Dynamic Stress Concentration in Annular Disks Under Transient Loading

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ABSTRACT

Elastic waves in bounded regions represent a problem of dynamic stress concentration closer to practical applications than corresponding investigations on the scattering of waves by a discontinuity in unbounded media. Supplementing recent considerations of steady-state stress concentrations in an annular disk, here transient waves in such structures are considered. In particular, a disk subjected to a locally concentrated impact in the form of a radially directed Heaviside step function is discussed in detail.

KEYWORDS

Dynamic stress concentration; transient loading; annular disk; modal expansion technique.

INTRODUCTION

One of the major applications of the theory of elastostatics is the determination of stress concentrations in a body with holes or other similar discontinuities. Since the late 1950s, many researchers have extended such investigations to the dynamic case, e.g., Jimbo and Nishimura (1954), Pao (1962) or Itou (1983). Common to all of the mentioned contributions is the fact that they discuss the scattering of waves in unbounded regions.

As far as the authors are aware, dynamic stress concentrations in finite media - including damping - were first analyzed by Wauer (1983), who considered the simplest case of an annular disk under a circumferentially constant, radially directed, time-harmonic loading at the outer surface (a related paper by Boroskowsky et al. (1982) discussed only the vector field of displacements for a cracked disk and a contribution by Atluri et al. (1979) considered propagating cracks in finite bodies).

Maass (1986) extended the results to more general disk geometries and arbitrarily distributed circumferential load configurations. He also presented the first results on transient stress concentrations. But since the expense of calculations increases rapidly, in the latter case he restricted his examples to annular disks under rotationally symmetric, radially directed loads again. As prototypes of transient sources, Heaviside and Dirac functions were introduced. It will be noticed that about the same time a paper by Kraft (1985) studied similar problems but was purely numerical, using finite-element methods.

The aim of this contribution is to extend Maass's fundamental results on transient stress concentrations in annular disks to circumferentially nonuniform impact loads, here in the form of Heaviside step functions. As a practical solution technique, a modal expansion into a series will be used.

ELASTODYNAMIC FUNDAMENTALS AND THE APPLICATION TO A THIN ANNULAR DISK

It will be assumed that the material is homogeneous and isotropic, and that all displacements are so small that a linear theory is sufficient. This linear theory of elastodynamics is embodied in a set of equations (Eringen and Suhubi (1985), Achenbach (1975), for instance) defined for a body of volume V enclosed by a surface $S = S_1 + S_2$. In particular, we have the equations of motion, the strain-displacement relations and Hooke's law for an elastic body. There are added corresponding boundary conditions on S_1 (prescribed displacements) or S_2 (prescribed stresses) and initial conditions.

Eliminating the strain and stress tensor leads to Lamé- Navier's classical equation of motion for an isotropic, homogeneous, 3-dimensional elastic body expressed in displacement quantities only. It can be stated here that an extension to a viscoelastic solid is possible without fundamental difficulties (Maass, 1986).

In the case of thin disks to be studied here, the governing vector equation of motion can be modified in the sense of a generalized stress state (Mußchelischwilli, 1971)

$$\left[\frac{2\mu\lambda}{\lambda+2\mu} + \mu\right]\nabla\nabla\cdot\underline{u} + \mu\nabla^2\underline{u} + \underline{k} = \rho\,\frac{\partial^2}{\partial t^2}\,\underline{u}. \tag{1}$$

The original boundary and initial conditions remain formally unchanged. ∇ denotes the well-known Nabla operator, $\underline{u}$ is the vector of displacements and ρ the mass density. λ,μ are Lamé's constants characterizing an isotropic, homogeneous elastic material, and by the vector $\underline{k}$ volume forces are taken into consideration.

The elastodynamic basic equations will be applied to an annular disk (see Fig. 1) with constant thickness b, inner and outer radii R_i and R_a, excited by a radially directed space-and-time-dependent stress per unit length $\sigma_0(\varphi,t)/R_a$ at the outer surface $r = R_a$. The inner edge will be free of stress. An adequate co-ordinate system describing the governing boundary value problem is a frame of polar co-ordinates r,φ with the unit base vectors $\underline{e}_r$, $\underline{e}_\varphi$. The corresponding space-and-time-dependent displacements are $u_r(r,\varphi,t)$ and $u_\varphi(r,\varphi,t)$, combined in the vector $\underline{u} = (u_r,u_\varphi)^T$, where the superscript T denotes transposition.

It is assumed that the disk is at in rest for $t \leq 0$. First of all, it follows that the initial conditions are homogeneous:

$$\underline{u}(r,\varphi,0) = \underline{0}, \qquad \frac{\partial}{\partial t}\,\underline{u}(r,\varphi,0) = \underline{0}. \tag{2}$$

Next, the excitation has to be incorporated. Here, it will be done by

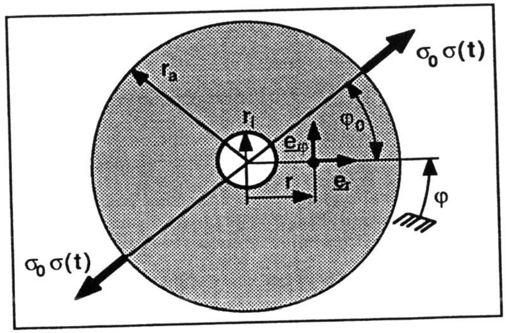

Fig. 1. Physical model

homogeneous dynamic boundary conditions

$$\sigma_{rr}(R_i,\varphi,t) = \sigma_{rr}(R_a,\varphi,t) = 0, \qquad \sigma_{r\varphi}(R_i,\varphi,t) = \sigma_{r\varphi}(R_a,\varphi,t) = 0 \tag{3}$$

(surface parts S_1, where displacements are prescribed, will not exist). The elements of the plane stress tensor can be expressed as functions of the displacements u_r, u_φ:

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[\frac{\partial u_r}{\partial r} + \nu\left(\frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi}\right)\right],$$

$$\sigma_{\varphi\varphi} = \frac{E}{1-\nu^2}\left[\frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\varphi}{\partial\varphi} + \nu\,\frac{\partial u_r}{\partial r}\right], \tag{4}$$

$$\sigma_{r\varphi} = \frac{E}{2(1+\nu)}\left[\frac{1}{r}\frac{\partial u_r}{\partial u_\varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{1}{r}u_\varphi\right].$$

Instead of Lamé's constants λ,μ, here and in the following the usual technical properties of elasticity E, ν (Young's modulus, Poisson's ratio) will be used.

Since the boundary conditions have been assumed to be homogeneous, the excitation has to be included in the body force $\underline{k}$ using the Dirac function $\delta(r-R_a)$:

$$\underline{k}(r,\varphi,t) = \left[\sigma_o(\varphi,t)\delta(r-R_a),0\right]^T. \tag{5}$$

Finally, carrying out the derivatives in operator form within eq. (1) leads to the field equations

$$M\frac{\partial^2}{\partial t^2}\underline{u} - K\underline{u} = \underline{k}(r,\varphi,t), \qquad M = \rho\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \tag{6}$$

$$K = \frac{E}{1-\nu^2}\left[\begin{array}{c|c} \frac{\partial^2}{\partial r^2} - \frac{1}{r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1-\nu}{2}\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} & -\frac{3-\nu}{2}\frac{1}{r^2}\frac{\partial}{\partial\varphi} + \frac{1+\nu}{2}\frac{1}{r}\frac{\partial^2}{\partial r\partial\varphi} \\ \hline \frac{3-\nu}{2}\frac{1}{r^2}\frac{\partial}{\partial\varphi} + \frac{1+\nu}{2}\frac{1}{r}\frac{\partial^2}{\partial r\partial\varphi} & \frac{1-\nu}{2}\left[\frac{\partial^2}{\partial r^2} - \frac{1}{r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} \end{array}\right]$$

for an elastic body described in polar co-ordinates. Eq. (6) together with the relations (2) to (5) define the complete initial boundary value problem for plane stress vibrations in a bounded annular disk.

In order to limit the expense of calculations, the excitation $\sigma_o(\varphi,t)$ in eq. (5) will be specified here in the form of the simplest circumferentially non-uniform, transient load function

$$\sigma_o(\varphi,t) = \sigma_o\left[\delta(\varphi-\varphi_o) + \delta(\varphi-\varphi_o-\pi)\right]\, h(t) \tag{7}$$

($h(t)$ = Heaviside step function), which means that a concentrated step pulse with the intensity σ_o acts on opposite sides of the disk, at an arbitrary position φ_o and at $\varphi_o+\pi$.

STRATEGY OF SOLUTION

Well-established in the dynamics of vibrating finite beams, plates, shells, etc. is the solution of the governing boundary value problems by eigenfunction expansions. Also, for the problem discussed here of transient elastic waves in a finite region it will be used.

The expansion theorem states (see Eringen and Suhubi (1985), for instance) that the general solution to eqs. (2) to (5) is

$$\underline{u}(r,\varphi,t) = \sum_m \underline{U}_m(r,\varphi)\, T_m(t), \tag{8}$$

where the vector functions

$$\underline{U}_m(r,\varphi) = \left[U_{rm}(r,\varphi),\ U_{\varphi m}(r,\varphi)\right]^T \tag{9}$$

are the corresponding infinitely many eigen functions, which satisfy the reduced elastic wave equation

$$K\underline{U}(r,\varphi) + \omega^2 M\underline{U}(r,\varphi) = 0 \tag{10}$$

with formally unchanged homogeneous boundary conditions (3) ($\Gamma(r,\varphi)$ will then identify the stress amplitudes free of time). The eigenvalues ω_m^2 are real and non-negative (vanishing eigenvalues can be omitted here), and the eigenfunctions satisfy the orthonormal conditions.

$$\int_{\varphi=o}^{2\pi}\int_{r=R_i}^{R_a} \underline{U}_m^T\, M\, \underline{U}_n\, r\, dr\, d\varphi = \delta_{mn}, \tag{11}$$

where δ_{mn} denotes Kronecker's delta.

The time functions $T_m(t)$ are given by

$$T_m(t) = \omega_m \int_o^t \sin\omega_m(t-\tau)\, \epsilon_m h(\tau)\, d\tau \tag{12}$$

(for homogeneous initial conditions (2)), where the weight factors ϵ_m can be easily evaluated in closed form (see Chapter 6).

EIGENFUNCTIONS, EIGENVALUES

The main problem in studying the transient stress concentration problem is obviously the solution of the corresponding eigenvalue problem.

For this purpose, first the general solution to eq. (10) is sought by resolving $\underline{U}(r,\varphi)$ into a dilatational field $\nabla\Phi$ and a rotational field $\nabla\times\underline{\Psi} = \nabla\times\Psi\underline{e}_z$, where $\underline{e}_z$ is the unit base vector in the direction of the outer normale of the middle surface. The scalar potential Φ and the co-ordinate Ψ of the vector potential $\underline{\Psi}$ satisfy the decoupled scalar potential equations

$$c_P^2\nabla^2\Phi + \omega^2\Phi = 0, \qquad c_S^2\nabla^2\Psi + \omega^2\Psi = 0,$$

$$c_P^2 = \frac{E}{\rho(1-\nu^2)}, \qquad c_S^2 = \frac{E}{2\rho(1-\nu)}, \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{13}$$

Evaluating the relation $\underline{U} = \nabla\Phi + \nabla\times\underline{\Psi}$ in polar co-ordinates,

$$U_r = \frac{\partial\Phi}{\partial r} + \frac{1}{r}\frac{\partial\Psi}{\partial\varphi}, \qquad U_\varphi = \frac{1}{r}\frac{\partial\Phi}{\partial\varphi} - \frac{\partial\Psi}{\partial r} \tag{14}$$

and using the relations (4) between stresses and displacements leads to the associated boundary conditions

$$\Gamma_{rr}(r,\varphi)\Big|_{r=R_i,R_a} = \frac{E}{1+\nu}\left[\frac{\nu}{1+\nu}\nabla^2\Phi + \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\Psi}{\partial r\partial\varphi} - \frac{1}{r^2}\frac{\partial\Psi}{\partial\varphi}\right]\Bigg|_{r=R_i,R_a} = 0$$

$$\Gamma_{r\varphi}(r,\varphi)\Big|_{r=R_i,R_a} = \frac{E}{1+\nu}\left[\frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\varphi} - \frac{1}{r^2}\frac{\partial\Phi}{\partial\varphi} + \frac{1}{2r^2}\frac{\partial^2\Psi}{\partial\varphi^2} - \frac{1}{2}\frac{\partial^2\Psi}{\partial r^2} + \right.$$

$$+ \left.\frac{1}{2r}\frac{\partial \Psi}{\partial r}\right]\Bigg|_{r=R_i,R_a} = 0 \qquad (15)$$

expressed in potential properties.

In the next step, a solution of the eigenvalue problem (13), (15) in product form

$$\begin{aligned} \Phi\,(r,\varphi) &= P(r)e^{\mp in\varphi}, \\ & \qquad\qquad n = 0(1)\infty, \\ \Psi\,(r,\varphi) &= R(r)e^{\mp in\varphi}, \end{aligned} \qquad (16)$$

which satisfies the conditions of 2π-periodicity with respect to φ, is assumed.

Substitution into the eqs. (16) leads to two single Bessel differential equations

$$\begin{aligned} (\alpha x)^2 \frac{\partial^2 P}{\partial x^2} + (\alpha x)\frac{\partial P}{\partial x} + (\alpha^2 x^2 - n^2)P &= 0, \\ (\beta x)^2 \frac{\partial^2 R}{\partial x^2} + (\beta x)\frac{\partial R}{\partial x} + (\beta^2 x^2 - n^2)R &= 0, \qquad n = 0(1)\infty, \\ x = \frac{r}{R_a}, \quad \alpha^2 = \frac{\omega^2 R_a^2}{c_P^2}, \quad \beta^2 &= \frac{\omega^2 R_a^2}{c_S^2} = \frac{2}{1-\nu}\,\alpha^2 \end{aligned} \qquad (17)$$

and boundary conditions for $P(x)$, $R(x)$ at the surfaces $x = R_i/R_a$, 1, not explicitly written down here.

The solution of the Bessel differential equations (17) can be written in terms of the Bessel functions J_n and I_n of the first and second kind of order n, so that for a certain order n the solution for the potenials, Φ,Ψ can be given as

$$\begin{aligned} \Phi\,(r,\varphi) &= \left[A_{1n}J_n(\alpha x) + A_{2n}I_n(\alpha x)\right](B_{1n}\sin n\varphi + B_{2n}\cos n\varphi), \\ \Psi\,(r,\varphi) &= \left[C_{1n}J_n(\beta x) + C_{2n}I_n(\beta x)\right](D_{1n}\sin n\varphi + D_{2n}\cos n\varphi). \end{aligned} \qquad (18)$$

Fitting these solutions to the boundary conditions yields a homogeneous system of algebraic equations to determine the integration constants $A_{1n},\dots,D_{2n}$. The vanishing determinant is a necessary condition for non-trivial solutions for $A_{1n},\dots,D_{2n}$, and represents the eigenvalue equation to be solved numerically.

In a last step, by means of the relations (14), the related displacements U_{rm}, $U_{\varphi m}$, $m = 1(1)\infty$ as eigenfunctions can be found.

In Fig. 2 the first four sets of eigenfunctions and the corresponding eigenvalues α_m are plotted for a selected set of data, namely $R_i/R_a = 0.2$, $\nu = 0.3$.

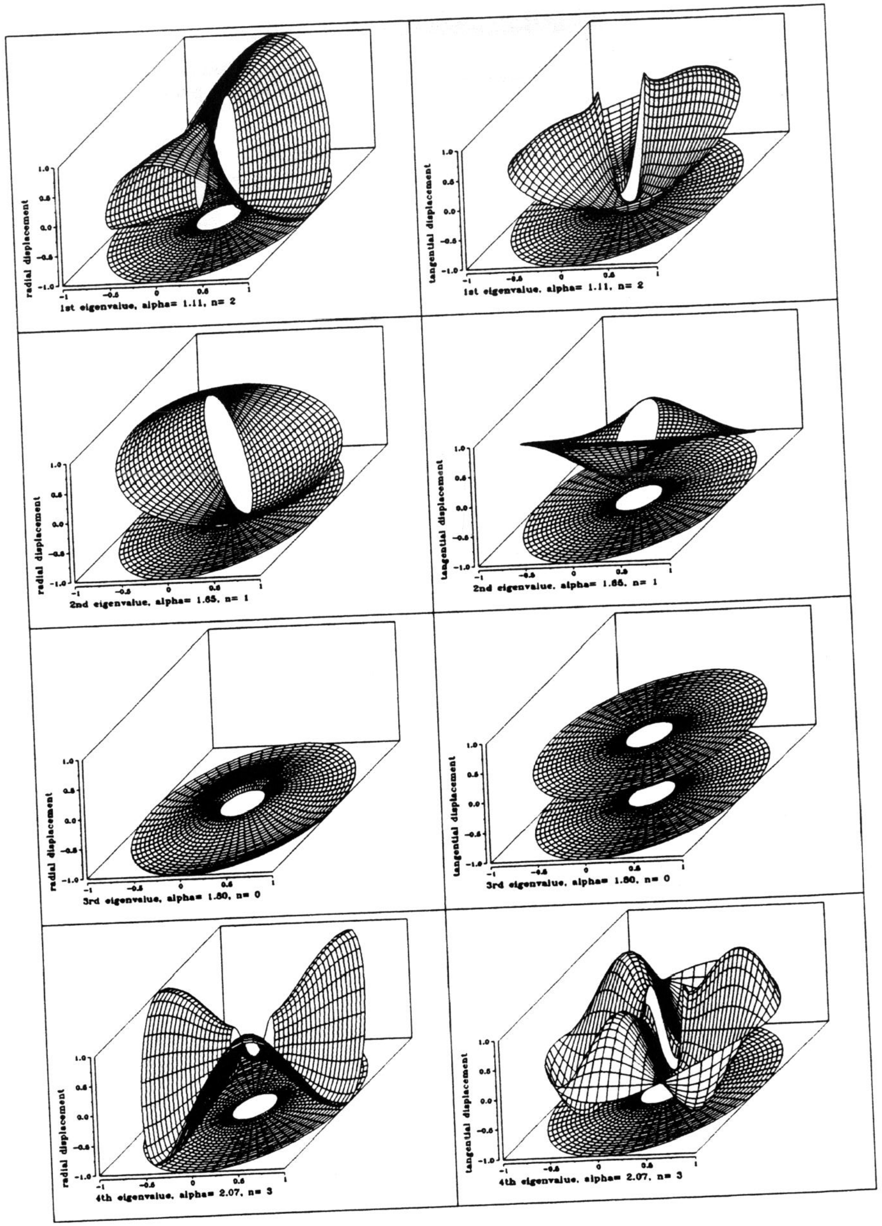

Figure 2. Eigenfunctions

It can easily be understood that not only the potentials Φ_m, Ψ_m (see eq. (16)), but also the displacements are representable in product form, e.g., as

$$\underline{U}_m(r,\varphi) := \begin{bmatrix} U_{rm} \\ U_{\varphi m} \end{bmatrix} = \begin{bmatrix} Y_{rm}(r) \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ Y_{\varphi m}(r) \end{bmatrix}, \; n = 0,$$

$$\underline{U}_m(r,\varphi) := \begin{bmatrix} U_{rm} \\ U_{\varphi m} \end{bmatrix} = \begin{bmatrix} Y_{rm}(r)\cos n\varphi \\ -Y_{\varphi m}(r)\sin n\varphi \end{bmatrix} \text{ and } \begin{bmatrix} Y_{rm}(r)\sin n\varphi \\ Y_{\varphi m}(r)\cos n\varphi \end{bmatrix}, \; n = 1(1)\infty,$$

$$m = 1(1)\infty. \quad (19)$$

Using eq. (19), the weight factor ϵ_m (see eq. (12)),

$$\epsilon_m = \int_{\varphi=0}^{2\pi} \int_{r=R_i}^{R_a} \underline{U}_m^T \left(\hat{\sigma}_o\left[\delta(\varphi-\varphi_o) + \delta(\varphi-\varphi_o-\pi)\right]\delta(r-R_a),\; 0\right)^T r\,dr\,d\varphi \quad (20)$$

can be specified now:

$$\epsilon_m = Y_{rm}(R_a)\left[\cos n\varphi_o + \cos n(\varphi_o+\pi)\right] \text{ and } Y_{rm}(R_a)\left[\sin n\varphi_o + \right.$$

$$\left. + \sin n(\varphi_o+\pi)\right], \quad n = 0(1)\infty \quad (21)$$

It follows that for rotationally symmetric eigenfunctions U_{rm} (n=0) we obtain

$$\epsilon_m = 2\hat{\sigma}_o\, Y_{rm}(R_a) \quad (22)$$

(if the eigenfunction is a purely tangential mode, then ϵ_m will vanish identically). For all other cases the result depends on whether n is an even or an odd number:

$$\epsilon_m = \begin{cases} 0, n = 1(2)\infty \\ 2\hat{\sigma}_o\, Y_{rm}(R_a)\cos n\varphi_o \text{ and } 2\hat{\sigma}_o\, Y_{rm}(R_a)\sin n\varphi_o, \quad n = 2(2)\infty. \end{cases} \quad (23)$$

The subsequent procedure to find the transient solutions of displacement and stress according to eq. (8) and (4), respectively, has to be carried out in computer-aided form. Some essential results in form of a sequence of plots for $\sigma_{rr}(r,\varphi,t)$, $\sigma_{\varphi\varphi}(r,\varphi,t)$ and $\sigma_{r\varphi}(r,\varphi,t)$, respectively, for different time steps are shown in Fig. 3 (Naumann, 1988). The evolution of the stress state for increasing time can be observed in this way. It will be noticed that in all cases, due to a limited number of terms of a series no singularities can occur. As known from the case of a uniformly loaded annular disk (Maass, 1986), at different points of the disk a significant strengthening of the stresses appears in comparison with the static case (Carmine, 1988).

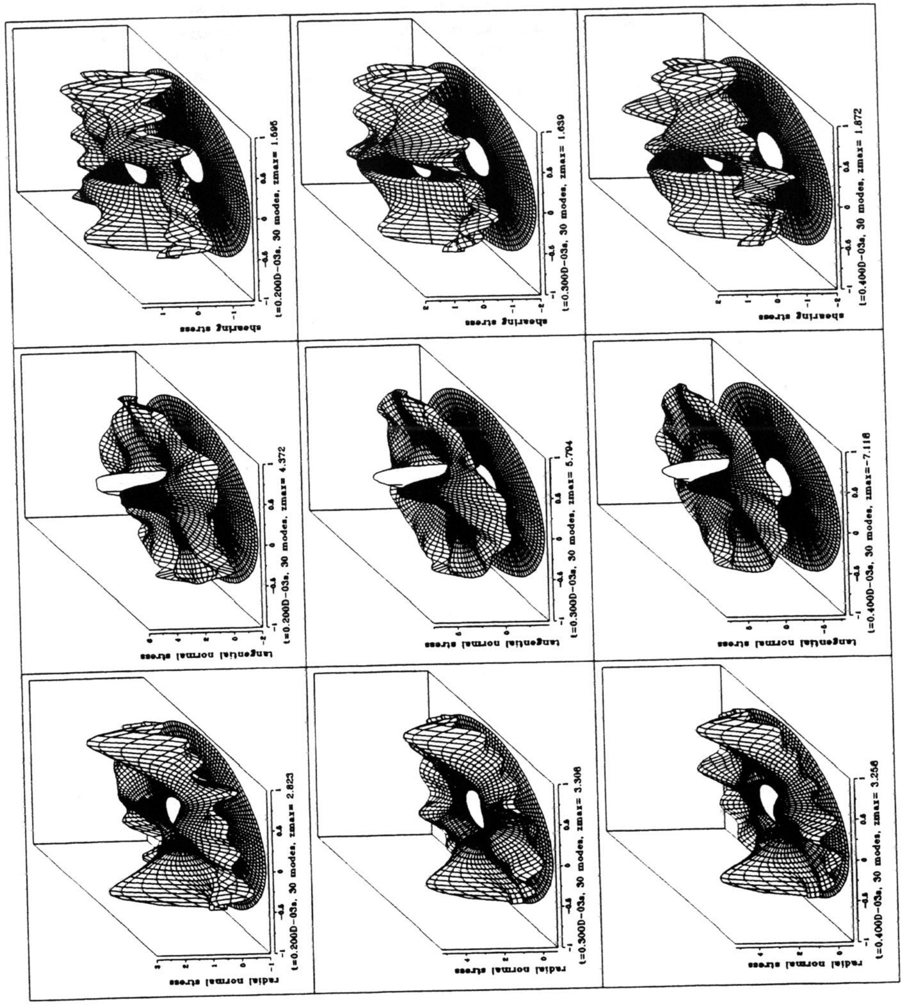

Figure 3. Transient stress

CONCLUSIONS

Dynamic stress concentrations in finite structural members with holes or conclusions etc. comprise an important topic in fracture mechanics. In many cases, eigenfunction expansion techniques seem to be an efficient instrument to treat such transient wave phenomena in finite solids. With an insignificant additional expense of theoretical work in comparison with finite element methods, for example, the evaluation time by computer is comparatively short.

Results for transient stress concentrations in annular disks have now been obtained for circumferentially uniform and locally concentrated step pulses. In both cases, significant deviations occur in relation to the corresponding purely static case, in the form of considerable enhancements

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