Crack-tip Field at Steady Crack Growth and Vanishing Linear Strain-hardening

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ABSTRACT

A solution to the quasi statically propagating mode III crack in an elastic perfectly plastic solid is obtained as the limit solution for vanishingly small linear strain hardening. A point above or below the crack plane will experience, first a centered fan slip line field followed at the angle ±32.8° ahead of the crack tip by elastic unloading and finally, from the angle ±179.7° to the crack surfaces, plastic reloading occurs at a constant stress. The limit solution performs a discontinuity in the radial stress rate across the elastic plastic boundary at ±32.8°.

KEYWORDS

Fracture Mechanics; elastic-plastic fracture; quasi-static fracture; asymptotic field.

INTRODUCTION

The region of dominance for single term asymptotic fields often is often extremely small, and one observes, when numerical solutions are studied, that the effects of hardening generally is confined to a small region surrounding the crack tip. Outside this region the solution is effectively one of perfect plasticity. Thus solutions for perfectly plastic materials becomes very important and the question of uniqueness arise. One may for instance ask whether a solution depends on the definition of perfect plasticity, e.g. one such definition may be the limit of a vanishing strain hardening. An asymptotic solution for perfect plasticity has been obtained by Chitaley and McClintock (1971). This solution involves three regions of different material behaviour. A particle slightly above or below the crack plane experiences, as the crack tip pass, first a plastic centered fan slip line field, followed by a linear elastic unloading field and finally a plastic constant stress field near the crack surfaces. The displacement rate was shown to be proportional to ln(r) where r is the distance from the crack tip.

For linearly hardening solids Amazigo and Hutchinson (1977) studied asymptotic solutions using a von Mises yield condition and its associated flow rule. They showed that the displacement rate in the asymptotic field should be proportional to r^s. The solution was, for the angular distribution of stresses and strains, obtained after neglecting the possibility of plastic flow on the crack flanks, as however had been observed for perfectly plastic solids. A corrective solution was recently given by Ponte Castaneda (1987) who included the possibility of reloading on the crack flanks. Such plastic reloading was found to occur, except for, unreasonably high hardening rates. The plastic reloading sector was however found to be very small, i.e. less than about 0.3° and the inclusion of this sector did not in general change the old results. In the study by Ponte Castaneda (1987) the numerical solutions goes to a few very small values of the hardening parameter in order to try to make a connection with the perfectly plastic problem. Whether the limiting solutions for the hardening materials can in fact be related to the perfectly plastic materials solutions have been discussed by many authors. Due to a hypothetical assumption as regards the angular distribution of stresses for the infinitesimally linear hardening material Dunayevski and Achenbach (1982), comes to the conclusion that the radial dependence in the limit is $r^{\pm\sqrt{\alpha}}$. A related discussion is of course also one of uniqueness for the established perfectly plastic materials solution. The present paper is a contribution to that discussion. It concerns the asymptotic stress and velocity fields for a mode III crack steadily quasi statically propagating in an elastic plastic linearly hardening material. Plasticity that follows the von Mises yield condition and the associated flow rule is considered. Further we have restricted ourselves to small strains and deformations. The solution in the perfectly plastic limit for a linearly hardening material is studied and found to be different from the solution for the perfectly plastic materials. It is thus argued that the asymptotic solution for elastic perfectly plastic materials is not unique. As a matter of more practical significance, the difficulties as regard the identification of the asymptotic solution at numerical investigations of elastic perfectly plastic boundary value problems is recognized.

ANALYSIS

The study considers a large body, cut by a plane crack. Only a small region surrounding the right crack tip is analysed. A moving cartesian coordinate system is introduced such that the crack occupies the region x<0, y=0 (see Fig.1). An elastic linearly hardening material, obeying von Mises yield criterion and its associated flow rule is assumed. The deformation is assumed to be anti-plane and to be anti-symmetric with respect to the plane y=0. Thus it is sufficient to consider only the upper half of the body (y≥0). The crack tip moves with the velocity V with respect to a stationary coordinate system. Within the frames of quasi static and steady crack growth V is constant and the material derivative is given by

$$\dot{( )} = -V\partial( )/\partial x \quad , \qquad (1)$$

where the dot denotes differentiation with respect to time. Now a polar coordinate system r, θ is attached to the crack tip such that the crack is situated at θ=±π. (see Fig.1) The incremental components of stress and strain for the linearly hardening material at plasticity are related through

$$G_t\dot{\gamma}_{rz} = \alpha\dot{\tau}_{rz} + (1-\alpha)\tau_{rz}\dot{\tau}\tau^{-1} \quad , \qquad (2)$$

$$G_t\dot{\gamma}_{\theta z} = \alpha\dot{\tau}_{\theta z} + (1-\alpha)\tau_{\theta z}\dot{\tau}\tau^{-1} \quad , \qquad (3)$$

where $\tau=(\tau_{rz}{}^2+\tau_{\theta z}{}^2)^{1/2}$ is the effective stress (see Fig.2). The parameter α denotes the ratio G_t/G where G and G_t is the tangent modulus at elastic and plastic deformations respectively. The stress rate components are assumed to be in equilibrium, i.e.

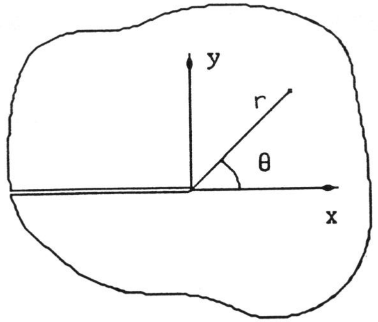

Fig. 1 The crack tip geometry.

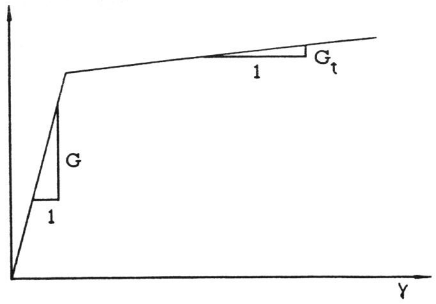

Fig. 2 The stress and strain curve.

$$\partial\dot{\tau}_{xz}/\partial x + \partial\dot{\tau}_{yz}/\partial y = 0 \quad . \qquad (4)$$

Inertia and body forces are assumed to be negligible. Now a stress rate function Φ(x,y) is introduced, which will readily maintain equilibrium stress rates

$$\dot{\tau}_{xz} = \partial\Phi/\partial y \quad , \qquad (5)$$

$$\dot{\tau}_{yz} = -\partial\Phi/\partial x \quad . \qquad (6)$$

The compatibility equation for the polar components of strain rates reads

$$\partial\dot{\gamma}_{rz}/\partial\theta - \partial(r\dot{\gamma}_{\theta z})/\partial r = 0 \quad . \qquad (7)$$

After elimination of strain rates one obtains the following equation for the stress rates in the plastic regions:

$$\alpha r\Delta\Phi + (1-\alpha)[\partial(\tau_{rz}\dot{\tau}\tau^{-1})/\partial\theta - \partial(r\tau_{\theta z}\dot{\tau}\tau^{-1})/\partial r] = 0 \quad . \qquad (8)$$

and in the unloading region

$$\Delta\Phi = 0 \quad . \qquad (9)$$

Equation (8) is of degree one in the stress rate which invites us to look for solutions proportional to r^s. Further a plastic centered fan field develops ahead of a crack tip in a perfectly plastic material (Chitaley and McClintock, 1971), suggesting the following substitution

$$\Phi = K\tau_o V r^s[f(\theta) + \cos\theta] \quad . \qquad (10)$$

K is an amplitude factor and the yield stress τ_o is introduced for convenience when the function f(θ) and the parameter s is determined. Now the polar stress rates may be written

$$\dot{\tau}_{rz} = K\tau_o V r^{s-1}(f' - \sin\theta) \quad , \qquad (11)$$

$$\dot{\tau}_{\theta z} = K\tau_o V s r^{s-1}(f + \cos\theta) \quad . \qquad (12)$$

Integration of $\dot{\tau}_{yz}$ using (1) and (6) gives

$$\tau_{yz} = K\tau_o r^s(f + \cos\theta) \quad . \qquad (13)$$

It is easily shown, by insertion into the equation of equilibrium that the function of y appearing after integration must be a constant which can be neglected without loss of generality.

Now the following notation is introduced

$$\tau_{rz} = K\tau_o r^s F(\theta) \quad , \qquad (14)$$

where F is an unknown function of θ. After using the relation (1) the following relation between F and f is obtained

$$F'\sin\theta\cos\theta + F(\sin^2\theta - s\cos^2\theta) = f'\cos\theta + f\sin\theta \quad . \qquad (15)$$

The polar stress component $\tau_{\theta z}$ is given by

$$\tau_{\theta z} = -\tau_{rz}\tan\theta + \tau_{yz}\cos^{-1}\theta = K\tau_o r^s(1 - F\tan\theta + f\cos^{-1}\theta) \quad . \qquad (16)$$

Further the effective stress rate is given by

$$\dot{\tau} = (\tau_{rz}\dot{\tau}_{rz} + \tau_{\theta z}\dot{\tau}_{\theta z})\tau^{-1} \quad . \qquad (17)$$

The boundary condition due to the anti symmetry of displacements across θ=0 implies symmetry for Φ and thus

$$f'(0) = 0 \quad . \qquad (18)$$

At the traction free crack surfaces τ_{yz}=0 and thus

$$f(\pi) = 0 \quad . \qquad (19)$$

The asymptotic field has at numercal analyses (Ponte Castaneda, 1987) been found to consist of a primary plastic zone, followed at an angle θ_p by elastically unloading and finally at an angle θ_s a second plastic zone appears during reloading adjacent to the crack surface. Let [ ] denote a jump in a quantity across a discontinuity line. According to the work of Drugan and Rice (1984) the following continuity conditions have to be fulfilled

$$[\gamma_{rz}] = 0 \quad . \qquad (20)$$

$$[\tau_{rz}] = [\tau_{\theta z}] = 0 \quad . \qquad (21)$$

Continuous plastic strains then implies that

$$[\dot{w}] = 0 \quad . \qquad (22)$$

At $\theta = \theta_p$ it is necessary that

$$\dot{\tau} = 0 \qquad (23)$$

and hence according to (22)

$$[\dot{\tau}_{rz}(\theta_p)] = 0 \quad . \qquad (24)$$

Further the equations (12) and (13) implies that

$$[\dot{\tau}_{\theta z}] = 0 \quad . \qquad (25)$$

A particle passing the crack tip retains the plastic strain state it had when it entered the elastic unloading sector. Thus plastic reloading occurs at θ_s if the effective stress of the particle regains its value at unloading value, i.e.

$$\tau(\theta_p)/\sin^s\theta_p = \tau(\theta_s)/\sin^s\theta_s \quad . \qquad (26)$$

Equations (8) to (9) together with the boundary conditions (18) to (19), the continuity conditions (24) and (25) and the condition (26) constitutes a one dimensional boundary value problem for a general value of α. Somewhat differently formulated, this problem was earlier treated numerically by Ponte Castaneda (1987).

THE SMALL α LIMIT

We assume for a moment that

$$s = O(\alpha^{1/2}) \qquad \text{as} \qquad \alpha \to 0 \qquad (27)$$

A solution which is singular at the crack tip is looked for and thus we assume that s<0. An examination reveals that equation (8) is regular at θ=0 and F=0. For sufficiently small values of F and θ, one obtains by insertion the following relation

$$\alpha f''\cos^{-1}\theta - F's - \alpha + s^2 = 0 \qquad (28)$$

Since according to (15) $F'f' \geq 0$, F' has to fall off immediately towards $s-\alpha/s$ when θ is O(s). The conclusion is that

$$F'' = O(\alpha^{1/2}), \quad F' = O(\alpha^{1/2}) \quad \text{and} \quad F = O(\alpha^{1/2}) \qquad (29)$$

in the limit as $\alpha \to 0$ and for sufficiently small but finite values of θ. Equation (8) reduces to

$$F'(2F\sin\theta + s\cos\theta) - 2sF\sin\theta + (F^2 + \alpha - s^2)\cos\theta = 0 \qquad (30)$$

One readily observes that (as long as F stays regular) F is monotonically increasing for $0 \leq \theta < \pi/2$ if $s^2 < \alpha$. One equally observes that F is monotonically decreasing for $0 \leq \theta < \pi/2$ if $s^2 > \alpha$.

Considering (29) one obtains

$$\dot{\tau} = -K\tau_o V r^{s-1}(F\sin\theta + s\cos\theta) \qquad (31)$$

By inspection of this condition one comes to the conclusion that if s^2 is larger than $O(\alpha)$ then unloading cannot occur for $\theta < \pi/2$. On the other hand if $s^2 = o(\alpha)$ unloading has to occur immediately at θ=0. Since non of these cases are acceptable the assumption (28) is justified. The cases where $s^2 > \alpha$, has to be rejected since the condition (23) cannot be fulfilled for $\theta < \pi/2$. For $s^2 < \alpha$ one finds that $F'/s \to -\infty$ as $F \to -(s/2)\cot\theta$ the following notation is convenient

$$F \to F_o = -(s/2)\cot\theta_o \qquad \text{as} \quad \theta \to \theta_o \qquad (32)$$

It is concluded that unloading cannot occur for $0 < \theta < \theta_o$ if $s \neq 0$, as long as condition (29) is valid. One may write for $\theta \to \theta_o$, $\theta < \theta_o$

$$F'(F-F_o)+(1/2)(\alpha+F_o^2)\cot\theta_o=0 \tag{33}$$

The solution is

$$F=F_o-[(\alpha+F_o^2)\cot\theta_o(\theta_o-\theta)]^{1/2} \tag{34}$$

This solution is valid under the condition θ_o-θ>$O(\alpha^{1/3})$. When θ_o-θ=$O(\alpha^{1/3})$ the following relations will according to (34) replace (29)

$$F''=O(1), \quad F'=O(\alpha^{1/3}) \quad \text{and} \quad F=O(\alpha^{2/3}) \tag{35}$$

and thus (30) does no longer represent (8) as $\alpha\to0$. Equation (8) is now reduced to the following

$$(\alpha+F_o^2)(F''-\cot\theta_o)-2F'(F-F_o)=0 \tag{36}$$

After integration one obtains

$$(\alpha+F_o^2)[F'-(\theta_o-\theta)\cot\theta_o]-(F-F_o)^2=0 \tag{37}$$

Note that the integration constant may be dropped without any loss of generality. The boundary condition is found from (34)

$$F=F_o-[(F_o^2+\alpha)\cot\theta_o(\theta_o-\theta)]^{1/2} \quad \text{as} \quad (\theta_o-\theta)\alpha^{-1/2}\to-\infty \tag{38}$$

It is readily seen that F' is greater than or equal to zero and monotonically increasing. In the limit for large values of F one obtains from (38) that

$$(\alpha+F_o^2)F'-(F-F_o)^2\to0 \quad \text{as} \quad F\to\infty \tag{39}$$

After integration this reads

$$F\to F_o+(\alpha+F_o^2)(\theta_1-\theta)^{-1} \quad \text{as} \quad \theta\to\theta_1 \tag{40}$$

where θ_1 is a constant such that θ_1-θ_o=$O(\alpha^{1/2})$. This solution holds as long as θ_1-θ=$o(\alpha^{1/2})$. When θ_1-θ=$O(\alpha^{1/2})$ then

$$F''=O(\alpha^{-1/2}), \quad F'=O(1) \quad \text{and} \quad F-F_o=O(\alpha^{1/2}) \tag{41}$$

Now (8), when second order terms are neglected, reads

$$F''(\alpha+F^2)+F'(2F'F-2F-s\cot\theta_o)=0 \tag{42}$$

This equation may be integrated and consideration of the boundary condition for $(\theta_1-\theta)\alpha^{-1/2}\to-\infty$ permits determination of the constant appearing after integration and thus the following equation results

$$F'(\alpha+F^2)-(F-F_o)^2=0 \tag{43}$$

In the elastic unloading region the solution for f can be written

$$f=A\sin(s\theta+\delta)-\cos\theta \tag{44}$$

where A and δ are arbitrary constants. The boundary condition of traction free crack surfaces

implies that

$$\delta=s\,\theta_s \tag{45}$$

The continuity conditions (24) and (25) allows elimination of A. Thus it is necessary that

$$F'(\theta_p)=1+\cot\theta/(\theta_p-\theta_s) \tag{46}$$

Now the only remaining condition at θ_p is (23) to ensure unloading. After consideration of (41) one obtains

$$Ff'-F\sin\theta_p-s\cos\theta_p=0 \tag{47}$$

which after simplification and insertion of (46) gives

$$\theta_s=\theta_p+[\tan\theta_p-(s^2/4\alpha)\cot\theta_p]^{-1} \ . \tag{48}$$

In the remaining calculations only (30) and (48) has to be solved under the condition of (26) which in the limit of vanishing hardening rates reads

$$\tau(\theta_p)=\tau(\theta_s) \ . \tag{49}$$

RESULT

For a suggested value of α inserted into (30) the angle θ_p was calculated. The angle θ_s at which τ_{yz} equals zero is found through (48). Then the condition (49) is checked and as long as this condition is not fulfilled within acceptable limits the procedure is repeated with a new value of α.

The result was found to be

$$s=-0.812228\,\alpha^{1/2} \tag{50}$$

$$\theta_p=32.845^o \tag{51}$$

and

$$\theta_s=179.7191^o \tag{52}$$

The result can be compared with the result of Ponte Castaneda (1987) who numerically attempted to approach the perfectly plastic limit. The lowest value chosen for α in his analysis was 10^{-6}. The reported results where s=-0.81$\alpha^{1/2}$, θ_p=33.88° and θ_s=179.731° which in the light of the present paper seems to be rather accurate. When comparing with the Chitaley and McClintock (1971) result for a perfectly plastic material, the differences are substantial, e.g. the results are θ_p=19.71° and θ_s=179.634° . Examination of the solution presented in this paper reveals that the plastic field ahead of the crack tip in the limit is a centered fan field. The deviations are here proportional to $\alpha^{1/2}$. Due to the different unloading angle, the deviations from the Chitaley and McClintock (1971) solution are finite in the trailing elastic unloading region. The solution in the plastic re-loading region near the crack surface approach in the limit a constant stress field. Figure 3 shows a somewhat surprising difference, namely that the limit solution allows for a discontinuity for the radial stress rate in spite of the condition of continuous stress rates for both hardening materials (see eqn. (24)) and for perfectly plastic materials when

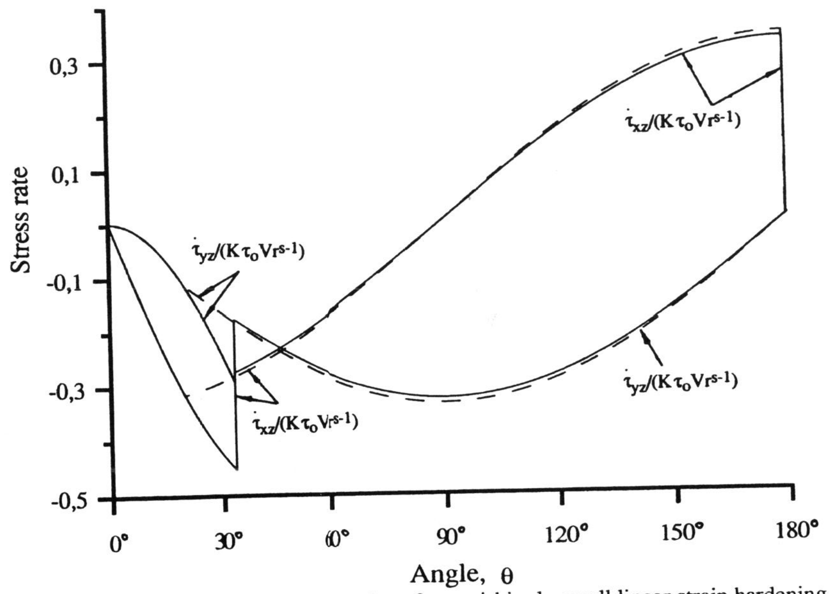

Fig. 3 Stress rate distribution τ_{rz}, and $\tau_{\theta z}$ for vanishingly small linear strain hardening. The dashed lines show the result obtained by Chitaley and McClintock.

perfect plasticity is assumed a priori (cf. Chitaley and McClintock, 1971).

It is believed that the appearance of two alternative solutions for the perfectly plastic material is an artifact of the condition of unloading at the boundary between the leading plastic and the unloading region. This condition is approached at any angle, as the perfectly plastic limit for a linearly hardening material is approached, but as long as hardening is present it is completely fulfilled only at one specific angle. It is an open question whether other solutions will be found for perfect plasticity approached from other hardening assumptions. Further it seems probable that solutions provided from numerical methods such as the finite element method cannot be expected give reliable estimations of for instance θ_p, due to the extreme sensitivity to only slight hardening effects.

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