A New Nonlinear Line-spring Model for Elastic–Plastic Analysis of Surface Cracks

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ABSTRACT

A new nonlinear line-spring model for elastic-plastic analysis of surface cracks is proposed by considering the yielding at back surface. The numerical examples are given and plausible.

KEYWORDS

Surface crack, nonlinear line-spring model, elastic-plastic analysis, back surface yielding.

INTRODUCTION

A nonlinear line-spring model for elastic-plastic analysis of surface cracks based on the D-M model was developed in previous paper (Lu and Jiang, 1983). It does not consider the yielding at back surface of crack. However, this yielding is observed in experiments and has a Considerable effect on the field intensity and opening displacement of the crack tip. Therefore, A new model is proposed by incorporating the yielding at back surface shown in Fig. (a) and (b) in this paper. The nonlinear Constitutive relations of line-spring can be derived from the improved D-M model Solutions for Single edge cracked strip by Considering the yielding at back surface, as shown in Fig. 1(c). To model a surface crack in plate or shell, the distributed line-springs are then embedded between the two surfaces of a through crack in plate or shell, as shown in Fig. (d). The solution of the through cracked plate is based on Reissner plate theory. Finally the numerical examples are given.

THE IMPROVED D-M MODEL SOLUTIONS FOR SINGLE EDGE CRACKED STRIPS

The D-M model of a single edge cracked strip can be improved by incorporating the yielding at back surface, as shown in Fig. 1(c). It can be considered as a double edge cracked strip with yielding stresses acting on the crack surfaces. For this plane strain double edge cracked strip, the equilibrium equations in terms of the displacements are

$$\beta\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+(\beta-1)\frac{\partial^2 v}{\partial x\partial y}=0 \tag{1a}$$

$$\frac{\partial^2 v}{\partial x^2}+\beta\frac{\partial^2 v}{\partial y^2}+(\beta-1)\frac{\partial^2 u}{\partial x\partial y}=0 \tag{1b}$$

where $\beta=\frac{2(1-\nu)}{1-2\nu}$ and ν is poisson's ratio.

Solving eqs. (1) with the help of the Fourier transform and Satisfying the boundary conditions, we get a singular integral equation for the dislocation density function $\phi(y)$

$$\int_L\left[\frac{1}{t-y}+k(y,t)\right]\phi(t)\,dt=-\frac{\pi}{m_1}p(y)\quad,\qquad L\in(0,a_{1p})\cup(a_{2p},h) \tag{2}$$

where $\phi(y)=\frac{\partial}{\partial y}u(0,y)$, $m_1=4\mu/(1+\kappa)$, $\kappa=(\beta+1)/(\beta-1)$, is modulus of elasticity in shear.

Let $t=\frac{a_{1p}}{2}(\bar{t}+1)$, $y=\frac{a_{1p}}{2}(\bar{y}+1)$ as t and $y\in(0,a_{1p})$ and $t=\frac{1}{2}[(h-a_{2p})\bar{t}+(h+a_{2p})]$, $y=\frac{1}{2}[(h-a_{2p})\bar{y}+(h+a_{2p})]$ as t and $y\in(a_{2p},h)$, we can write equqtion (2) as

$$\int_{-1}^{+1}\left(\frac{1}{\bar{t}-\bar{y}}\{\phi(\bar{t})\}+h(\bar{y},\bar{t})\{\phi(\bar{t})\}\right)d\bar{t}=-\frac{\pi}{m_1}\{p(\bar{y})\} \tag{3}$$

where

$$\{\phi(\bar{t})\}=\begin{Bmatrix}\phi_1(\bar{t})\\ \phi_2(\bar{t})\end{Bmatrix}=\begin{Bmatrix}\phi\left(\frac{a_{1p}}{2}(\bar{t}+1)\right)\\ \phi\left(\frac{1}{2}[(h-a_{2p})\bar{t}+h+a_{2p}]\right)\end{Bmatrix}$$

$$\{p(\bar{y})\}=\begin{Bmatrix}\sigma_M+\sigma_B[(1-\zeta_{1p})-\zeta_{1p}\bar{y}]-\sigma_s H(\bar{y}-2\zeta_1/\zeta_{1p}-1)\\ \sigma_M+\sigma_B[-\zeta_{2p}-(1-\zeta_{2p})\bar{y}]+\sigma_s\end{Bmatrix}$$

$\sigma_M=N/h$, $\sigma_B=6M/h^2$, $\zeta_1=a_1/h$, $\zeta_{1p}=a_{1p}/h$, $\zeta_{2p}=a_{2p}/h$, $H(t)=1$ as $t>0$ and $H(t)=0$ as $t<0$, and σ_s is the yielding stress.

The solution of equation (3) is in form $\{\phi(\bar{t})\}=(1-\bar{t}^2)^{-\frac{1}{2}}\{G(\bar{t})\}$, and $\{G(\bar{t})\}$ can be numerically obtained from the Lobatta-chebyshev method (Theocaris and Ioakimidis, 1977). The stress intensity factors of the crack tips at $y=a_{1p}$ and a_{2p} are

$$K_I(a_{1p})=-m_1\sqrt{\frac{\pi a_{1p}}{2}}G_1(1)=\sqrt{h}\left[\sigma_M g_{M1}(\zeta_{1p},\zeta_{2p})+\sigma_B g_{B1}(\zeta_{1p},\zeta_{2p})-\sigma_s g_{s1}(\zeta_1,\zeta_{1p},\zeta_{2p})\right] \tag{4a}$$

$$K_I(a_{2p})=m_1\sqrt{\frac{\pi(h-a_{2p})}{2}}G_2(-1)=\sqrt{h}\left[\sigma_M g_{M2}(\zeta_{1p},\zeta_{2p})+\sigma_B g_{B2}(\zeta_{1p},\zeta_{2p})-\sigma_s g_{s2}(\zeta_1,\zeta_{1p},\zeta_{2p})\right] \tag{4b}$$

where $g_{M1}(\zeta_{1p},\zeta_{2p})=-G_{1M}(1)\sqrt{\frac{\pi}{2}\zeta_{1p}}$, $g_{B1}(\zeta_{1p},\zeta_{2p})=-G_{1B}(1)\sqrt{\frac{\pi}{2}\zeta_{1p}}$,

$g_{M2}(\zeta_{1p},\zeta_{2p})=G_{2M}(-1)\sqrt{\frac{\pi}{2}(1-\zeta_{2p})}$, $g_{B2}(\zeta_{1p},\zeta_{2p})=G_{2B}(-1)\sqrt{\frac{\pi}{2}(1-\zeta_{2p})}$,

$g_{s1}(\zeta_1,\zeta_{1p},\zeta_{2p})=-G_{1s}(1)\sqrt{\frac{\pi}{2}\zeta_{1p}}$, $g_{s2}(\zeta_1,\zeta_{1p},\zeta_{2p})=G_{2s}(-1)\sqrt{\frac{\pi}{2}(1-\zeta_{2p})}$.

The fact that the stresses at the crack tips are finite requires $K_I(a_{1p})=0$

and $K_I(a_{2p})=0$. We have

$$\sigma_M g_{M1}(\zeta_{1p},\zeta_{2p})+\sigma_B g_{B1}(\zeta_{1p},\zeta_{2p})-\sigma_s g_{s1}(\zeta_1,\zeta_{1p},\zeta_{2p})=0 \tag{5a}$$

$$\sigma_M g_{M2}(\zeta_{1p},\zeta_{2p})+\sigma_B g_{B2}(\zeta_{1p},\zeta_{2p})-\sigma_s g_{s2}(\zeta_1,\zeta_{1p},\zeta_{2p})=0 \tag{5b}$$

The plastic zone sizes ζ_{1p} and ζ_{2p} can be obtained from solving eqs.(5) if the crack size ζ_1 is given.

The crack tip opening displacement is

$$\delta_t=2u(0,a_1)=\int_{a_{1p}}^{a_1}2\phi(t)\,dt=-2a_{1p}\int_{\bar{y}_1}^{1}\phi_1(\bar{t})\,d\bar{t}=-2a_{1p}\int_{-1}^{+1}\phi_1(\bar{t})H(\bar{t}-\bar{y}_1)\,d\bar{t} \tag{6}$$

where $\bar{y}_1=2\frac{a_1}{a_{1p}}-1$ and $\phi_1(\bar{t})=(1-\bar{t}^2)^{-\frac{1}{2}}\frac{1}{m_1}\left[\sigma_M G_{1M}(\bar{t})+\sigma_B G_{1B}(\bar{t})-\sigma_s G_{1s}(\bar{t})\right]$.

THE NONLINEAR CONSTITUTIVE RELATIONS OF LINE-SPRINGS

For the strain energy U of the double edge cracked strip shown in Fig.1(c), we have

$$U=U_0+\int_{U_0}^{U}dU=U_0+\int_{(0,1)}^{(\zeta_{1p},\zeta_{2p})}\left(\frac{\partial U}{\partial\zeta_{1p}}d\zeta_{1p}+\frac{\partial U}{\partial\zeta_{2p}}d\zeta_{2p}\right) \tag{7}$$

where U_0 is the strain energy of the strip without the crack.

Using the relation between the potential energy release rate G_I and the stress intensity factor K_I, Eq.(7) becomes

$$U=U_0+\frac{h(1-\nu^2)}{E}\int_{(0,1)}^{(\zeta_{1p},\zeta_{2p})}\left[K_I^2(a_{1p})d\zeta_{1p}-K_I^2(a_{2p})d\zeta_{2p}\right]$$

$$=U_0+\frac{h(1-\nu^2)}{E}\int_0^1\left[(\sigma_M g_{M1}+\sigma_B g_{B1}-\sigma_s g_{s1})^2\zeta_{1p}+(\sigma_M g_{M2}+\sigma_B g_{B2}-\sigma_s g_{s2})^2(1-\zeta_{2p})\right]dt \tag{8}$$

From relations $\delta_y=\frac{1}{h}\frac{\partial U}{\partial\sigma_M}$ and $\theta_y=\frac{6}{h^2}\frac{\partial U}{\partial\sigma_B}$, the constitutive relations of line-springs are derived

$$\begin{Bmatrix}\delta_y/h\\ \theta_y/6\end{Bmatrix}=\frac{2(1-\nu^2)}{E}\left(\begin{bmatrix}\alpha_{MM}, & \alpha_{MB}\\ \alpha_{BM}, & \alpha_{BB}\end{bmatrix}\begin{Bmatrix}\sigma_M\\ \sigma_B\end{Bmatrix}-\sigma_s\begin{Bmatrix}\alpha_{MS}\\ \alpha_{BS}\end{Bmatrix}\right) \tag{9}$$

where $\alpha_{\lambda\mu}=\int_0^1\left[g_{\lambda1}g_{\mu1}\zeta_{1p}+g_{\lambda2}g_{\mu2}(1-\zeta_{2p})\right]dt \qquad (\lambda,\mu=M,B)$

$\alpha_{\lambda s}=\int_0^1\left[g_{\lambda1}g_{s1}\zeta_{1p}+g_{\lambda2}g_{s2}(1-\zeta_{2p})\right]dt \qquad (\lambda=M,B)$

Introducing the dimensionless dislocation densities $\bar{\delta}_y'$ and $\bar{\theta}_y'$, Eq.(9) becomes

$$\begin{Bmatrix}\sigma_M\\ \sigma_B\end{Bmatrix}=\frac{E}{2(1-\nu^2)}\begin{bmatrix}\gamma_{MM}, & \gamma_{MB}\\ \gamma_{BM}, & \gamma_{BB}\end{bmatrix}\int_{-1}^{+1}\begin{Bmatrix}\bar{\delta}_y'(\bar{t})\\ \bar{\theta}_y'(\bar{t})\end{Bmatrix}H(\bar{x}-\bar{t})\,d\bar{t}+\sigma_s\begin{bmatrix}\gamma_{MM}, & \gamma_{MB}\\ \gamma_{BM}, & \gamma_{BB}\end{bmatrix}\begin{Bmatrix}\alpha_{MS}\\ \alpha_{BS}\end{Bmatrix} \tag{10}$$

where $\begin{bmatrix}\gamma_{MM}, & \gamma_{MB}\\ \gamma_{BM}, & \gamma_{BB}\end{bmatrix}=\begin{bmatrix}\alpha_{MM}, & \alpha_{BM}\\ \alpha_{MB}, & \alpha_{BB}\end{bmatrix}^{-1}$

When $a_{2p}=0$ in Eq.(10), we get the Constitutwe relations without the yielding at back surface, and when $a_1=0$ in Eq.(10), we obtain the Constitutive relations without the crack and with the yielding zone. These Constitutive relations are all necessary to elastic-plastic analysis of surface crack.

When the ligament yielding appear, the Constitutive relation in incremental form of line-spring is given as

$$d\{\sigma\} = [S_{ep}]d\{q\} \tag{11}$$

where $[S_{ep}]=[S_e]-[S_e]\left\{\frac{\partial\phi}{\partial\sigma}\right\}\left\{\frac{\partial\phi}{\partial\sigma}\right\}^T[S_e]\Big/\left\{\frac{\partial\phi}{\partial\sigma}\right\}^T[S_e]\left\{\frac{\partial\phi}{\partial\sigma}\right\}$, $\{\sigma\}=\begin{Bmatrix}\sigma_M\\ \sigma_B\end{Bmatrix}$,

$\left\{\frac{\partial\phi}{\partial\sigma}\right\}=\begin{Bmatrix}\frac{\partial\phi}{\partial\sigma_M}\\ \frac{\partial\phi}{\partial\sigma_B}\end{Bmatrix}$, $[S_e]=\frac{E}{2(1-\nu^2)}\begin{bmatrix}\alpha_{MM}, & \alpha_{MB}\\ \alpha_{BM}, & \alpha_{BB}\end{bmatrix}^{-1}$, $\{q\}=\begin{Bmatrix}\delta_y/h\\ \theta_y/6\end{Bmatrix}=\int_{-1}^{+1}\begin{Bmatrix}\bar{\delta}_y'(\bar{t})\\ \bar{\theta}_y'(\bar{t})\end{Bmatrix}H(\bar{x},\bar{t})d\bar{t}$

and $\phi(\sigma_M,\sigma_B,\zeta_1)$ is the generalized yielding function of the edge cracked strip. $\phi(\sigma_M,\sigma_B,\zeta_1)=0$ is the generalized yield surface, and it is obtained from eqs. (5) with $\zeta_{1p}=\zeta_{2p}=\zeta_p$.

To simplify the numerical calculations, the yield surface is linearized. Therefore, $[S_{ep}]$ is Constant, and the constitutive relation (11) can be integrated into the following form

$$\{\sigma\}=\{\sigma\}_0+[S_{ep}](\{q\}-\{q\}_0) \tag{12}$$

where $\{\sigma\}_0$ and $\{q\}_0$ are the generalized stress and displacement at which the ligament starts to yield.

THE BEHAVIOR EQUATIONS OF REISSNER PLATE WITH A THROUGH CRACK

For Reissner plate theory, we take the basic equations as follows

$$\nabla^4 W = 0 \tag{13$_a$}$$
$$\nabla^4 F = 0 \tag{13$_b$}$$
$$\nabla^2\psi - \frac{1}{\alpha^2}\psi = 0 \tag{13$_c$}$$

where W is deflection, F stress function, ψ additional function and $\alpha=h/\sqrt{10}$.

Having solved eqs.(13) for W, F and ψ, the stress resultants and couples and the midplane deformations of the plate can be expressed as

$$\beta_x=-\frac{\partial G^*}{\partial x}+\frac{\partial\psi}{\partial y},\quad \beta_y=-\frac{\partial G^*}{\partial y}-\frac{\partial\psi}{\partial x},\quad G^*=W+\frac{2\alpha^2}{1-\nu}\nabla^2 W,$$

$$N_{xx}=\frac{\partial^2 F}{\partial y^2},\quad N_{yy}=\frac{\partial^2 F}{\partial x^2},\quad N_{xy}=-\frac{\partial^2 F}{\partial x\partial y},$$

$$M_{xx}=D\left(\frac{\partial\beta_x}{\partial x}+\nu\frac{\partial\beta_y}{\partial y}\right),\quad M_{yy}=D\left(\frac{\partial\beta_y}{\partial y}+\nu\frac{\partial\beta_x}{\partial x}\right),\quad M_{xy}=M_{yx}=\frac{(1-\nu)D}{2}\left(\frac{\partial\beta_x}{\partial y}+\frac{\partial\beta_y}{\partial x}\right),$$

$$Q_x=\frac{(1-\nu)D}{2\alpha^2}\left(\beta_x+\frac{\partial W}{\partial x}\right),\quad Q_y=\frac{(1-\nu)D}{2\alpha^2}\left(\beta_y+\frac{\partial W}{\partial y}\right).$$

in which $D=Eh^3/12(1-\nu^2)$.

The solution of eqs.(13) by Fourier transform in the half plane $y\geqslant 0$ is given as

$$F(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(A_1+A_2 y)e^{-|s|y}e^{-isx}ds \tag{14$_a$}$$
$$W(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(A_3+A_4 y)e^{-|s|y}e^{-isx}ds \tag{14$_b$}$$
$$\psi(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A_5 e^{-\gamma|s|y}e^{-isx}ds \tag{14$_c$}$$

where $\gamma=\sqrt{1+\frac{1}{\alpha^2 s^2}}$.

Substituting eqs.(14) into the boundary Conditions, a system of Cauchy-type singular integral equations can be obtained

$$\frac{Eh}{4\pi C_{1p}}\int_{-1}^{+1}\frac{\bar{\delta}_y'(\bar{t})}{\bar{t}-\bar{x}}d\bar{t}=\sigma_M(\bar{x})-\sigma_M^{\infty} \tag{15$_a$}$$
$$\frac{3Eh}{4\pi C_{1p}}\int_{-1}^{+1}\bar{\theta}_y'(\bar{t})\left(\frac{1}{\bar{t}-\bar{x}}+k_1(\bar{x},\bar{t})\right)d\bar{t}=\sigma_B(\bar{x})-\sigma_B^{\infty} \tag{15$_b$}$$

where $\bar{t}=t/C_{1p}$, $\bar{x}=x/C_{1p}$, $k_1(\bar{x},\bar{t})=\frac{2}{1+\nu}\int_0^{\infty}(1+2\bar{\alpha}^2 s^2(1-\bar{\gamma}))\sin(\bar{t}-\bar{x})s\,ds$, $\bar{\gamma}=\sqrt{1+\frac{1}{\bar{\alpha}^2 s^2}}$

and $\bar{\alpha}=\frac{1}{\sqrt{10}}\frac{h}{C_{1p}}$.

NUMERICAL RESULTS AND CONCLUSIONS

The Combination of eqs.(16) and the Constitutive relations of line-springs results in the governing equations for the dislocation densities

$$\int_{-1}^{1}\left(\frac{1}{\bar{t}-\bar{x}}\begin{Bmatrix}\bar{\delta}_y'\\ \bar{\theta}_y'\end{Bmatrix}+[K(\bar{x},\bar{t})]\begin{Bmatrix}\bar{\delta}_y'(\bar{t})\\ \bar{\theta}_y'(\bar{t})\end{Bmatrix}\right)d\bar{t}=\frac{4\pi C_{1p}}{Eh}\{Q(\bar{x})\} \tag{16}$$

where

$$[K(\bar{x},\bar{t})]=\begin{cases}\begin{bmatrix}0, & 0\\ 0, & k_1(\bar{x},\bar{t})\end{bmatrix}-\frac{2\pi}{1-\nu^2}\frac{C_{1p}}{h}\begin{bmatrix}1, & 0\\ 0, & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\gamma_{MM}, & \gamma_{MB}\\ \gamma_{BM}, & \gamma_{BB}\end{bmatrix}H(\bar{x}-\bar{t}) & |\bar{x}|\geqslant|\bar{x}^*|\\ \begin{bmatrix}0, & 0\\ 0, & k_1(\bar{x},\bar{t})\end{bmatrix}-\frac{4\pi}{E}\frac{C_{1p}}{h}\begin{bmatrix}1, & 0\\ 0, & \frac{1}{3}\end{bmatrix}\begin{bmatrix}S_{ep}\end{bmatrix}H(\bar{x}-\bar{t}) & |\bar{x}|<|\bar{x}^*|\end{cases}$$

$$\{Q(\bar{x})\}=\begin{cases}\frac{4\pi C_{1p}}{Eh}\begin{bmatrix}1, & 0\\ 0, & \frac{1}{3}\end{bmatrix}\left(\sigma_s\begin{bmatrix}\gamma_{MM}, & \gamma_{MB}\\ \gamma_{BM}, & \gamma_{BB}\end{bmatrix}\begin{Bmatrix}\alpha_{Ms}\\ \alpha_{Bs}\end{Bmatrix}-\begin{Bmatrix}\sigma_M^{\infty}\\ \sigma_B^{\infty}\end{Bmatrix}\right) & |\bar{x}|\geqslant|\bar{x}^*|\\ \frac{4\pi C_{1p}}{Eh}\begin{bmatrix}1, & 0\\ 0, & \frac{1}{3}\end{bmatrix}\left(\begin{Bmatrix}\sigma_M\\ \sigma_B\end{Bmatrix}_0-[S_{ep}]\{q\}_0-\begin{Bmatrix}\sigma_M^{\infty}\\ \sigma_B^{\infty}\end{Bmatrix}\right) & |\bar{x}|<|\bar{x}^*|\end{cases}$$

As the plastic zone sizes a_{1p}, a_{2p}, c_{1p} and c_{2p} are given, the eqs.(16) is a system of cauchy type singular integral equations and it can be numerically solved with Gauss-Chebyshev quadrature (Theocaris and Ioakimidis, 1977). Because the plastic zone sizes are unknown, it is necessary to use four additional relationships, that is eqs.(5) for a_{1p} and a_{2p}, the third equation for c_{2p} obtained from making $a_{2p}=0$ in eqs.(5) and elimination a_{1p}, and the fourth one $c_{1p}=c+\frac{1}{2\pi}\left[\frac{K_I(\beta=0)}{\sigma_s}\right]^2$. Then, the solutions are obtained from iterative program. with these solutions, the line-spring generalized forces $\{\sigma\}$ can be obtained from eq.(10) or eq.(12), and thus the crack tip opening displacement δ_t can be calculated from eq.(6).

The numerical results of surface cracked plate under tension loadings is given in Fig 2-5 for a/h=0.2, 0.4, 0.6 and 0.8. The curves in these figures give the variations of the dimensionless crack tip opening displacement $\delta_t^* = \frac{\pi}{4}\frac{E}{(1-\nu^2)\sigma_s}\frac{\delta_t}{h}$ with tension loading σ_M^∞/σ_s for a/c=0.2, 0.4, 0.6 and 0.8. It can be found that the numerical results are plausible.

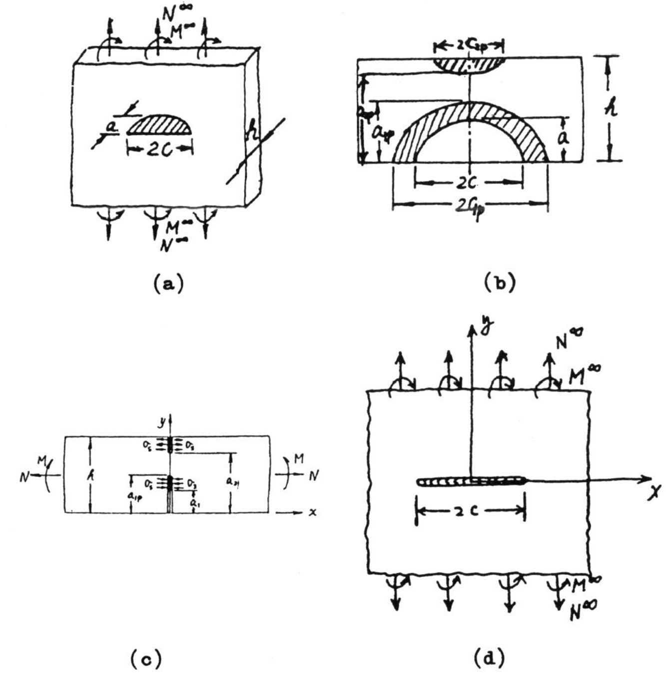

Fig. 1

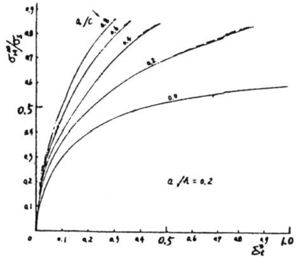

Fig. 2

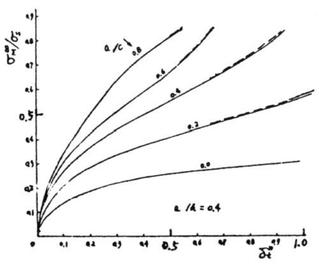

Fig. 3

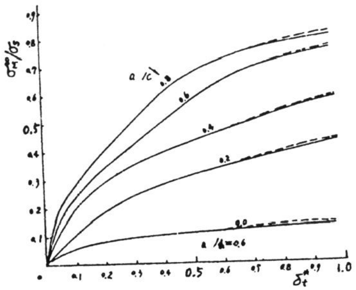

Fig. 4

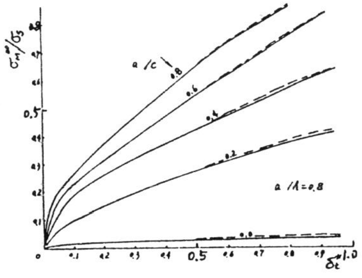

Fig. 5

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