown by the calculation using CM and elasto-plastic theory.

Table 2. J $(\times\sigma_s^2 b\times10^{-5})$ values for the mixed mode plate (plane stress)

σ/σ_s \ r/a		0.05	0.075	0.1	0.125	0.15	0.175	0.2	average value
0.45	(e)	0.5229	0.5218	0.5198	0.5162	0.5100	0.5002	0.4853	0.5109
	(p)	0.6921	0.6870	0.6739	0.6572	0.6375	0.6149	0.5896	0.6503
0.6075		1.034	1.105	1.114	1.102	1.078	1.046	1.007	1.0694
0.6413		1.097	1.198	1.217	1.212	1.190	1.159	1.119	1.1703
0.675		1.156	1.281	1.310	1.310	1.294	1.263	1.222	1.2623
0.6856		1.202	1.351	1.387	1.388	1.371	1.338	1.294	1.3330

Note: when $\sigma/\sigma_s = 0.45$, J values have been calculated at the first step (elastic) and the last step (plastic). Both groups are listed.

(2) Plane Strain Case

For the plane strain case J-integral has also been evaluated and shown in Table 3. At the first step, i.e. elastic calculation, the average J-value is 0.5728×10^{-5} while it is 0.5882×10^{-5} by eq. (35) using the boundary collocation method. The difference between them is about 2.5%. From the table it is also seen that the J-integral can be evaluated by the CM and elasto-viscoplastic theory, and the path independence of J-integral can also be shown by the calculation using this method.

Table 3. J $(\times\sigma_s^2 b\times10^{-5})$ values for the mixed mode plate (plane strain)

σ/σ_s \ r/a		0.05	0.075	0.1	0.125	0.15	0.175	0.2	average value
0.5	(e)	0.5874	0.5586	0.5838	0.5793	0.5718	0.5598	0.4143	0.5728
	(p)	0.7739	0.7688	0.7556	0.7385	0.7183	0.6951	0.6691	0.7313
0.675		1.152	1.212	1.215	1.193	1.171	1.140	1.099	1.1689
0.7312		1.268	1.362	1.377	1.361	1.342	1.310	1.267	1.3267
0.7736		1.383	1.511	1.540	1.529	1.513	1.482	1.436	1.4849
0.7854		1.403	1.536	1.569	1.560	1.545	1.515	1.469	1.5139

Note: when $\sigma/\sigma_s = 0.5$, J values have been calculated at the first step (elastic) and the last step (plastic). Both groups are listed.

CONCLUSION

The collocation method and elasto-viscoplasstic theory is effective for studying elasto-plastic problems, such as calculating J-integral, plastic zone. When compared with FEM, this method has comparable accuracy, but requires smaller computer capacity and less computational time. The method is quite versatile, and can be used to study other crack problems, with a suitable change in displacement functions.

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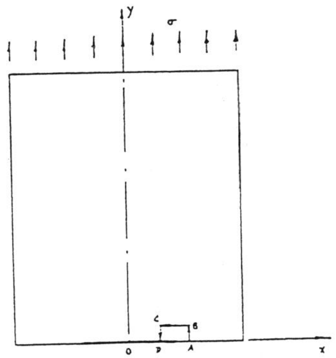

Fig. 1 Calculation path of J-integral for center-cracked plate

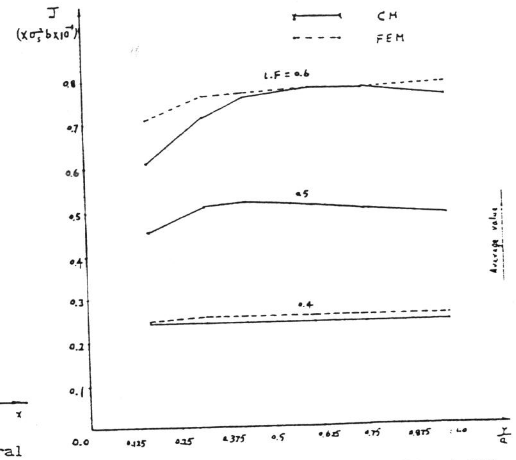

Fig. 2 Comparison of J values by CM and FEM

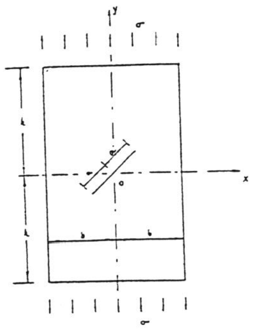

Fig. 3 Center slant cracked plate

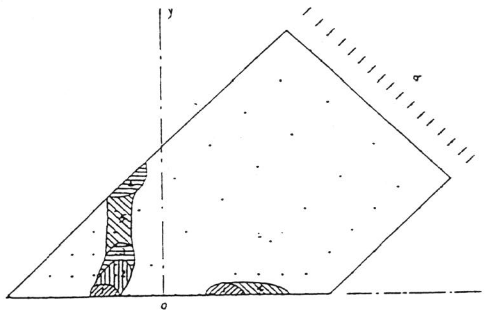

Fig. 4 The collocation points and plastic zone