SIMPLE FORMULAS FOR DYNAMIC FRACTURE MECHANICS PARAMETERS OF ELASTIC AND VISCOELASTIC THREE-POINT BEND SPECIMENS BASED ON TIMOSHENKO'S BEAM THEORY

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ABSTRACT

Simple formulas for the dynamic fracture mechanics parameters of a three-point elastic or viscoelastic bend specimen are derived by using the Timoshenko's beam theory. A numerical analysis is performed for a steel or PMMA specimen impacted by a falling steel cylinder or sphere. The results are compared with those obtained by using the formulas which were obtained previously using the Euler-Bernoulli's beam theory, and the effects of rotatory inertia and transverse-shear deformation of specimen are discussed.

KEYWORDS

Dynamic fracture mechanics; stress intensity factor; strain intensity factor; energy release rate; three-point bend specimen; viscoelasticity

INTRODUCTION

A variety of works (e.g. Bush, 1970; Costin and co-workers, 1977; Shoemaker and Rolfe, 1969; Turner, 1970) has been conducted to measure the dynamic fracture toughness K_{Id} of materials. However, the standard procedure for K_{Id} determination has not been established. The main difficulties in the test would come from the fact that the inertia effects influence the stress state in the specimen and the quasi-static stress estimation of the dynamic stress intensity factor leads to erroneous results (e.g. Aberson and co-workers, 1977; Kalthoff, 1982).

Because of the complexities in evaluating the interactions between the crack and the boundary of the body, the existing analytical solutions have restricted applicability to specimens of finite dimensions. Although the time variation of the fracture mechanics parameters of the specimen can be computed by the finite element method (e.g. Aoki and co-workers, 1978), this method requires a large computer time and is not appropriate to use in material testing.

In a previous paper the authors (Kishimoto and co-workers, 1980) derived a simple formula for the dynamic stress intensity factor $K_I(t)$ of an elastic three-point bend specimen by making use of Nash's solution (1969) for a notched Euler-Bernoulli beam. The purpose of the present paper is to devolop an improved formula for the dynamic stress intensity factor by taking account of the effects of the rotatory inertia and the shear deformation of the specimen. The derivation of simple formulas for the dynamic fracture mechanics parameters of viscoelastic solids is also intended.

SIMPLE FORMULA FOR DYNAMIC STRESS INTENSITY FACTOR

Cracked Timoshenko Beam

We consider the cracked three-point bend specimen as shown in Fig. 1, where O-x,y is the Cartesian co-ordinate system. The coupled equations for the deflection u_y and the bending slope ψ of the Timoshenko beam subjected to a concentrated load $F(t)$ are given by

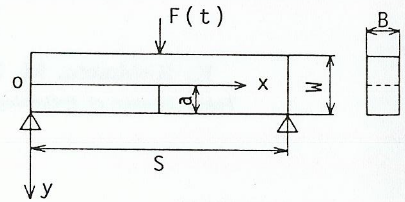

Fig. 1. Three-point bend specimen.

$$\rho A\frac{\partial^2 u_y}{\partial t^2} = \frac{1}{S}\frac{\partial Q}{\partial \xi} + \frac{1}{S}F(t)\,\delta(\xi - \tfrac{1}{2}) \quad (1a)$$

$$\rho I\frac{\partial^2 \psi}{\partial t^2} = Q - \frac{1}{S}\frac{\partial M}{\partial \xi} \quad (1b)$$

where A and I are the cross-sectional area and the area moment of inertia of the gross section, respectively, ρ is the density, ξ is x-co-ordinate normalized by the span length S, and $\delta(\xi)$ is the Dirac delta function. The bending moment M and the vertical shear force Q are expressed as

$$M = -\frac{E}{S\Phi}\frac{\partial \psi}{\partial \xi} \quad (2a) \qquad Q = kGA\left(\frac{1}{S}\frac{\partial u_y}{\partial \xi} - \psi\right) \quad (2b)$$

where E and G are the Young's and shear modulus, respectively. For a rectangular cross section, the shear coefficient k is given by $10(1+\nu)/(12+11\nu)$ (Cowper, 1966) where ν is the Poisson's ratio. Following Nash (1969) we consider the influence of the crack by specifying Φ as

$$\Phi = 1/I + (D/S)\delta(\xi - \tfrac{1}{2}) \quad (3)$$

To determine the coefficient D, we analyse the static deflection at the midspan, $u_{ys}(1/2)$. By solving the time independent version of Eqn. (1) under the following boundary conditions

$$[u_y]_{\xi=0,1} = 0\,, \qquad [\partial\psi/\partial\xi]_{\xi=0,1} = 0 \quad (4)$$

we obtain

$$u_{ys}(\tfrac{1}{2}) = \frac{F_s S^3}{48EI}\left(1 + \frac{12EI}{kGAS^2}\right) + \frac{F_s D S^2}{16E} \quad (5)$$

The last term of the above equation corresponds to the displacement due to introducing the crack Δ_{crack}, which has also been obtained by G-compliance method as

$$\Delta_{crack} = \frac{(1-\nu^2)}{8EI}F_s W S^2 V(a/W) \qquad \text{(plane strain)} \quad (6)$$

where $V(a/b)$ is the shape function (Tada and co-workers, 1973). Thus from

Eqns. (5) and (6) the value of D is determined as

$$D = 2(1-\nu^2)W\,V(a/W)/I \quad (7)$$

Returning to the dynamic equation (1), we next analyse the eigenvectors and eigenvalues. Assuming the solutions to the homogeneous equations in the form;

$$u_y(\xi,t) = Y(\xi)e^{j\omega t}\,, \quad \psi(\xi,t) = \Psi(\xi)e^{j\omega t} \quad (j = \sqrt{-1}) \quad (8)$$

and solving the resulting differential equations by using Laplace transforms [see Huang (1961) for references], we finally obtain the normal modes $Y_m(\xi)$ and $\Psi_m(\xi)$, and natural frequencies ω_m:

$$Y_m(\xi) = \alpha_m\cos\frac{b_m\beta_m}{2}\sin b_m\alpha_m\xi - \beta_m\cos\frac{b_m\alpha_m}{2}\sin b_m\beta_m\xi \quad (9a)$$

$$\Psi_m(\xi) = \frac{b_m}{S}\left\{(\alpha_m^2 - q^2)\cos\frac{b_m\beta_m}{2}\cos b_m\alpha_m\xi - (\beta_m^2 - q^2)\cos\frac{b_m\alpha_m}{2}\cos b_m\beta_m\xi\right\} \quad (9b)$$

$$(0 < \xi < \tfrac{1}{2})$$

and

$$\omega_m = \frac{b_m}{S^2}\sqrt{\frac{EI}{\rho A}} \quad (9c)$$

where

$$r^2 = I/(AS^2) \quad (10a) \qquad q^2 = EI/(kGAS^2) \quad (10b)$$

$$\begin{pmatrix}\alpha_m^2\\ \beta_m^2\end{pmatrix} = \frac{(q^2+r^2) \mp [(q^2-r^2)^2 + (4/b_m^2)]^{1/2}}{2} \quad (10c)$$

and b_m is the mth root of the following characteristic equation:

$$1 + \frac{DI}{2(\beta_m^2-\alpha_m^2)}\left[-\frac{1+r^2b_m^2(\beta_m^2-q_m^2)}{b_m\beta_m}\tan\frac{b_m\beta_m}{2} + \frac{1+r^2b_m^2(\alpha_m^2-q_m^2)}{b_m\alpha_m}\tan\frac{b_m\alpha_m}{2}\right] = 0 \quad (11)$$

In case $(q^2+r^2) < [(q^2-r^2)^2 + (4/b_m^2)]^{1/2}$, α_m becomes imaginary quantity. However, Eqns. (9a) and (9b) can be transformed to the real equations by making use of the relations $\cos jx = \cosh x$, etc.

With use of the orthogonality condition of the normal modes given by

$$\int_0^1 \{Y_m(\xi)Y_n(\xi) + r_g\Psi_m(\xi)\Psi_n(\xi)\}d\xi = 0 \quad (n \neq m,\ r_g = I/A) \quad (12)$$

the displacement of the specimen subjected to a concentrated load $F(t)$ can be expressed as

$$u_y(\xi,t) = \sum_{m=1}^{\infty} Y_m(\xi)\frac{Y_m(1/2)}{\omega_m W_m S}\int_0^t F(\tau)\sin\{\omega_m(t-\tau)\}d\tau \quad (13)$$

where

$$W_m = 2\rho A\int_0^{1/2}[Y_m^2(\xi) + r_g\Psi_m^2(\xi)]d\xi \quad (14)$$

The impulse response function for deflection at the midspan is thus given by

$$h_B^e(t) = \sum_{m=1}^{\infty}\frac{Y_m^2(1/2)}{\omega_m W_m S}\sin\omega_m t \quad (15)$$

Dynamic Stress Intensity Factor

Since the dynamic stress intensity factor is proportional to the magnitude of displacements in the vicinity of a crack-tip, we set

$$K_I(t) = k_e u_y(\tfrac{1}{2}, t) = k_e \int_0^t F(\tau) h_B^e(t-\tau) d\tau \tag{16}$$

where the proportional factor k_e can be determined by the static solution, because the displacement field near a crack-tip is similar to that for the static problem.

The static deflection at the midspan can be expressed as

$$u_{ys}(\tfrac{1}{2}, t) = F(t) \cdot \lim_{t' \to \infty} \int_0^{t'} \frac{\tau}{t'} h_B^e(t'-\tau) d\tau \tag{17}$$

Thus, the proportional factor k_e is obtained as

$$k_e = K_s/u_{ys}^e(1/2, t)$$

$$= K_s/[F(t)\sum_{m=1}^{\infty} \frac{Y_m^2(1/2)}{\omega_m^2 W_m S}] \tag{18}$$

where K_s represents the static stress intensity factor and is given by

$$K_s = 3SF(t)\sqrt{\pi a}\ \Lambda(a/W)/2BW^2 \tag{19}$$

where $\Lambda(a/b)$ is the shape function (Tada and co-workers; 1973).

VISCOELASTIC PROBLEM

Impulse Response Function of Viscoelastic Beam

According to the correspondence principle, the Laplace transform of viscoelastic solutions can be obtained directly from the transform of elastic solutions by replacing the elastic moduli with the transform parameter multiplied transform of the appropriate viscoelastic relaxation functions if the type of boundary condition prescribed at points of the boundary remains invariant with time (Cristensen, 1971). Thus, assuming that the solid can be modeled as a three-parameter linear standard solid (Fig. 2) and that Poisson's ratio is constant, we obtain the following impulse response function of the viscoelastic beam from Eqn. (15):

$$h_B^v(t) = \sum_{m=1}^{\infty} \frac{Y_m^2(1/2)}{W_m S} [C_{1m} e^{-P_{1m}t} + e^{-P_{2m}t}(C_{2m}\cos P_{3m}t + C_{3m}\sin P_{3m}t)] \tag{20}$$

where C_{1m}, C_{2m}, C_{3m}, P_{1m}, P_{2m}, and P_{3m} are constants determined by the values of E_1, E_2, μ, and ω_m (Sakata and co-workers, 1980). It is noted that $Y_m(1/2)$ and W_m are the same as in elastic case, while ω_m^2 is defined by

$$\omega_m = \frac{b_m}{S^2}\sqrt{\frac{E'I}{\rho A}} \tag{21}$$

where $E' = E_1E_2/(E_1+E_2)$.

Simple Formulas

It has been reported by the authors (Aoki and co-workers, 1980) that the near-tip fields of the viscoelastic body are characterized by three parameters: the stress intensity factor $K_I(t)$, the strain intensity factor $T_I(t)$, and the energy release rate $\hat{J}$. Since the strain intensity factor is proportional to the magnitude of displacements near the crack-tip, we set

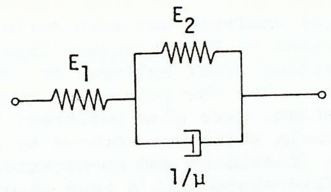

Fig. 2. Model of three-parameter solid.

$$T_I(t) = k_v \int_0^t F(\tau) h_B^v(t-\tau) d\tau \tag{22}$$

where the proportional factor k_v is determined by the static analysis as in the elastic case.

Owing to the elastic-viscoelastic correspondence principle, the static solution for the deflection at the midspan is derived from Eqn. (17) as

$$u_{ys}(\tfrac{1}{2}, t) = \sum_{m=1}^{\infty} \frac{Y_m^2(1/2)}{W_m S} \cdot \frac{E'}{\omega_m^2}\{F(t)J(0) + \int_0^t F(t-\tau)\frac{\partial J(\tau)}{\partial \tau} d\tau\} \tag{23}$$

where $J(t)$ denotes the creep compliance and is given, in this case, by

$$J(t) = \frac{E_1+E_2}{E_1E_2} - \frac{1}{E_2} e^{-\mu E_2 t} \tag{24}$$

Since the static stress intensity factor in viscoelastic problems is also represented by Eqn. (19), with use of the relation (Aoki and co-workers, 1980)

$$T_I(t) = (1+\nu)\{K_I(t)J(0) + \int_0^t K_I(t-\tau)\frac{\partial J(\tau)}{\partial \tau} d\tau\} \tag{25}$$

the static strain intensity factor can be expressed as

$$T_s = \frac{(1+\nu)K_s}{F(t)}\{F(t)J(0) + \int_0^t F(t-\tau)\frac{\partial J(\tau)}{\partial \tau} d\tau\} \tag{26}$$

Thus, we obtain the following result from Eqns. (23) and (26):

$$k_v = T_s/u_{ys}\ (1/2, t)$$

$$= \frac{3}{2}\frac{1+\nu}{E'}\frac{S^2}{BW^2}\sqrt{\pi a}\ \Lambda(a/W)\left[\sum_{m=1}^{\infty}\frac{Y_m^2(1/2)}{W_m \omega_m^2}\right]^{-1} \tag{27}$$

The dynamic strain intensity factor and energy release rate are given by

$$K_I(t) = 2\{T_I(t)\mu(0) + \int_0^t T_I(t-\tau)\frac{\partial \mu(\tau)}{\partial \tau} d\tau\} \tag{28a}$$

$$\hat{J} = (1-\nu)T_I(t)\ K_I(t) \qquad \text{(plane strain)} \tag{28b}$$

where $\mu(t)$ is the relaxation modulus for pure shear (Aoki and co-workers, 1980). Thus, we can estimate the dynamic fracture mechanics parameters after evaluating the impact load $F(t)$.

NUMERICAL EXAMPLES

Numerical analysis has been performed for a steel or PMMA specimen impacted by a falling steel cylinder or sphere (Fig. 3). For the purpose of comparison, computations were also performed by using the formula which was derived by the authors (Kishimoto and co-workers, 1980) using Euler-Bernoulli's beam theory.

In order to evaluate the dynamic fracture mechanics parameters, the contact force between the specimen and impactor is estimated by applying the Hertz's theory to the local deformation near the contact point. With $h_A(t)$ and $h_B(t)$ denoting respectively the impulse response functions of the impactor and specimen at the contact point, the force-displacement relation may be expressed as (Goldsmith, 1960)

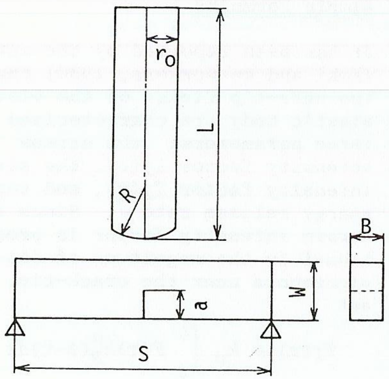

Fig. 3. Falling cylinder and specimen (S=80mm, W=20mm, B=10mm, r_0=10mm, R=100mm. L=1.5 or 0.75m).

$$F(t) = k_c[v_0 t - \eta_A(t) - \eta_B(t)]^{3/2} \qquad (29)$$

where

$$\eta_A(t) = \int_0^t F(\tau)h_A(t-\tau)d\tau \qquad (30a) \qquad \eta_B(t) = \int_0^t F(\tau)h_B(t-\tau)d\tau \qquad (30b)$$

Here v_0 is the initial relative velocity and k_c is the spring constant of the non-linear restoring effect in Hertz's contact theory. The impulse response functions of a rigid body and that of a slender rod were used for $h_A(t)$ of a sphere and cylinder, respectively (Endo and co-worker, 1981).

In solving Eqn. (29), time increment of 2.5 μs is adopted for the cylinder of 1.5m length and sphere, and that of 1.25 μs for the cylinder of 0.75m length. The first 22 and 17 normal modes were retained for the cylinder and specimen, respectively. Numerical values employed are listed in Table 1.

TABLE 1 Numerical Values Used in Computation

Steel	PMMA	
E = 206 GPa	E_1 = 4.58 GPa	ρ = 1.15 Mg/m^3
ρ = 7.8 Mg/m^3	E_2 = 45.9 GPa	ν = 0.38
ν = 0.3	μ = 148 (GPas)$^{-1}$	

Figure 4 shows the impact force generated by the falling sphere of 15 mm radius. It is found that the magnitude of the force and the contact time are affected by the rigidity of the specimen. The impact forces by the falling cylinder with various impact velocities are shown in Fig. 5. The magnitude of the force increases with the impact velocity, while the contact time is almost constant.

The time variation of the dynamic stress intensity factor of the steel specimen and the dynamic strain intensity factor of PMMA specimen are respectively shown in Figs. 6 and 7, where the results obtained by the simple formulas based on Euler-Bernoulli's beam theory are also depicted. It is noted that the fracture mechanics parameters are overestimated unless the effects of rotatory inertia and transverse-shear deformation of specimen are considered.

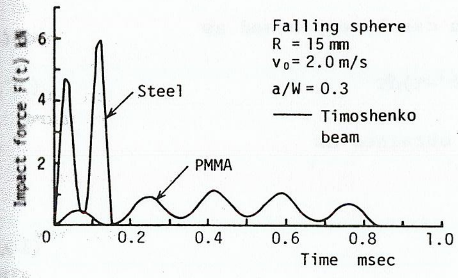

Fig. 4. Influence of specimen rigidity on impact force.

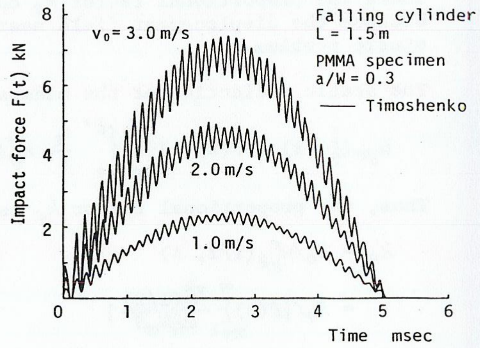

Fig. 5. Influence of impact speed on impact force.

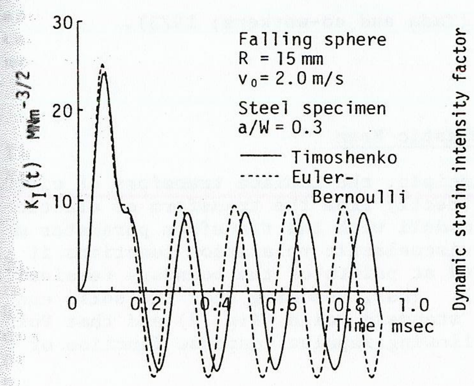

Fig. 6. Time variation of dynamic stress intensity factor of steel specimen.

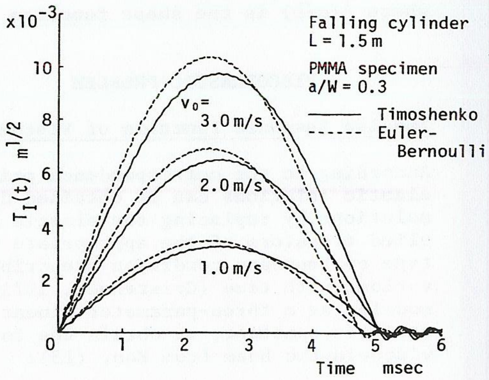

Fig. 7. Time variation of dynamic strain intensity factor of PMMA specimen.

CONCLUSIONS

An improved formulas for the dynamic fracture mechanics parameters of a three-point elastic or viscoelastic bend specimen were derived by employing the Timoshenko's beam theory. Numerical studies suggest that the rotatory inertia and transverse-shear deformation of specimen should be considered in estimating the fracture mechanics parameters.

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