Equilibrium Criterion for a Nonlinear Elastic Slited Body

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ABSTRACT

An energy criterion of equilibrium for a nonlinear elastic body containing a slit is formulated. From this criterion statical equations and equilibrium conditions on exterior and interior boundaries are obtained. The theory is illustrated by a simple example.

KEYWORDS

Energy, nonlinear, slit, variation, inequality.

AN ENERGY CRITERION

Let us consider a nonlinear elastic body containing a defect in its natural state. The defect being modeled by a surface of discontinuity will be called a slit. This slit is assumed to settle on a smooth surface Ω, being bounded by a contour $\partial\Omega$.The body occupies a region $V_\Omega=V\setminus(\Omega\cup\partial\Omega)$ of Euclidean three space, having the exterior boundary ∂V and the interior boundary $\Omega\cup\partial\Omega$. This natural configuration is choosen to be a reference configuration. The Cartesian coordinates of a particle are denoted by X_a , a=1,2,3. In a deformed state this particle will have the Cartesian coordinates x_i given by

$$x_i=x_i(X_1,X_2,X_3)\quad , \; i=1,2,3 \tag{1}$$

The coordinates x_i run through a region v, occupied by the deformed body. If the deformed state is equilibrium, then the functions $x_i(X_a)$ perform a one-to-one continuously differentiable mapping from V_Ω to v with the condition that $\det|\partial x_i/\partial X_a|>0$, $X_a\in V_\Omega$. Tracks of $x_i(X_a)$ on two sides of Ω draw the slit surfaces in the deformed configuration. The problem is to find an equilibrium criterion for the configuration $x_i(X_a)$. For this purpose, we formulate a variational principle:a necessary and sufficient equilibrium condition for a configuration of a body containing a slit is that the variation of its energy at this configuration should be more than or equal to zero in the class of admissible configurations. A configuration will be called admissible if its surface of discontinuity contains or coinci-des with Ω.

If the body does not contain the slit, then the above stated variational principle in the class of the continuously differen tiable configurations is reduced to the classical variational principle of nonlinear elasticity (Gibbs,1876;Sedov,1968 and Berdichevski,1983). The first step of generalization to frac-ture mechanics was made by Griffith (Griffith,1920 and 1924). For determining a critical length of an equilibrium slit (in a case of plane deformations) Griffith differentiated the body energy, expressed in terms of the slit length, and equated it to zero. The further reduction of his ideas has been developed in many publications (Cherepanov,1967;Rice,1968;Hutchinson,1968 Carlsson,1974;Bui,1977;Atluri and others,1984). The present paper is aimed at showing the consequencies from the above stated variational principle, particularly the statical equati-ons and the equilibrium conditions on the exterior and interior boundaries. The physical sense of the equilibrium condition on the slit tip is that the module of the transversal energy flux entering into it should be less than, or equal to, the doubled surface energy density. In the linearized theory this condition can be reduced to the well-known Cherepanov's condition (Chere-panov,1967). It is important to note that the restriction for the admissible configurations forbids the body to be back in the state with the "healed" slit. Therefore the theory has evident nonholonomicity.

BASIC EQUATIONS OF EQUILIBRIUM

In order to derive equilirium equations, according to the energy criterion, one must define an energy functional of a body on its arbitrary admissible configuration. Let a configu-ration $x_i(X_a)$ have a surface of discontinuity Σ, which may differ from Ω. By analogy with Griffith's theory we postulate the following expression for the energy functional

$$E[x_i(X_a)] = \int_{V_\Sigma} U(x_{i,a},K_B)\,dX + \int_{\Sigma} 2\gamma\,dA + \int_{V_\Sigma} \rho_o \Phi\,dX - \int_{\partial V_T} T_i x_i\,dA \qquad (2)$$

Here $V_\Sigma = V\setminus(\Sigma \cup \partial\Sigma)$, $\partial V = \partial V_x \cup \partial V_T$, dX and dA denote the element of volume and surface area respectively, ρ_o is the mass density of the material in its natural state, U and γ are the internal energy per unit volume and the surface energy per unit area (the last one is supposed to be constant). The tensor $x_{i,a} = \partial x_i/\partial X_a$ corresponds to the gradient of deformation, while $K_B(X_a)$, the material characteristics. The potential of the body force is denoted by $\Phi(x_i)$, and T_i is the dead traction,prescri-bed on the part ∂V_T of the exterior boundary. On the rest part ∂V_x of the exterior boundary the values of x_i are given $x_i = r_i(X_a)$. Here and afterwards the comma is used to denote partial differentiation and the repeated suffix to denote summation.

According to the formulated energy criterion, the configuration $x_i(X_a)$ with the surface of discontinuity Ω will be equilibrium

if for all admissible configurations $y_i = y_i(X_a,\varepsilon)$ with surfaces of discontinuity Ω^ε, satisfying the restrictions $\Omega^\varepsilon \supseteq \Omega$, $y_i(X_a, 0) = x_i(X_a)$ and $y_i(X_a,\varepsilon) = r_i(X_a)$ when $X_a \in \partial V_x$, we have the inequality

$$\delta E = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E[y_i(X_a,\varepsilon)] \geqslant 0 \qquad (3)$$

From the variational inequality (3), being held for all admissible configurations, one can show that the equilibrium configuration should satisfy the following relations

$$T_{ai,a} + \rho_o F_i = 0 \ , \qquad T_{ai} = \partial U/\partial x_{i,a} \qquad \text{in } V_\Omega \qquad (4)$$

$$x_i = r_i(X_a) \quad \text{on } \partial V_x \ , \quad T_{ai} N_a = T_i \qquad \text{on } \partial V_T \qquad (5)$$

$$T^{\pm}_{ai} N_a = 0 \quad \text{on } \Omega\setminus\Omega^{\pm} \ , \quad T^{\pm}_{ai} N_a x_{i,\alpha} = 0 \quad \text{on } \Omega^{\pm} \qquad (6)$$

$$T^{+}_{ai} N_a n_i \sqrt{a}\big|_{\eta_\alpha} = T^{-}_{ai} N_a n_i \sqrt{a}\big|_{\theta_\alpha} = -p \leqslant 0 \quad \text{on } \Omega^{+} \qquad (7)$$

$$|J_\alpha| = \sqrt{J_a J_a - J_3^2} \leqslant 2\gamma \quad \text{on } \partial\Omega \qquad (8)$$

Here T_{ai} is the Piola stress tensor, $F_i = -\partial\Phi/\partial x_i$ is the body force per unit mass, the indexes +,- indicate the limit values of quantities on two sides of Ω, N_a is the outward unit normal on surfaces (on the interior surface Ω it is in the direction of +). Let the sub-areas of Ω be denote by Ω^+ and Ω^- , whose points after deformation will be in contact with each other. If the surface Ω is referred to a curvilinear two-coordinates system, then the contact means that $x^+_i(\eta_\alpha) = x^-_i(\theta_\alpha)$, where $\eta_\alpha \in \Omega^+$, $\theta_\alpha \in \Omega^-$, $\alpha = 1,2$. In the boundary conditions (6) and (7) $x^\alpha_{i,\alpha} = \partial x^\alpha_i/\partial\eta_\alpha$, $a = \det|a_{\alpha\beta}|$, $a_{\alpha\beta} = X_{a,\alpha} X_{a,\beta}$, $X_{a,\alpha} = \partial X_a/\partial\eta_\alpha$. The vector n_i is the common unit normal on two deformed contact surfaces in the direction of +. The second condition of (6) expresses the fact, that the friction force between the contact surfaces is not taken into account. Lastly, in the condition (8) J_a is the vector of the energy flux entering into the slit tip (Cherepanov,1967;Rice,1968) to be calculated by

$$J_a = \lim_{|\Gamma|\to 0} \int_\Gamma (-T_{bi} x_{i,a} \kappa_b + U\kappa_a)\,dS \qquad (9)$$

where the closed contour Γ,settling on the transversal to $\partial\Omega$ plane surface, surrounds the point X_a on $\partial\Omega$ and shrinks to it when the contour length $|\Gamma|$ tends to zero. The vector κ_b is the unit outward normal on Γ , J_3 denotes the component $J_a \tau_a$, where τ_a is the tangential vector on $\partial\Omega$.

The relations (4)-(8) compose the system of statical equations and boundary conditions, which must be satisfied by any equili-brium configuration. It is of interest to note that the condition (8) is separated from the rest of the relations. Therefore in practice one can solve firstly the system (4)-(7) to find the deformed configuration $x_i(X_a)$ and the stress field T_{ai} , and then, by (9),calculate J_a and verify the inequality (8)

As an illustration consider a simple example of an anti-plane shear of an infinite slab containing a slit, when it is deformed at infinity to a state of simple shear (Knowles,1977) (Fig. 1)

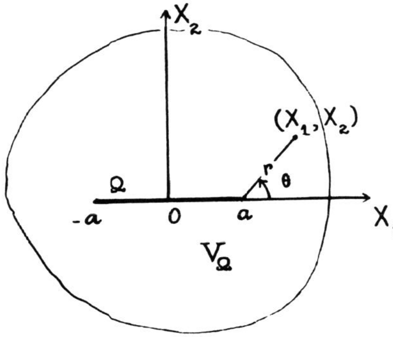

Figure 1.

The material of the slab is supposed to be homogeneous, incompressible and neo-Hookean, for which

$$U = \frac{\mu}{2}(x_{i,a}x_{i,a}-3), \quad \det|x_{i,a}|=1 \tag{10}$$

where $\mu > 0$ is the (constant) shear modulus. For the exterior boundary condition instead of (5) one has to give the asymptotic field of simple shear at infinity

$$x_1 = X_1 \; ; \quad x_2 = X_2 \; , \quad x_3 = X_3+kX_2+o(1) \text{ as } X_1^2+X_2^2\to\infty \tag{11}$$

In the case to be considered here, the contact of the slit surfaces takes place and $\Omega^+= \Omega^-=\Omega$, where $\Omega=\{X_a, X_2=0, |X_1|<a\}$.
But one can show that p=0 on Ω , so the slit surface are traction-free. Not going into problem solution details (Knowles ,1977), one demonstrates the asymptotic behaviour of x_i and T_{ai} near the slit tip

$$x_1=X_1 \; , \; x_2=X_2 \; , \quad x_3= X_3 + k\sqrt{2ar}\,\sin\frac{\theta}{2}$$
$$T_{13}=T_{31}=- ka\sqrt{2ar}\,\sin\frac{\theta}{2} \; , \; T_{23}=T_{32}= ka\sqrt{2ar}\,\cos\frac{\theta}{2} \tag{12}$$
$$T_{\alpha\beta}=0 \; ,\alpha\beta=1,2 \; , \quad T_{33}=0$$

where r,θ are polar coordinates at the right slit tip (Fig.1). Using the formulae (9),(12) one can easily calculate J_a. At the right slit tip J_a is given by

$$J_1=\frac{1}{2}\mu k^2 a\pi \; , \quad J_2=J_3=0 \tag{13}$$

From (8) and (13) one can conclude that only slabs, containing a slit of length less than or equal to $8\gamma/(\mu k^2\pi)$ can be in the state of equilibrium under an anti-plane shear.

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