NEW APPROACH TO THE FRACTURE TOUGHNESS OF CONCRETE - PROBABILISTIC MODEL

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ABSTRACT

A statistical theory of failure considering creation of a random fracture surface and based on an approach of averaging over a set of all possible fracture surfaces is applied for the toughness characterization of concrete and formulation of global failure criteria. The size effect on the fracture toughness for brittle failure of concrete is accounted both macroscopically and microscopically. The proposed probabilistic model accounts for both the global stress state and the stochastic nature of the material's microstructure through the probability of failure and the probabilistic measure, respectively.

KEYWORDS

Aggregate; cement-mortar; concrete; crack trajectory; diffusion process; energy release rate; fracture mechanics; failure criteria; heterogeneity; microcracking; notched beam ; probabilistic measure; radius of correlation; size effect; statistical analysis; stochastic; toughness; Weibull distribution.

INTRODUCTION

Fracture toughness for brittle failure of a homogeneous material can be determined utilizing the Griffith's criterion for crack instability. Considering a heterogeneous material, however, such as concrete where a random strength field is encountered, a more appropriate fracture toughness model has to be formulated. To accomplish this task the effect of size viewed both macroscopically and microscopically has to be determined and accounted for. Not only the accumulated strain energy released at failure but also the inherent stochastic nature of a heterogeneous material will help establish reasonable similitude criteria.

The similitude of stress fields can be expressed in terms of the stress intensity factor K, a known function of the applied stress σ_∞ and a geometrical factor (ℓ/h), where ℓ is the crack length and h the specimen size. The use of either the critical stress intensity factor K_c or the critical energy release rate J_c may only partially describe the similarity conditions. They do not describe, for example, the well established experimental dispersion of key parameters, i.e. the critical value of applied stress and crack length at failure and their dependency on aggregate size. Test results in brittle fracture are usually inconsistent and show the widest scatter compared with other macroscopical studies, such as irreversible or elastic deformations, mass and heat transfer, etc. Therefore, the stochastic nature of the microstructure of concrete has to be accounted for in order to establish realistic similarity criteria. This fact raises serious questions regarding the applicability of the Griffith-Irwin's fracture mechanics approach to aggregative materials (Griffith, 1924; Irwin, 1957) and the validity of direct extrapolation to larger concrete structures.

Weibull's statistical theory of strength for brittle materials (Weibull, 1939) associated with the "weakest link" approach and later modified by Bolotin (1961) and Freudenthal (1968) is, of course, a pioneering work of immense importance. It is based on criteria of failure at a point other than on the entire fracture surface. However, failure of aggregative materials like concrete does not depend solely on the largest flaw but it is a result of sequential local failures and microcracks that merge. With this respect, a statistical theory of failure which considers creation of a random fracture surface resulting from merging of local defects and a method of averaging over a set of possible fracture surfaces has been proposed by Chudnovsky (1973, 1977). An application of this approach for the toughness characterization of concrete and formulation of global failure criteria is presented.

STATISTICAL THEORY OF GLOBAL FAILURE

A statistically representative sample set Ω_s of observed fracture surfaces from an ensemble of a large number of macroscopically identical four-point bending notched cement-mortar specimens is shown in Fig. 1. It represents a set of crack trajectories $\omega(x)$ for plane-strain conditions. Statistical homo-

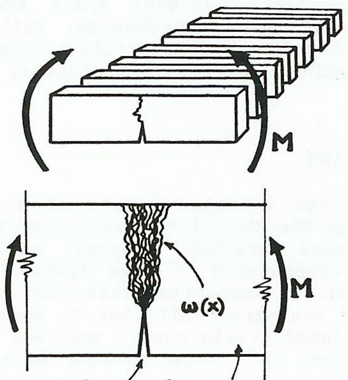

Fig. 1. Representative sample set of crack trajectories $\omega(x)$ for pure bending.

geneity in the lateral direction is reasonably satisfied. For a given notch depth to height ratio ℓ/h and sand size d, the set Ω_s can be considered as a statistical sample out of a set Ω of all possible fracture surfaces $(\omega_1, \omega_2, \ldots, \omega_n, \ldots)$ for each specimen in the considered ensemble.

In a homogeneous medium, bands $\Delta\Omega$ of crack trajectories (quasi-intervals according to Wiener's terminology) are considered instead of single crack trajectories. The band width is equal to or larger than a representative volume size sufficiently large compared to the characteristic aggregate size d. These bands can only describe the "global" roughness of the random cracks. To distinguish the "local" roughness of a crack trajectory ω, an appropriate probabilistic measure $\mu(\Omega)$ of $\Delta\Omega$ is introduced.

Then, the probability of occurrence of the random event of failure $P\{F\}$ for a continuum can be written as (Chudnovsky, 1973):

$$P\{F\} = \int_{\Omega} P[\omega] \, d\mu(\Omega) \tag{1}$$

The probability of failure $P[\omega]$ along a given crack trajectory ω can be defined as a functional of stress, strain, strength etc. along ω, based on a realistic criterion of failure for a heterogeneous material (see next section). The probabilistic measure $\mu(\Omega)$ derived from statistical analysis of the observed crack trajectories quantitatively describes their "local" roughness. Statistical analysis of the preliminary experimental results (Chudnovsky and Perdikaris, 1983) indicates that Wiener's measure and integral may be used for calculation of (1). It is worth noting that Wiener's measure for any set of smooth functions representing crack trajectories is equal to zero. In other words, Wiener's measure is concentrated exclusively on locally rough diffusion-type crack trajectories. Thus, it selects a certain class of random funtions which realistically reflect the nature of fracture process in aggregative materials.

FRACTURE TOUGHNESS MODEL

In the case of brittle failure of a homogeneous material, Griffith's criterion for crack propagation can be described by the necessary condition

$$J \geq 2\gamma \tag{2a}$$

and the sufficient condition

$$\partial J/\partial \ell > 0 \quad , \tag{2b}$$

where J is the energy release rate and γ the surface energy. Since, for the considered toughness tests of four-point bending specimens, condition (2b) is always satisfied, condition (2a) becomes the necessary and sufficient condition for crack instability in a homogeneous material.

Due to fluctuation, however, of the random field of strength (surface energy) γ in a heterogeneous material it is possible that criterion (2a) is not met at all points along a possible fracture trajectory (see Fig. 2). Therefore, for a heterogeneous material Griffith's criterion is not adequate and a more appropriate crack instability criterion has to be formulated.

The random event of failure, $\{F\}$, along a given crack trajectory $\omega(x)$ can be defined as follows:

$$\{F \cap \omega\} = \bigcap_{k=1}^{N} \{J(x_k, \omega(x_k)) \geq 2\gamma(x_k, \omega(x_k))\} \tag{3}$$

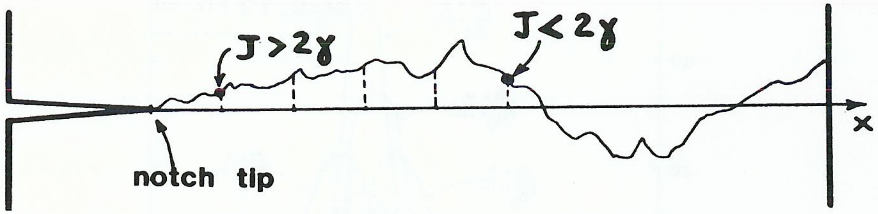

Fig. 2. Possible fracture trajectory.

for all $x \in [\ell, h]$, where ℓ is the existing crack length and h the height of the beam specimen. If the interval $[\ell, h]$ is divided into steps Δx larger than the radius of correlation r_o of the random field γ, the elementary events $J \geq 2\gamma$ in Eq. 3 are mutually independent.

Since only one continuous fracture surface ω_k has been observed in each specimen (Chudnovsky and Perdikaris, 1983) (see Fig. 3), appearance of two or more fracture surfaces in each specimen can be considered as mutually

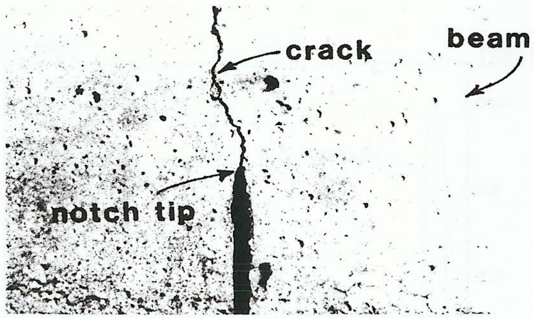

Fig. 3. Side view of fracture surface in a notched mortar beam specimen.

exclusive events for the given testing conditions. Therefore, according to Eq. 3, the probability of failure along a given trajectory

$$P\{F \cap \omega\} = P[\omega] = P \bigcap_{k=1}^{N} \left\{(J(x_k, \omega(x_k)) \geq 2\gamma(x_k, \omega(x_k))\right\}$$
$$= 1 - P \bigcap_{k=1}^{N} \left\{(J(x_k, \omega(x_k)) \leq 2\gamma(x_k, \omega(x_k))\right\} \tag{4}$$

can be expressed as follows:

$$P[\omega] = 1 - \left\{\sum_{k=1}^{N} P\,(J \leq 2\gamma) - \frac{1}{2!}\sum_{k=1}^{N} [P(J \leq 2\gamma)]^2 + \frac{1}{3!}\sum_{k=1}^{N} [P(J \leq 2\gamma)]^3 + \ldots\right\}$$
$$= 1 - \left\{1 - [\exp - \sum_{k=1}^{N} P(J \leq 2\gamma)]\right\} \tag{5}$$

Substituting summation by integration we get:

$$P[\omega] = \exp - \int_{\ell}^{h} P\,\{J(x, \omega(x)) \leq 2\gamma\,(x, \omega(x))\}\,\frac{dx}{r_o}, \tag{6}$$

where $J(x, \omega(x))$ and $\gamma(x, \omega(x))$ are both random quantities.

If $J(x, 0)$ stands for the energy release rate for the rectilinear path and we assume small deviations of the crack trajectory from the rectilinear path, J may be approximated as (Chudnovsky, Loginova and Shariber, to be published):

$$J(x, \omega(x)) \cong J(x, 0)\,[1 - \lambda \frac{\omega^2(x)}{r_o^2}], \tag{7}$$

where λ is a constant. For pure bending, $J(x, 0)$ in Eq. 7 is given by the following expression:

$$J(x, 0) = \frac{36M^2}{b^2h^3}\,\pi\,\psi^2\left(\frac{\ell+x}{h}\right), \tag{8}$$

where b and h are the specimen's width and height, M is the applied moment and $\psi(\frac{\ell+x}{h})$ a known geometrical factor (Tada, Paris and Irwin, 1973).

To completely characterize the random surface energy field $\gamma(x, \omega(x))$, which is the main source of roughness in the crack trajectories, its multidimensional joint probability density distribution function has to be determined. However, since crack propagation follows the minimal values of γ-field, the "Weibull" distribution (Weibull, 1939) for the minimum values of γ may be justified in this case.

If the location parameter γ_{min} (minimum value of strength) of the "Weibull" distribution is taken as zero, the one-dimensional cummulative distribution function of the γ-field along a crack trajectory is given by:

$$F(\gamma) = \begin{cases} 0 & , \text{ for } \gamma < 0 \\ 1 - e^{-(\frac{\gamma}{\gamma_o})^{\alpha}} & , \text{ for } \gamma \geq 0 \end{cases} \tag{9}$$

The other two Weibull parameters, that is γ_o and α (scale and shape parameter, respectively) can be determined from the experimental results.

Utilizing expressions (7) and (9), the probability of failure $P[\omega]$ from Eq. 6

reduces to:

$$P[\omega] = \exp\{-\int_{\ell}^{h} [\exp - (\frac{J(x,0)}{2\gamma_o})^{\alpha}][1 + \alpha\lambda(\frac{J(x,0)}{2\gamma_o})^{\alpha-1} \frac{\omega^2(x)}{r_o^2}]\frac{dx}{r_o} \qquad (10)$$

Substituting now the expression for $P[\omega]$ in Eq. 1, the total probability of failure $P\{F\}$ is written as:

$$P\{F\} = \exp\{-\int_{\ell}^{h} [\exp - (\frac{J(x,\ 0)}{2\gamma_o})^{\alpha}] \frac{dx}{r_o}\} \cdot \int_{\Omega} [\exp - \int_{\ell}^{h} \beta(x) \frac{\omega^2(x)dx}{r_o^2\ r_o}]d\mu(\Omega), \qquad (11)$$

$$\text{where} \quad \beta(x) = \alpha\lambda(\frac{J(x,0)}{2\gamma_o})^{\alpha-1} [\exp - (\frac{J(x,\ 0)}{2\gamma_o})^{\alpha}] \qquad (11a)$$

The Wiener-type integral $\int_{\Omega}$ in Eq. 11 can be calculated in closed form (Gelfand and Jaglom, 1956).

Finally, substituting expression (8) in (11) and performing the integration over Ω, the total probability of failure as a function of the applied moment M, the statistical parameters α, M_o and the microscopical and macroscopical geometrical characteristics of the specimen is expressed as follows:

$$P\{F\} = \begin{cases} 0 & , \text{ for } M<0 \\ \exp - \frac{h}{r_o}\int_{1}^{\infty} \xi'(\eta)\ [\exp - (\frac{M^2\eta}{M_o^2})^{\alpha}]d\eta & , \text{ for } M \geq 0, \end{cases} \qquad (12)$$

$$\text{where} \qquad \xi = \frac{\ell+x}{h} \quad \text{and} \quad \eta(\xi) = \frac{\psi^2(\xi)}{\psi^2(\ell/h)} \qquad (12a)$$

The probability density distribution function for failure f(M) can be calculated from expression (12) after determining the Weibull parameters α and M_o. At this point, utilizing the rather limited available test results it seems adequate to assume for the four-point bending cement-mortar specimens a shape parameter $\alpha = 4$. For M_o = 14.3 Nm(0.127 K-in), which corresponds to a notch depth to height ratio of $\ell/h = 0.25$, f(M) is plotted versus M for selected values of α and h/r_o in Fig. 4. The effect of the characteristic size of aggregate d is shown through the ratio h/r_o. For given α and h, decreasing the size of aggregate (that is increasing h/r_o) results in increasing the mean and decreasing the variance of M, as expected. Finally, the density function f(M) is plotted in Fig. 5 for $\alpha = 4$, $h/r_o = 35$ and various values of M_o showing the effect of notch depth on the fracture toughness of cement-mortar specimens.

CONCLUSIONS

A new approach with very promising results is described towards the formulation of a fracture toughness model characterizing the brittle failure of a heterogeneous material, in particular concrete. More extensive experimental data on fracture toughness tests of concrete are definitely needed to determine the appropriate Weibull parameters on the basis of the statistical ana-

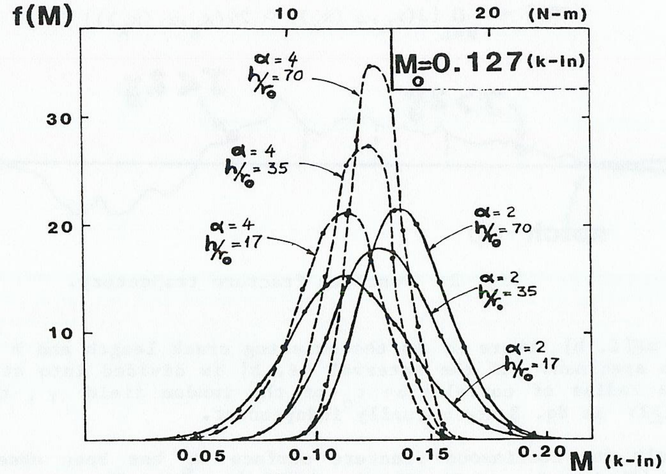

Fig. 4. Effect of α and h/r_o on f(M) ($\ell/h = 0.25$).

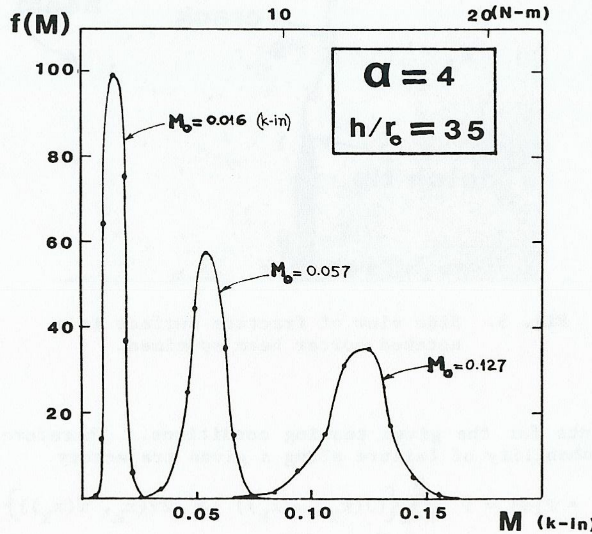

Fig. 5. Effect of M_o on f(M) ($\alpha = 4$, $h/r_o = 35$).

lysis of the test results.

It is important to emphasize that the proposed model accounts for both the global stress state through the probability of failure $P[\omega]$ and the statistical nature of the material's microstructure through the probabilistic measure $\mu(\Omega)$. The apparent diffusion features of the observed crack trajectories (Chudnovsky and Perdikaris, 1983) indicate that the "Weiner" measure can adequately describe their local roughness.

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