Discussion on the Singular Character of the Stress Components at the Crack Tips in Static and Dynamic Fracture

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ABSTRACT

In this paper, it is shown that the singular factor $1/\sqrt{r}$ appearing in static fracture problems is correct only under the assumption that couple stresses are neglected. Furthermore, using some fundamental results from the theory of partial differential equations it is argued that for dynamic fracture problems the stresses can not be singular. This would imply that traditional dynamic fracture analyses are in error.

Problems in the Theory of Static Fracture - Classical Solutions and Order 2 Hyperelasticity

It is well-known that the governing equations for the canonical fracture problems are elliptic in nature in case the Airy stress function ϕ defining type I and II fracture problems may be decomposed as equation (1) (see e.g. [1,2]. Thus, plane problems (Modes I and II) are governed by the biharmonic equation, anti-plane problems (Mode III) are governed by Laplace's equation, and the fundamental three-dimensional problem (penny-shaped crack) is governed by harmonic equations through Papkovitch-Neuber potentials. It is known further that a biharmonic function, ϕ, can be expressed as the sum of two harmonic functions ϕ_1 and ϕ_2 as follows [2]:

$$\phi(x, y) = x\phi_1(x, y) + \phi_2(x, y)$$
$$\phi(r, \theta) = (r^2-r_o^2)\phi_1(r, \theta) + \phi_2(r, \theta) \quad (1)$$

Thus, in essence the governing equations are all partial differential equations of the elliptic type.

It is known that solutions of elliptic equations are global solutions. Since there is only one constraint condition (boundary condition known as the Dirichlet or Neumann problem) for each elliptic equation, it is only possible to impose a condition of analyticity on either an inner or outer boundary with possible singularities appearing at the opposite boundary. For example, in the case of Laplace's equation on a circular region the solution may be written as

$$u(r, \theta) = \sum_{n=o}^{\infty} r^n \ (A_n \cos n\theta + B_n \sin n\theta) \quad ; r<a$$
$$u(r, \theta) = \sum_{n=o}^{\infty} r^{-n} \ (A_n \cos n\theta + B_n \sin n\theta) \quad ; r>a \quad (2)$$

Note that if the solution is required to be bounded at infinity, it will be singular at the origin, and vice versa. It is therefore possible to infer the singular nature of the solutions in fracture problems, since in these cases one requires bounded stresses at infinity but still includes the origin (where a singularity will exist) in the body.

These observations only hold in the case of classical elasticity in which there are no couple stresses. In the theory of hyper-elasticity, couple stresses are included so as to annihilate the singular character at the crack tip. To illustrate this, consider the hyper-elasticity plane problem of order 2. It can be shown [3] that the governing equations for this problem are as follows:

$$\nabla^4 \phi(x, y) = 0 \tag{3}$$

$$l^2 \nabla^4 \phi(x, y) - \nabla^2 \psi(x, y) = 0 \tag{4}$$

in which $l^2 = 2(1 + \nu)\eta / E$, where η is the flexure modulus. The functions ϕ and ψ are stress potentials analogous to the Airy stress function from which the stresses may be determined as follows:

$$\sigma_x = \phi_{,yy} - \psi_{,xy} \tag{5}$$

$$\sigma_y = \phi_{,xx} + \psi_{,xy} \tag{6}$$

$$\tau_{xy} = -\phi_{,xy} - \psi_{,yy} \tag{7}$$

$$\tau_{yx} = -\phi_{,xy} - \psi_{,xx} \tag{8}$$

$$\mu_x = \psi_{,x} \tag{9}$$

$$\mu_y = \psi_{,y} \tag{10}$$

It is seen that the shear stresses are not symmetric due to the presence of the couple stresses μ_x and μ_y, defined in Fig. 1 below.

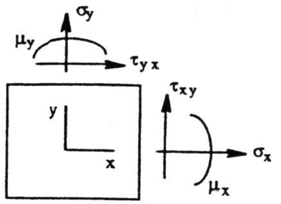

Fig. 1 Basic Couple Stresses

More importantly for the present discussion, the governing equations are no longer purely elliptic. Equation (4) is in fact hyperbolic, and as will be shown shortly, this does not allow singular behavior in the solution. Thus, the singular nature of the stresses in classical elasticity fracture problems can be viewed as depending on the elliptic nature of the governing equations and the imposition of analytic behavior at infinity. Including couple stresses changes the nature of the governing equations, and thereby removes the singularity at the crack tip. If

we try to apply this couple stress theory defined by equations (3)-(10) to static fracture problems of type I and II, we are forced to abandon the boundary condition of giving stress exerted at infinity, since the governing equation (4) permits only local solution, as the case of the next section. The couple stresses theory is better than the classic one. Of course, it can be applied to solve static fracture problems.

Classical Elasticity and Fracture Dynamics

It is well known that the governing equations for elastodynamics are as follows:

$$c_L^2 \nabla^2 \phi = \phi_{,tt} \quad ; \quad c_T^2 \nabla^2 \psi = \psi_{,tt} \tag{11}$$

in which c_L and c_T represent the wave speeds for pressure and shear waves, respectively. Since these equations are hyperbolic, each equation will have two constraint conditions, in contrast to the elliptic problem in which there is only a single constraint. In effect, for hyperbolic equations one must specify both boundary data and initial conditions.

The existence and uniqueness of solutions to hyperbolic equations is proved by the Cauchy-Kowalewsky theorem [4], which demonstrates that the solution can be expanded into a convergent power series in the neighborhood of a given point. This provides a local solution, as the series is convergent only in a small part of the domain. There can be no singular terms existing in the expression of the solution. Nevertheless it is common in dynamic fracture problems to consider stresses applied at infinity in analogy with static problems. If one requires bounded stresses at infinity, then the solution can exist only in the neighborhood of infinity: it is therefore meaningless to consider the effect of dynamically applied stresses at infinity on material near the origin. Another way to think of this is to note that for finite propagation speeds, it will take infinite time for a disturbance from infinity to reach the origin. Thus one should always work with stresses applied in the vicinity (generally on) of the crack itself.

A further consequence of the Cauchy-Kowalewsky theorem is that for problems considering stresses applied on the crack faces, there can be no singularities in the solution. An example of such a non-singular solution can be found in [5]. This is of course counter to the prevalent approach, which is based on the initial assumption of singular stresses at the crack tip. Based on the theory of partial differential equations, however, the traditional approach appears to be incorrect.

Discussions

It is mentioned in Section 3.5 of [5] that the method of coordinate stretching or shrinking had been successfully applied to get the asymptotic solutions of some nonlinear hyperbolic partial differential equations but failed to that of elliptic equations. Furthermore, in this reference, the governing equation of a hydrodynamic problem which is the equation (2.3.16) in [5] has been defined to be an elliptic equation:

$$\frac{\partial^4 \phi}{\partial y^4} = \alpha R \left[\frac{\partial^3 \phi}{\partial y^2 \partial t} + \left(U + \frac{\partial \phi}{\partial y}\right) \frac{\partial^3 \phi}{\partial x \partial y^2} - \left(\frac{d^2 U}{dy^2} + \frac{\partial^3 \phi}{\partial y^3}\right) \frac{\partial \phi}{\partial x} \right] - 2\alpha^2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2}$$
$$+ \alpha^3 R \left[\frac{\partial^3 \phi}{2x^2 \partial t} + \left(U + \frac{\partial \phi}{\partial y}\right) \frac{\partial^3 \phi}{\partial x^3} - \frac{\partial \phi}{\partial x} \frac{\partial^3 \phi}{\partial x^2 \partial y} \right] - \alpha^4 \frac{\partial^4 \phi}{\partial x^4} \qquad (2.3.16)\ [5]$$

Another is the following one which is the equation (2.3.37) of [5] defined to be a hyperbolic equation:

$$\frac{\partial h}{\partial t} + 2h^2 \frac{\partial h}{\partial x} + \alpha [- \frac{2}{3} h^3 \left(\cot\theta - \frac{4}{5} Rh^3\right) \frac{\partial^2 h}{\partial x^2} + \frac{2}{3} T\csc\theta h^3 \frac{\partial^4 h}{\partial x^4}$$

$$- 2h^2 \left(\cot\theta - \frac{8}{5} Rh^3\right) \left(\frac{\partial h}{\partial x}\right)^2 + 2T\csc\theta h^2 \frac{\partial h}{\partial x} \frac{\partial^3 h}{\partial x^3}] + 0(\alpha^2) = 0 \qquad (2.3.37)\ [5]$$

The equation (4) of this paper is just like the type of the above equation (2.3.37) [5], thus, Eq. (4) is an hyperbolic equation in nature.

In shell theory [6], a shell of revolution with its genertrix $r = \lambda z^{\mu}$, here λ and μ are parameters, z is the axis of revolving. In momentless theory, the governing equation is:

$$\frac{\partial^2 \phi}{\partial z^2} + \frac{2\mu}{z} \frac{\partial \phi}{\partial z} - \frac{\mu(\mu-1)}{z^2} \frac{\partial^2 \phi}{\partial \beta^2} = 0 \qquad (2.3)\ [6]$$

The shell is defined in the closed interval $0 \leq z \leq \zeta$ (a certain positive constant). Thus, if $0 < \mu < 1$ the equation (2.3) [6] is of elliptic type which is defined by positive Gaussian curvature. In that case, this equation has an unique solution under given load and proper boundary condition. If $\mu < 0$ or $\mu > 1$, the above equation is of hyperbolic type which is defined by negative Gaussian Curvature, in this case, the stresses of the shell may adopt indefinite value under no external load.

All the above statements illustrate that there exists radical distinction between different type of partial differential equations, we must consider seriously in all branches of mechanics this distinction otherwise entirely different results would be caused, i.e., the incorrect results.

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